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Strong Convergence Theorems for Coincident Points of Banach Operator Pair

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Abstract—We obtain results concerning strong convergence of coincident fixed points of asymptotically I-nonexpansive map T for which (T, I) is a Banach operator pair in a Banach space with uniformly Gateaux differentiable norm. Several coincident point and best approximation results for this newly defined class of maps are proved.

Keywords—*Banach, Gateaux, Approximation*

I. INTRODUCTION

We first give a brief overview of definitions and specific results that will be used throughout this paper. Let M be a subset of a normed space $(X, \|.\|)$. The set $P_M(u) = \{x \text{ e } M : \|x \in u\| = dist(u, M)\}$ is called the *set of best approximants* to $u \in X$ out of M, where $dist(u,M) = \inf\{\|y - u\| : y \in M\}$. We shall use N to denote the set of positive integers, *cl(M)* to denote the closure of a set M and *wcl(M)* to denote the weak closure of a set M. Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called an I-*contraction* if there exists $0 \leq k < 1$ such that $\|Tx - Ty\| \leq k\|Ix - Iy\|$ for any $x,y \in M$. If k = 1, then T is called I-*nonexpansive.* The map T is called *asymptotically I-nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_n k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|Ix - Iy\|$ for all $x,y \in M$ and n = 1,2,3, … . The map T is called *uniformly asymptotically regular* on M, if for each n > 0, there exists N(η) = N such that $\|T^n x - T^{n+1}x\| < \eta$ for all $n \geq N$ and all $x \in M$. The set of invariant points of T (resp. I) is denoted by *F(T)* (resp. *F(I)*). A point $x \in M$ is a *coincidence point* of I and T if Ix = Tx (x = Ix = Tx). The set of coincidence points of I and T is denoted by C(I, T). The pair {I, T} is called:

(1) *commuting* if TIx = ITx for all $x \in M$;

(2) *R-weakly commuting* if for all $x \in M$, there exists R > 0 such that $\|ITx - TIx\| \leq R\|Ix - Tx\|$. If R = 1, then the maps are called *weakly commuting;*

(3) *compatible* if $\lim_n \|TIx_n - ITx_n\| = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some *t* in M;

(4) *weakly compatible* if they commute at their coincidence points, i.e., if ITx = TIx whenever Ix = Tx.

The set M is called *q-starshaped* with $q \in M$, if the segment $[q, x] = \{(1 - k)q + kx : 0 < k < 1\}$ joining q to x is contained in M for all $x \in M$. Suppose that M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then T and I are called:

(5) C_q-*commuting* if ITx = TIx for all $x \in C_q(I, T)$,

where $C_q(I, T)= \bigcup \{C(I, T_k) : 0 \leq k \leq 1\}$ where $T_k x = (1 - k)q + kTx$;

(6) *R-subweakly commuting* on M if for all $x \in M$, there exists a real number R > 0 such that $\|ITx - TIx\| \leq R \, dist(Ix, [q,Tx])$;

(7) *uniformly R-subweakly commuting* on M \ {q} if there exists a real number R > 0 such that $\|IT^n x - T^n Ix\| \leq R \, dist(Ix, [q,T^n x])$, for all $x \in M \setminus \{q\}$ and $n \in N$.

C_q-commuting maps are weakly compatible but not conversely in general and uniformly R-subweakly commuting maps are R-subweakly commuting and R-subweakly commuting maps are C_q-commuting but the converse does not hold in general.

The *normal structure coefficient* *N*(X) of a Banach space X is defined by $N(X) = \inf\{\frac{diam(C)}{r_C(C)} : C$ is nonempty bounded convex subset of X with *diam(C)* > 0}, where $r_C(C) = \inf_{x \in C}\{\sup_{y \in C} \|x - y\|\}$ is the Chebyshev radius of C relative to itself and $diam(C) = \sup_{x,y \in C} \|x - y\|$ is diameter of C.

The space X is said to have the uniformly normal structure if

$N(X) > 1$

A *Banach limit LIM* is a bounded linear functional on l^∞ such that

$$\lim_{n\to\infty} \inf t_n \le LIMt_n \le \limsup_{n\to\infty} t_n \text{ and } LIMt_n = LIMt_{n+1},$$

for all bounded sequences $\{t_n\}$ in l^∞.

Let $\{x_n\}$ be a bounded sequence in X. Then we can define the real- valued continuous convex function f on X by $f(z) = LIM\|x_n - z\|^2$ for all $z \in X$.

The following lemmas are well known.

Lemma 1.1 Let X be a Banach space with uniformly Gateaux differentiable norm and $u \in X$. Let $\{x_n\}$ be a bounded sequence in X. Then $f(u) = \inf_{z\in X} f(z)$ if and only if $LIM(z, J(x_n - u)) = 0$ for all $z \in X$, where $J : X \to X^*$ is the normalized duality mapping and **(.,.)** denotes the generalized duality pairing.

Lemma 1.2 Let C be a convex subset of a smooth Banach space *X*, *D* be a nonempty subset of *C* and *P* be a retraction from *C* onto *D*. Then *P* is sunny and nonexpansive if and only if $(x - Px, J(z - Px)) \le 0$ for all $x \in C$ and $z \in D$.

Definition. Let M be a nonempty closed subset of a Banach space X, I, T : M → M be mappings and $C = \{x \in M : f(x) = \min_{z\in M} f(z)\}$. Then I and T are said to satisfy the *property* (S) if the following holds:

for any bounded sequence $\{x_n\}$ in M, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ implies $C \cap F(I) \cap F(T) \ne \phi$.

We have strong convergence theorems in the framework of Hilbert spaces with implicit and explicit iteration, respectively. These results have been extended in various directions. The following extension is in this direction :

Theorem 1.4. Let *M* be a bounded closed convex subset of a uniformly smooth Banach space *X*. Let *T* be a nonexpansive self-map on *M*. Fix $u \in M$ and define a net $\{y_\alpha\}$ in *M* by $y_\alpha = (1-\alpha)Ty_\alpha + \alpha u$ for $\alpha \in (0,1)$. Then $\{y_\alpha\}$ converges strongly to $Pu \in F(T)$ as a tends to $+0$, where *P* is the unique sunny nonexpansive retraction from *M* onto *F(T)*.

Theorem 1.5. *Let* X, M, T, P *and* u be as in *Theorem 1.4. Define a sequence* $\{y_n\}$ *in* M *by* $y_1 \in M$, $y_{n+i} = (1 - \alpha n)Ty_n + \alpha_n u$, where $\{\alpha_n\}$ is *a real sequence in* (0,1) *satisfying*

(C_1) $\lim_{n\to\infty} \alpha_n = 0$

(C_2) $\sum_{n=1}^{\infty} \boldsymbol{\alpha_n} = \infty$

and $\lim_{n\to\infty} \dfrac{\alpha_n + 1}{\alpha_n} = 1$. Then $\{y_n\}$ converges strongly to *Pu*.

Further, generalizations of the above mentioned results were studied in various papers.

In this paper, we improve and extend the recent coincident point results to the class of asymptotically I-nonexpansive maps T for which (T, I) are Banach operator pairs and as an application, we establish more general approximation results without the condition of linearity or affinity of I which is key assumption in the results of many authors. We also study strong convergence of coincident points of asymptotically I-nonexpansive map T for which (T, I) is a Banach operator pair in a Banach space with uniformly Gateaux differentiable norm.

II. MAIN RESULTS

The ordered pair (T, I) of two self maps of a Banach space X is called a *Banach operator pair,* if the set F(I) is T-invariant, namely $T(F(I)) \subseteq F(I)$. Obviously any commuting pair (T, I) is a Banach operator pair but not conversely in general. If (T, I) is a Banach operator pair then (I, T) need not be Banach operator pair. If the self-maps T and I of X satisfy

$$\|ITx - Tx\| \le k\|Ix - x\|,$$

for all $x \in X$ and $k \ge 0$, then (T, I) is a Banach operator pair. In particular, when I = T and X is a normed space, the above inequality can be rewritten as

$$\|T^2x - Tx\| \le k\|Tx - x\|,$$

for all $x \in X$. Such T is called *Banach operator of type* k.

The Banach Contraction Mapping Principle states that if X is a Banach space, C is a nonempty closed subset of X and T : C → C is a self-mapping satisfying $\|Tx - Ty\| \le \lambda\|x - y\|$ for all $x, y \in C$, where $0 < \lambda < 1$, then T has a unique invariant, say z in C, and the Picard iteration $\{T^n x\}$ converges to z for all $x \in C$. Further we have the following extension

$\|Tx - Ty) \le \lambda \max \{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|y-Tx\|\},$

where $0 < \lambda < 1$.

The following result is a consequence of the above extension.

Theorem 2.1. Let M be a subset of a Banach space X, and I and T be weakly compatible self-maps of M. Assume that $clT(M) \subset I(M)$, $clT(M)$ is complete, and T and I satisfy for all $x,y \in M$ and $0 \leq h < 1$,

$$\|Tx-Ty\| \leq h \max\{\|Ix-Iy\|, \|Ix-Tx\|, \|Iy-Ty\|, \|Ix-Ty\|, \|Iy-Tx\|\}.$$

Then $M \cap F(I) \cap F(T)$ is a *singleton.*

The following result is being important in our discussion.

Lemma 2.2. Let M be a nonempty subset of a Banach space X and (T, I) be a Banach operator pair on M. Assume that $clT(M)$ is complete, and T and I satisfy for all $x, y \in M$ and $0 \leq h < 1$,

$$\|Tx - Ty\| \leq h\max\{\|Ix-Iy\|, \|Tx-Ix\|, \|Ty-Iy\|, \|Tx-Iy\|, \|Ty-Ix\|\} \quad (2.1)$$

If I is continuous and $F(I)$ is nonempty, then there exists a unique coincident point of T and I.

Proof. By our assumptions, $T(F(I)) \subseteq F(I)$ and $F(I)$ is nonempty and closed. Moreover, $cl(T(F(I)))$ being subset of $cl(T(M))$ is complete. Further for all $x, y \in F(I)$, we have by inequality (2.1),

$$\begin{aligned}\| Tx-Ty\| &\leq h.\max\{\|Ix-Iy\|, \|Ix-Tx\|, \|Iy-Ty\|, \|Iy-Tx\|, \|Ix-Ty\|\}\\ &= h.\max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|y-Tx\|, \|x-Ty\|\}.\end{aligned}$$

Hence T is a generalized contraction on $F(I)$ and $cl(T(F(I))) \subseteq cl(F(I)) = F(I)$. T has a unique fixed point z in $F(I)$ and consequently $F(I) \cap F(T)$ is singleton.

The following result presents the analogue of the above result for Banach operator pair without the linearity of I.

Lemma 2.3. Let I and T be self-maps on a nonempty q-starshaped subset M of a normed space X. Assume that I is continuous and $F(I)$ is q-starshaped with $q \in F(I)$, (T, I) *is* Banach operator pair on M and satisfy for each $n \geq 1$

$$\| T^n x - T^n y \| \leq k_n \max\left\{\begin{matrix}\| Tx - Iy \|, \text{dist}(Ix,[q,T^n x]), \text{dist}(Iy,[q,T^n y]),\\ \text{dist}(Ix,[q,T^n y]), \text{dist}(Iy,[q,T^n x])\end{matrix}\right\} \quad (2.2)$$

for all $x,y \in M$, *where* $\{k_n\}$ *is* a sequence of real numbers with $k_n \geq 1$ and $\lim_n k_n = 1$. *For each* $n \geq 1$, define a mapping T_n *on* M by

$$T_n x = (1 - \mu_n)q + \mu_n T^n x,$$

where $\mu_n = \dfrac{\lambda_n}{k_n}$ $\{\lambda_n\}$ *is* a sequence of numbers in $(0,1)$ such that $\lim_n \lambda_n = 1$. *Then* for each $n \geq 1$, T_n and I have exactly one common fixed point x_n in M such that

$$Ix_n = x_n = (1 - \mu_n)q + \mu_n T^n x_n,$$

provided one of the following conditions hold;

(i) $cl(T_n(M))$ is *complete for each* n,

(ii) *for each* n, $wcl(T_n(M))$ is *complete.*

Proof. By definition,

$$T_n x = (1 - \mu_n)q + \mu_n T^n x.$$

As (T, I) is Banach operator pair, for each $n \geq 1$, $T^n(F(I)) \subseteq F(I)$ and $F(I)$ is nonempty and closed. Since $F(I)$ is q-starshaped and $T^n x \in F(I)$, thus for each $x \in F(I)$, $T_n x = (1 - \mu_n)q + \mu_n T^n x \in F(I)$. Thus (T_n, I) is Banach operator pair for each n. Also by (2.2),

$$\|T_n x - T_n y\| = \mu_n\|T^n x - T^n y\|$$

$\leq \lambda_n \max\{\|Ix - Iy\|, \text{dist}(Ix, [q,T^n x]), \text{dist}(Iy, [q,T^n y]), \text{dist}(Ix, [q, T^n y]), \text{dist}(Iy, [q, T^n x])\} \leq \lambda_n \max\{\|Ix - Iy|, \|Ix - T_n x\|, \|Iy - T_n y\|, \|Ix - T_n y|, \|Iy - T_n x|\}$,

for each $x, y \in M$.

(i) By Lemma 2.2, for each $n \geq 1$, there exists a unique $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. Thus for each $n \geq 1$, $M \cap F(T_n) \cap F(I) \neq \phi$.

(ii) Conclusion follows from Lemma 2.2.

The following result is the extension for asymptotically I-nonexpansive maps.

Theorem 2.4. Let I and T *be* self-maps on a q-starshaped subset M of a normed space X. Assume that (T, I) is Banach operator pair on M, $F(I)$ *is* q-starshaped with $q \in F(I)$, I *is* continuous, T *is* uniformly asymptotically regular and asymptotically I-nonexpansive. Then $F(T) \cap F(I) \neq 0$, provided one of the following conditions hold:

(i) $cl(T(M))$ *is* compact and T *is* continuous;

(ii) *X is* complete, *I is* weakly continuous, *wcl(T(M)) is* weakly compact and either *I − T is* demiclosed at **0** *or X* satisfies Opial's condition.

Proof. (i) Notice that compactness of cl(T(M)) implies that $clT_n(M)$ is compact and thus complete. From Lemma 2.3, for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Ix_n = (1 - \mu_n)q + \mu_n T^n x_n$. As T(M) is bounded, so $\|x_n - T^n x_n\| = (1 - \mu_n)\|T^n x_n - q\| \to 0$ as $n \to \infty$. Since (T, I) is Banach operator pair and $Ix_n = x_n$, so $IT^n x_n = T^n Ix_n = T^n x_n$, thus we have

$$\|x_n - Tx_n\| = |x_n - T^n x_n\| + \|T^n x_n - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\|$$

$$\leq |x_n - T^n x_n| + |T^n x_n - T^{n+1}x_n\| + k_1 \|IT^n x_n - Ix_n\|$$

$$= \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1}x_n\| + k_1 \|T^n x_n - x_n\|.$$

Further, T is uniformly asymptotically regular, therefore we have

$$\|x_n - Tx_n\| \leq |x_n - T^n x_n\| + \|T^n x_n - T^{n+1}x_n\| + k_1\|T^n x_n - x_n\| \to 0,$$

as $n \to \infty$. Since cl(T(M)) is compact, there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to y$ as $m \to \infty$. By the continuity of I and T and the fact $\|x_m - Tx_m\| \to 0$, we have $y \in F(T) \cap F(I)$. Thus $F(T)\cap F(I) \neq \phi$.

(iii) The weak compactness of wclT(M) implies that $wclT_n(M)$ is weakly compact and hence complete due to completeness of X. From Lemma 2.3, for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Ix_n = (1-\mu_n)q+\mu_n T^n x_n$. The analysis in (i), implies that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. The weak compactness of wclT(M) implies that there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to $y \in M$ as $m \to \infty$. Weak continuity of I implies that Iy = y. Also we have, $Ix_m - Tx_m = x_m - Tx_m \to \mathbf{0}$ as $m \to \infty$. If I − T is demiclosed at **0,** then Iy = Ty. Thus $F(T) \cap F(I) \neq 0$.

If X satisfies Opial's condition and $y \neq Ty$, then

$$\liminf_{m\to\infty} \|xm - y\| < \liminf_{m\to\infty} \|x_m - Ty\|$$

$$\leq \liminf_{m\to\infty} \|x_m - Tx_m\| + \liminf_{m\to\infty} \|Tx_m - Ty\|$$

$$= \liminf_{m\to\infty} \|Tx_{\mathbf{m}} - Ty\| \leq \liminf_{m\to\infty} k_{\mathbf{m}}\|Ix_{\mathbf{m}} - Iy\|$$

$$= \liminf_{m\to\infty} \|x_m - y\|.$$

which is a contradiction. Thus Iy = Ty = y and hence $F(T) \cap F(I) \neq 0$.

Corollary 2.5. Let *I* and *T be* self- maps on a q-starshaped subset *M* of a normed space *X*. Assume that *(T, I) is* Banach operator pair on *M*, *F(I) is* q-starshaped with $q \in F(I)$, *I is* continuous, *T is I*-nonexpansive. Then $F(T) \cap F(I) = 0$, provided one of the following conditions holds;

(i) cl(T(M)) is *compact*;

(ii) *X is* complete, *I is* weakly continuous, *wcl(T(M)) is* weakly compact and either *I − T is* demiclosed at **0** *or X* satisfies Opial's condition.

Corollary 2.6. Let *I* and *T* be self-maps on a *q*-starshaped subset *M* of a normed space *X*. Assume that *(T, I)* is commuting pair on *M*, *F(I)* is *q*-starshaped with $q \in F(I)$, *I is* continuous, *T is I*-nonexpansive. Then $F(T) \cap F(I) = 0$, provided one of the conditions in Corollary *2.5* holds.

Theorem 2.7. Let *M* be a subset of a normed space *X* and *I*, $T : X \to X$ be mappings such that $u \in F(I) \cap F(T)$ for some $u \in X$ and $T(dM \cap M) \subseteq M$. Suppose that $P_M(u)$ *is* nonempty and q-starshaped, *I is* continuous on $P_M(u)$, $\|Tx - Tu\| \leq \|Ix - Iu\|$ for each $x \in P_M(u)$ and $I(P_M(u)) \subseteq P_M(u)$. If *(T, I) is* Banach operator pair on $P_M(u)$, *F(I) is* nonempty and q-starshaped for $q \in F(I)$, *T* is uniformly asymptotically regular and asymptotically *I*-nonexpansive then $P_M(u) \cap F(I) \cap F(T) \neq 0$, provided one of the following conditions is satisfied;

(i) *T* is continuous and $cl(T(P_M(u)))$ *is* compact*;*

(ii) *X is* complete, $wcl(T(P_M(u)))$ *is* weakly compact, *I is* weakly continuous and either *I − T* is demiclosed at **0,** or *X* satisfies Opial's condition.

Proof. Let $x \in P_M(u)$. Then for any $h \in (0,1)$, $\|hu+(1-h)x-u\| = (1-h)\|x-u\| < dist(u, C)$. It follows that the line segment $\{hu + (1 - h)x : 0 < h < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subseteq M$, Tx must be in M. Also $Ix \in P_M(u)$, $u \in F(I) \cap F(T)$ and I and T satisfy $\|Tx - Tu\| \leq \|Ix - Iu\|$, thus we have

$$\|Tx - u\| = \|Tx - Tu\| \leq \|Ix - Iu\| = \|Ix - u\| = dist(u,M).$$

It further implies that $Tx \in P_M(u)$. Therefore T is a self map of $P_M(u)$. The result now follows from Theorem 2.4.

Now, we are ready to prove strong convergence to coincident points of asymptotically I-nonexpansive mapping for which (T, I) is Banach operator pair.

Theorem 2.8. Let M be a closed convex subset of a reflexive Banach space X with uniformly Gateaux differentiable norm. Let I and T be continuous self-maps on M such that $F(I)$ *is* q-starshaped with $q \in F(I)$. Suppose that T is uniformly asymptotically regular and asymptotically I-nonexpansive with a sequence $\{k_n\}$. Let $\{\lambda_n\}$ be sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} \frac{k_n - 1}{k_n - \lambda_n} = 0$. If (T, I) *is a* Banach operator pair on M, then we have the following:

(a) For each $n \geq 1$, there is exactly one x_n in M *such* that

$$Ix_n = x_n = (1 - \mu_n)q + \mu_n T^n x.$$

(b) If $\{x_n\}$ is bounded and I and T satisfy property *(S)*, then $\{x_n\}$ converges strongly to $Pq \in F(T) \cap F(I)$, where P is the sunny nonexpansive retraction from M onto $F(T)$.

Proof . (a): Follows from Lemma 2.3.

(b): Since $\{x_n\}$ is bounded, we can define a function $f : M \to R^+$ by

$$f(z) = \mathrm{LIM}\, \|x_n - z\|^2$$

for all $z \in M$. As f is continuous and convex, $f(z) \to \infty$ as $\|z\| \to \infty$ and X is reflexive, $f(z_0) = \min_{z\in M} f(z)$ for some $z_0 \in M$. Clearly, the set $C = \{x \in M : f(x) = \min_{z\in M} f(z)\}$ is nonempty. Since $\{x_n\}$ is bounded and I and T satisfy property (S), it follows that $C \cap F(I) \cap F(T) \neq 0$. Next suppose that $v \in C \cap F(I) \cap F(T)$, then by Lemma 1.1, we have

$$\mathrm{LIM}(x - v, J(x_n - v)) \leq 0 \text{ for all } x \in M.$$

In particular, we have

$$\mathrm{LIM}(q - v, J(x_n - v)) \leq 0.$$

From (2.3), we have

$$x_n - T^n x_n = (1 - \mu_n)q - T^n x_n = \frac{1-\mu_0}{\mu_n}(q - x_n).$$

Now, for any $v \in F(I) \cap F(T)$, we have

$$(x_n - T^n x_n, J(xn - v)) = (x_n - v - T^n v - T^n x_n, J(x_n - v))$$

$$\geq -(k_n - 1)\|x_n - v\|^2$$

$$\geq -(k_n - 1)K^2,$$

for some $K > 0$. It follows from (2.5) that

$$(x_n - q, J(x_n - v)) \leq \frac{k_n - 1}{k_n - \lambda_n} = 0 \; K^2.$$

Thus we have

$$\mathrm{LIM}(x_n - q, J(x_n - v)) \leq 0.$$

This together with (2.4) implies that $\mathrm{LIM}(x_n - v, J(x_n - v)) = \mathrm{LIM}\|x_n - v\|^2 = 0.$ (2.3)

Therefore there is a subsequence $\{x_m\}$ of $\{x_n\}$ which converges strongly to v. Suppose there is another subsequence $\{xj\}$ of $\{x_n\}$ which converges strongly to y (say). Since T is continuous and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, y is a fixed point of T. It follows from (2.6) that

$$(v - q, J(v - y)) \leq 0 \text{ and } (y - q, J(y - v)) \leq 0.$$

Adding these two inequalities, we get $(v - y, J(v - y)) = \|v - y\|^2 \leq 0$ and thus $v = y$. Consequently, $\{x_n\}$ converges strongly to $v \in F(I) \cap F(T)$. We can define now a mapping P from M onto F(T) by $\lim_{n\to\infty} x_n = Pq$. From (2.6), we have $(q - Pq, J(v - Pq)) \leq 0$ for all $q \in M$ and $v \in F(T)$. Thus by Lemma 1.2, P is the sunny nonexpansive retraction on M. Notice that $x_n = Ix_n$ and $\lim_{n\to\infty} x_n = Pq$, so by the continuity of I, $Pq \in F(I)$.

Example. Let X = R with usual norm and $M = [1, \infty)$. Let $T(x) = x^2$ and $I(x) = 2x - 1$, for all $x \in M$. Let q = 1. Then M is convex with $q \in F(I)$, $F(I) = \{1\}$ and $C_q(I, T) = [1, \infty)$. Note that the pair (T, I) is Banach operator but T and I are not C_q-commuting maps and hence not R-subweakly and uniformly R-subweakly commuting maps.

(2.4)

Corollary 2.11. Let M be a closed convex subset of a reflexive Banach space X with uniformly Gateaux differentiable norm. Suppose that T is continuous self-map on M, T is uniformly asymptotically regular and asymptotically nonexpansive with a sequence $\{k_n\}$. Let $\{\lambda_n\}$ be a sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} \frac{k_n - 1}{k_n - \lambda_n} = 0 = 0$. Then we have the following:

(2.5)

(a) For $q \in M$ and each $n \geq 1$, there is exactly one x_n in M such that

$$x_n = (1 - \mu_n)q + \mu_n T^n x_n.$$

(b) If $\{x_n\}$ is bounded and T satisfies the property *(S)*, then $\{x_n\}$ converges strongly to $Pq \in F(T)$, where P is the sunny nonexpansive retraction from M onto $F(T)$.

Corollary 2.12. Let M be a closed convex subset of a reflexive Banach space X with uniformly Gateaux differentiable norm. Suppose that (T,I) is a continuous Banach operator pair on M such that $F(I)$ is q-starshaped with $q \in F(I)$ and T is I-nonexpansive. Let $\{\lambda_n\}$ be a sequence of real numbers in $(0,1)$ such that $lim_{n\to\infty} \lambda_n = 1$. Then we have the following:

(a) For each $n \geq 1$, there is exactly one x_n in M such that

$$Ix_n = x_n = (1 - \mu_n)q + \mu_n T^n x_n.$$

(b) If $\{x_n\}$ is bounded and I and T satisfy the property *(S)*, then $\{x_n\}$ converges strongly to $Pq \in F(T) \cap F(I)$, *where* P is the sunny nonexpansive retraction from M onto $F(T)$.

Proof.

(a) Define a mapping T_n on M by

$$T_n x = (1 - \lambda_n)q + \lambda_n Tx$$

Then following the lines of Lemma 2.5, we get the result.

(c) Since $\{x_n\}$ is bounded, $\lambda_n \to 1$, and $\|x_n - Tx_n)\| = \|Ix_n - Tx_n)\| \leq (\lambda_n^{-1} - 1)(\|q\| + \|x_n\|) \to 0$. The result now follows from Theorem 2.8.

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