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Common Fixed Point of Coincidently Commuting Mappings in 2 Non-Archimedean Menger PM-Space

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Abstract—In the present paper, we prove a fixed point theorem for quasi-contraction pair of coincidentally commuting mappings in a 2 non-Archimedean Menger PM-space sings idea of Achari [1] and Chamola et.al.[2].

Keywords—2 Non-Archimedean Menger probabilistic metric space, Common fixed points, Compatible maps, coincidentally commuting maps, quasi-contraction pairs.

I. Introduction

The notion of non-Archimedean Menger space has been established by Istrătescu and Crivat [5]. The existence of fixed point of mappings on non-Archimedean Menger space has been given by Istrătescu [4]. Recently, for quasi-contraction type mappings Achari [1] has proved some fixed point theorems in non-archimedean Menger space.

Sessa [10] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Using this idea, Singh and Pant [14] gave fixed point theorems for non-archimedeanMenger spaces. Jungck [7] soon enlarged the concept of Sessa [10] to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [9].

Recently, Chamola, Dimri and Pant [2] introduced the notion of weak com-mutativity in Menger spaces. Later on, Jungck and Rhoades [8] termed a pair of self-maps to be coincidentally commuting, or equivalently weakly compatible if they commute at their coincidence point.

II. Preliminaries.

Definition 1. A mapping $F : R \to R^+$ is called a distribution if it is non-decreasing left continuous with

$$\operatorname{Inf}\{F(t) \mid t \in R\} = 0 \text{ and } \sup\{F(t) \mid t \in R\} = 1.$$

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

Definition 2. A triangular norm * (shortly t-norm) is a binary operation on the unit interval [0, 1] such that for all a, b, c, d $\in$ [0, 1] the following conditions are satisfied :

(t-1) $a * 1 = a$,

(t-2) a * b = b * a,
(t-3) a * b ≤ c * d whenever a ≤ c and b ≤ d,
(t-4) a * (b * c) = (a * b) * c.
Examples of t-norms are a * b = max {a + b – 1, 0} and a * b = min {a,b} .

Proposition 1. If (X, d) is a metric space, then the metric d induces amapping $X \times X \rightarrow L$ defined by
$F_{p,q}(x) = H(x - d(p, q))$, $p, q \in X$ and $x \in R$.
Further, if the t-norm $t : [0,1] \times [0, 1] \rightarrow [0,1]$ is defined by
t(a, b) = min{a , b},

then(X, F, t) is a Menger space. It is complete if (X, d) is complete.
The space (X, F, t) so obtained is called the induced Menger space.
Following Achari [1], the induced non- archimedean Menger space is also well defined.

Definition 3. Let (X,F, t) be a 2 non- archimedean Menger space. Two map-pings f and g on X will be called a quasi-contraction pair (f;g) if and only if there exists a constant $h \in (0, 1)$ such that for all $u, v \in X$

a. $_{fu,fv}(hx) \geq \max \{F_{gu,gv}(x), F_{fu,gu}(x), F_{fv,gv}(x), F_{fu,gv}(x), F_{fv,gu}(x)\}$

holds for all $x \geq 0$.

Definition 4. Let X be a non-empty set and D be the set of all left-continuous distribution functions. An ordered pair (X, F) is called a non-Archimedean probabilistic metric space (shortly a N.A. PM-space) if F is a mapping from X×X×X into D satisfying the following conditions (the distribution function F(u,v,w) is denoted by $F_{u,v,w}$ for all $u, v, w \in X$) :
(PM-1) $F_{u,v,w}(x) = 1$, for all $x > 0$, if and only if at least two of the three points are equal,
(PM-2) $F_{u,v,w} = F_{u,w,v} = F_{w,v,u}$,
(PM-3) $F_{u,v,w}(0) = 0$,
(PM-4) If $F_{u,v,s}(x_1) = 1$, $F_{u,s,w}(x_2) = 1$ and $F_{s,v,w}(x_3) = 1$
then $F_{u,v,w}(\max\{x_1, x_2, x_3\}) = 1$
for all $u, v, w, s \in X$ and $x_1, x_2, x_3 \geq 0$.

Definition 5.A 2 N.A. Menger PM-space is an ordered triple (X, F,t), where (X,F) is a non-Archimedean PM-space and t is a t-norm satisfying the following condition:
(PM-5) $F_{u,w}(\max\{x,y\}) \geq t\,(F_{u,v}(x)\, F_{v,w}(y))$,

for all $u, v, w \in X$ and $x, y \geq 0$.

Definition 6.A 2 N.A. PM-space (X, F, t) is said to be of type $(C)_g$ if there exists a $g \in \Omega$ such that
$g(F_{x,y,z}(t)) \leq g(F_{x, y, a}(t)) + g(F_{x, a, z}(t)) + g(F_{a, y, z}(t))$
for all $x, y, z, a \in X$ and $t \geq 0$, where $\Omega = \{g \mid g : [0,1] \rightarrow (0,\infty)$ is continuous, strictly decreasing, $g(1) = 0$ and $g(0) < \infty \}$.

Definition 7. A 2 N.A. Menger PM-space (X, F, t) is said to be type $(D)_g$ if there exists a $g \in \Omega$ such that
$g(\Delta(t_1, t_2, t_3) \leq g(t_1) + g(t_2) + g(t_3)$
for all $t_1, t_2, t_3 \in [0,1]$.

Remark 1.
(1) If a 2 N.A. Menger PM-space (X, F, t) is of type $(D)_g$ then (X, F, Δ) is of type $(C)_g$.

(2) If a 2 N.A. Menger PM-space (X, F, t) is of type $(D)_g$, then it is metrizable, where the metric d on X is defined by d(x,y) = for all x, y, a ∈ X. (*)

Throughout this paper, suppose (X, F, t) be a complete 2 N.A. Manger PM-space of type $(D)_g$ with a continuous strictly increasing t-norm.

Letϕ: $[0,+\infty) \rightarrow [0,\infty)$ be a function satisfied the condition (Φ) :
(Φ) ϕ is upper-semi continuous from the right and $\phi(t)< t$ for all $t > 0$.

Definition 8. Self mapsS and T of a 2 non-archimedean Menger space (X,F, t) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if Sp = Tp for some p∈ X, thenSTp = TSp.

The following is an example of pair of self-maps in a 2 non- Archimedean Menger space which are weakly compatible but not compatible.

Example 1. Let (X, F, t) be the 2 N.A. Menger PM-space, where X = [0, 2] and the metric d on X is defined in condition (*) of remark 1. Define self-maps A and S as follows:

$$Ax = \begin{cases} 2-x, & \text{if} \quad 0 \le x < 1, \\ 2, & \text{if} \quad 1 \le x \le 2, \end{cases} \quad \text{and}$$

$$Sx = \begin{cases} x, & \text{if} \quad 0 \le x < 1, \\ 2, & \text{if} \quad 1 \le x \le 2. \end{cases}$$

Take $x_n = 1- 1/n$. Now

$F_{Axn,1}(\varepsilon) = H(\varepsilon- (1/n))$

Therefore, $\lim_{n\to\infty} (F_{Axn,1}(\varepsilon)) =1$

Then $Ax_n \rightarrow 1$ as $n \rightarrow \infty$ Similarly, $Sx_n \rightarrow 1$ as $n \rightarrow \infty$

Also

$F_{ASxn,SSxn,\,a}(\varepsilon) = H(\varepsilon - (1\text{-}1/n))$,

$F_{ASxn,SSxn,\,a}(\varepsilon) = H(\varepsilon - 1)\neq 1, \quad \forall \varepsilon > 0$

Hence, the pair (A,S) is not compatible.

Also, set of coincidence point of A and S is [1,2].

Now for any x ∈ [1,2],

Ax = Sx = 2 and

AS(x) = A(2) = 2 = S(2) = SA(x).

Thus A and S are weakly compatible but not compatible.

From the above example it is obvious that the concept of weak compatibility is more general than that of compatibility .

Lemma 1. Let$\{y_n\}$be a sequence in a 2 non- archimedean Menger space(X,F, t), where t is continuous and satisfies $t(x, x) \geq x$ for every x∈ [0, 1].If there exists an h ∈ (0,1) such that

$F_{y_n,y_{n+1},a}(hx) \geq F_{y_{n-1},y_{n,a}}(x), n = 1,2, \ldots$

for all x >0, then $\{y_n\}$ is a Cauchy sequence in X.

III. Main Results

Theorem 1. Let (X, **F**, t) be a 2 non-Archimedean Menger space, where t is continuous and satisfies $t(x, x) \geq x$ for every x∈ [0, 1]. Let (f, g) be a quasi-contraction pair of coincidentally commuting mappings on X satisfying

(i) There exists a sequence $\{u_n\}$ in X such that $fu_n = gu_{n+1}$,

(ii) The sequence $\{gu_n\}$ has a subsequence converging to a point in g(X):

Then f and g have a unique common fixed point and $\{gu_n\}$ converges to the fixed point.

Proof . Noting that

$F_{gu_{n+1},gu_{n-1},a}(x)=F_{gu_{n+1},gu_{n-1},a}(\max(xh, x))$ ¸
$\geq \max(F_{gu_{n+1},gu_{n},a}(xh),F_{gu_n,gu_{n-1},a}(x))$,

(a) and (i) give

$F_{gu_n,gu_{n+1},a}(xh) \geq F_{gu_{n-1},gu_{n},a}(x)$.

So, in view of the Lemma 1.$\{gu_n\}$ is a Cauchy sequence and by virtue of (ii), converges to a point p (say) in g(X). This implies the existence of a point z in X such that gz = P.
Now, let $U_{gz}(\varepsilon , \lambda)$ be a neighbourhood of gz. Then, for ε, λ>0,there exists an integer N such that

(b) $F_{gu_{n},gz,a}(\varepsilon / h) > 1- \lambda$ and $F_{gu_{n},gu_{n+1},a}(\varepsilon / h) > 1- \lambda$
For all $n \geq N$

Also by (a),

$F_{fz,gu_{n+1},a}(\varepsilon) \geq \max\{F_{gz,gu_{n},a}(\varepsilon / h), F_{fz,gz,a}(\varepsilon / h),$

$F_{gu_{n+1},gu_{n},a}(\varepsilon / h), F_{fz,gu_{n},a}(\varepsilon / h), F_{gu_{n+1},gz,a}(\varepsilon / h)),$

$\geq \max\{F_{gz,gu_{n},a}(\varepsilon / h), F_{fz,gu_{n+1},a}(\varepsilon),$

$F_{gu_{n+1},gz,a}(\varepsilon / h),F_{gu_{n+1},gu_{n},a}(\varepsilon / h), F_{fz,gu_{n+1},a}(\varepsilon /h)) , F_{gu_{n+1},gu_{n},a}(\varepsilon / h), F_{gu_{n+1},gz,a}(\varepsilon / h)\}$

$>1 - \lambda$ by (b).

Thus fz = gz. Since f and g are coincidentally commuting, we have fgz = gfz:

Also, fgz = gfz = ffz = ggz:

Now, the application of (a) gives
ffz = fz (= gz = p).
The uniqueness of fz as the common fixed point of f and g can be easily seen from (a).

Remark. The contraction conditions used by Achari [1] and Istrătescu [4] are special cases of (a). Thus the results of Achari [1] and Istrătescu [4] are obtained as special cases of the above result. Our result extends to the results of Sehgal and Bharucha-Reid [10] and Sherwood [12] to 2 non-archimedean menger pm space.

Corollary 3.1. Let (X, d) be a 2 non-archimedean metric space and (f,g) a coincidentally commuting pair of self mappings on X satisfying

a. $d(fu,fv) \leq h \max\{d(fu,gu), d(fv,gv), d(gu,gv), d(fv,gu), d(fu,gv)\}$
for all u, v X,
b. there exists a sequence $\{u_n\}$ in X such that
$gu_{n+1}= fu_n$, n = 1,2 …;

c. the sequence $\{gu_n\}$ has a subsequence converging to a point in g(X):

Then f and g have a unique common fixed point and $\{gu_n\}$converges to the fixed point.

Proof. The proof follows from Theorem 3.1 and by considering the induced2 non-archimedeanMenger space (X,F, t), where t(a, b) = min{a, b}
and
$F_{p,q}(x)$ = H(x-d(p, q)), H being the distribution function as given in Definition1.

IV. REFERENCES

[1] J. Achari, " *Fixed point theorems for a class of mappings on non-archimedean probabilistic metric space*, Mathematics, Vol.25, Issue.6, pp.5-9, 1983.

[2] KP.Chamola, Dimri R.G., Pant B.D., *"Onnon linearcontrac-tions on Menger spaces"*, Ganita, Vol.39, Issue.2, pp.49-50, 1988.

[3] Cho Y.J., Ha K.S., Chang S.S., "*Common fixed point theorems for compatible mappings of type (A) in non-Archimedean Menger PM-spaces*", Math. Japonica, Vol.48, Issue.1, pp.169-179, 1997.

[4] Istrătescu V.I., "*Fixed point theorems for some classes of contraction mappings on nonarchimedean probabilistic metric space*", Publ. Math. (Debrecen), Vol.25, Isssue.1-2, pp.29-34, 1978.

[5] Istrătescu V.I., Crivat N., "*On some classes of nonarchimedean Menger spaces, Seminar de spatii Metriceprobabiliste*", Univ. Timisoara Nr., Vol.1, Issue.2, pp.9-16, 1974.

[6] Jungck G., "*Compatible mappings and common fixed points*", Int. J. Math. and Math. Sci., Vol.9, Issue.4, pp.771-779, 1986.

[7] Jungck G., Rhoades B.E., "*Fixed points for set valued functions without continuity*", Indian J. Pure Appl. Math., Vol.29, Issue.1, pp.227-238, 1998.

[8] Menger K., "*Statistical metrics"*, Proc. Nat. Acad. Sci., USA., pp.535 -537, 1942.

[9] Mishra S.N., "*Common fixed points of compatible mappings in PM-spaces*", Math. Japon., Vol.36, Issue.2, pp.283-289, 1991.

[10] Sehgal VM., Bharucha-Reid A.T., "*Fixed points of contractionmappings on probabilistic matric spaces*", Math. Systems Theory, Vol.6, Issue.1, pp.97-102, 1972.

[11] Sessa S., "*On a weak commutativity condition in fixed point considerations*", Publ. Inst. Math., Vol.32, Issue.46, pp.149-153, 1982.

[12] Sherwood Howard, "*Complete probabilistic metric spaces*", *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, Vol.20, Issue.2, pp.117-128, 1971.

[13] Singh S.L., "*A note on recent fixed point theorems for commuting mappings*", Vijnana Parishad Anusandhan Patrika, Vol.26, Issue.4, pp.259-261, 1983.

[14] Singh, S.L., Pant B.D., Common fixed point theorems in probabilisticmetric spaces and extension to uniform spaces, Honam Math. J., 6(1984),1-12.

[15] Singh S.L., Pant B.D., "*Common fixed points of weakly commutingmappings on non-archimedean Menger spaces*", Vikram Math. J., Vol.6, Issue.2, pp.27- 31, 1986.

[16] Tiwari B.M.L., Pant, B.D., "*Fixed points of a pair of mappings onprobabilistic metric spaces*", Jnanabha, Vol.13, Issue.1, pp.13-25, 1983.