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Research Paper

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A Simple Sirs Mathematical Model with Mass Action Type Incidence

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Abstract— In this paper a simple SIRS mathematical model with mass action type incidence is formulated and studied. Steady state, equilibrium point and the basic reproduction number are obtained for the system of differential equation. Existence and stability of the diseases free and endemic stages are investigated. An example is also furnished which demonstrates validity of main result.

Keywords— *Mathematical Model, Equilibrium point, Stability, Incidence.*

I. INTRODUCTION

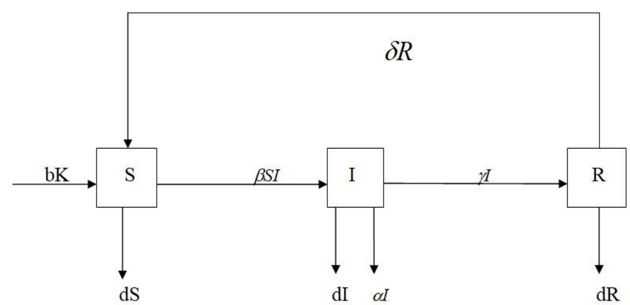
Mathematical modeling is an essential tool to know and predict the spread of communicable diseases. In this progression, rate of incidence plays a important role. The incidence in a mathematical model is pace at which susceptible become communicable. The first SIR epidemic model was studied by Kermack and Mc Kendric [16] in the year 1927. Mena Lorca and Hethcote [6] also analyzed. The SIRS epidemic model has been studied by various authors viz. Capaso and Serio [15], Porwal and Badshah [7,8,9,10,11], Hethcote [2,3,4], Anderson and May [12], Kumar et al. [9] studied modified SIRS epidemic model with immigration and saturated incidence rate.

Communicable diseases create a constant threat to human beings. Each person on the earth can be affected by a disease. The emergence and re-emergence of infectious diseases have become a noteworthy universal problem. Accurate understanding of transmission mechanisms of diseases caused by existing and new pathogens may facilitate devising prevention tools. Prevention tools against transmissions, including vaccines and drugs, need to be developed at a similar pace to that of the microbes. Implementation and proper use of these sophisticated tools against the microbes is one more challenge.

In this paper, we extend the model of Hethcote [5] by taking SIRS mathematical model with mass action type incidence in place of SIR mathematical model. Further we study the model and obtain diseases free and endemic equilibrium of the system and analyze for stability. Also give an example for verification of our results.

II. THE MATHEMATICAL MODEL

The proposed model describes a simple SIRS mathematical model with mass action type incidence. Here we adopt the following SIRS model:



$$\left.\begin{aligned} \frac{dS}{dt} &= bK - \beta SI - dS + \delta R \\ \frac{dI}{dt} &= \beta SI - (\gamma + \alpha + d)I \\ \frac{dR}{dt} &= \gamma I - (\delta + d)R \\ \frac{dN}{dt} &= bK - dN - \alpha I \end{aligned}\right\} \quad (2.1)$$

where $N = S + I + R$

and S = Number of Susceptible,
I = Number of Infectious,
R = Recovered Compartment,
N = Total Population Size

With other Parameters in the model are
b = Birth Rate,

K = Number of individual in the population
d = Natural Death Rate,
α = Disease Induced Death Rate,
β = Transmission Coefficient,
δ = Loss of Immunity Rate Constant,
γ = Recovery Rate,

III. STEADY STATE

The disease free equilibrium of the model (2.1) is obtained by setting right hand side of model (2.1) equals to zero and taking $I = 0$, we get.

$$\left.\begin{aligned} &bK - \beta SI - dS + \delta R = 0 \\ &\beta SI - (\gamma + \alpha + d)I = 0 \\ &\gamma I - (\delta + d)R = 0 \\ &bK - dN - \alpha I = 0 \end{aligned}\right\} \qquad (3.1)$$

From the third equation of system (3.1), we get

$$0 - (\delta + d)R = 0$$

$$R = 0.$$

and from first equation of system (3.1), we get

$$bK - 0 - dS + 0 = 0$$

$$S = \frac{bK}{d}.$$

Hence, the disease free equilibrium point E_0 is

$$E_0(S, I, R) = E_0\left(\frac{bK}{d}, 0, 0\right).$$

From Second equation of system (3.1)

$$\beta S - (\gamma + \alpha + d) = 0$$

$$S^* = \frac{(\gamma + \alpha + d)}{\beta}$$

by third equation of system (3.1)

$$\gamma I - (\delta + d)R = 0$$

$$R^* = \frac{\gamma I^*}{\delta + d}$$

Again by first equation of system (3.1) we get

$$bK - \beta S^* I - dS^* + \delta R^* = 0,$$

$$bK - dS^* = \left(\gamma + \alpha + d - \frac{\delta\gamma}{\delta + d}\right)I^*,$$

where $$I^* = \frac{bK - dS^*}{\left(\gamma + \alpha + d - \frac{\delta\gamma}{\delta + d}\right)}$$

$$I^* = \frac{b\beta K - d(\gamma + \alpha + d)}{\beta\left(\gamma + \alpha + d - \frac{\delta\gamma}{\delta + d}\right)}.$$

Hence endemic equilibrium point is $E_1(S^*, I^*, R^*)$,

where $$S^* = \frac{(\gamma + \alpha + d)}{\beta},$$

$$I^* = \frac{b\beta K - d(\gamma + \alpha + d)}{\beta\left(\gamma + \alpha + d - \frac{\delta\gamma}{\delta + d}\right)},$$

$$R^* = \frac{\gamma I^*}{\delta + d}$$

The Basic Reproduction number

$$R_0 = \frac{b\beta K}{d(\gamma + \alpha + d)}$$

IV. STABILITY ANALYSIS

To discuss the stability of the model (2.1) the governing dynamical system is

$$\left.\begin{aligned} &F_1 = bK - \beta SI - dS + \delta R \\ &F_2 = \beta SI - (\gamma + \alpha + d)I \\ &F_3 = \gamma I - (\delta + d)R \end{aligned}\right\}. \qquad (4.1)$$

The Jacobian matrix of system (4.1) is

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial S} & \frac{\partial F_1}{\partial I} & \frac{\partial F_1}{\partial R} \\ \frac{\partial F_2}{\partial S} & \frac{\partial F_2}{\partial I} & \frac{\partial F_2}{\partial R} \\ \frac{\partial F_3}{\partial S} & \frac{\partial F_3}{\partial I} & \frac{\partial F_3}{\partial R} \end{bmatrix}$$

$$= \begin{bmatrix} -\beta I - d & -\beta S & \delta \\ \beta I & \beta S - (\gamma + \alpha + d) & 0 \\ 0 & \gamma & -(\delta + d) \end{bmatrix}.$$

At equilibrium point E_0, the Jacobian matrix is

$$J_{E_0} = \begin{bmatrix} -d & \frac{-\beta bK}{d} & \delta \\ 0 & \frac{\beta bK}{d} - (\gamma+\alpha+d) & 0 \\ 0 & \gamma & -(\delta+d) \end{bmatrix}.$$

Its characteristics equation is

$$\left|J_{E_0} - \lambda I\right| = 0,$$

where

$$J_{E_0} = \begin{bmatrix} -d-\lambda & -\frac{\beta bK}{d} & \delta \\ 0 & \frac{\beta bK}{d} - (\gamma+\alpha+d) - \lambda & 0 \\ 0 & \gamma & -(\delta+d)-\lambda \end{bmatrix} = 0,$$

$$(-d-\lambda)\left(\frac{\beta bK}{d} - (\gamma+\alpha+d) - \lambda\right)\left(-(\delta+d)-\lambda\right) = 0$$

,

Whose eigen values are

$$\lambda_1 = -d \ , \lambda_2 = \frac{\beta bK}{d} - (\gamma+\alpha+d) = R_0 - 1$$

and $\lambda_3 = -(\delta+d)$.

All the eigen values are negative if $\lambda_2 < 0$

$$\frac{\beta bK}{d} - (\gamma+\alpha+d) < 0,$$

$$\frac{\beta bK}{d} < (\gamma+\alpha+d),$$

$$\frac{\beta bK}{d(\gamma+\alpha+d)} < 1$$

$$R_0 < 1,$$

where $$R_0 = \frac{b\beta K}{d(\gamma+\alpha+d)}.$$

Hence the equilibrium point E_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. It is also globally asymptotically stable if $R_0 < 1$.

At equilibrium point E_1 the Jacobian matrix is

$$J_{E_1} = \begin{bmatrix} -\beta I^* - d & -\beta S^* & \delta \\ \beta I^* & \beta S^* - (\gamma+\alpha+d) & 0 \\ 0 & \gamma & -(\delta+d) \end{bmatrix}.$$

Its characteristics equation

$$\left|J_{E_1} - \rho I\right| = 0,$$

$$= \begin{bmatrix} -\beta I^* - d - \rho & -\beta S^* & \delta \\ \beta I^* & \beta S^* - (\gamma+\alpha+d) - \rho & 0 \\ 0 & \gamma & -(\delta+d)-\rho \end{bmatrix}$$

$$= (\beta I^* - d - \rho)(\beta S^* - (\gamma+\alpha+d) - \rho)(-(\delta+d)-\rho) - \beta I^*(\beta S^*(\delta+d+\rho) - \gamma\delta) = 0$$

$$\Rightarrow (\beta I^* + d + \rho)(\beta S^* - (\gamma+\alpha+d) - \rho)(\delta+d+\rho) - \beta I^*(\beta S^*(\delta+d+\rho) - \gamma\delta) = 0$$

$$\Rightarrow -\rho^3 - \rho^2\left[\beta(S^*+I^*) + \delta + d - \gamma - \alpha - 2d\right] - \rho\begin{bmatrix} \beta I^*(\gamma+\alpha+d) + d(\gamma+\alpha+d) + (\delta+d)(\gamma+\alpha+2d) \\ +(\delta+d)(\beta S^* + \beta I^*) - d\beta S^* \end{bmatrix}$$

$$\Rightarrow \rho^3 + \rho^2\left[\beta(S^*+I^*) + \delta - (\gamma+\alpha+d)\right] + \rho\begin{bmatrix} (\beta I^*+d)(\gamma+\alpha+d) + (\delta+d)\beta(S^*+I^*) \\ +(\gamma+\alpha+2d)(\delta+d) - d\beta S^* \end{bmatrix} + \begin{bmatrix} (\beta I^*+d)(\gamma+\alpha+d)(\delta+d) \\ -d\beta S^*(\delta+d) - \beta I^*\delta d \end{bmatrix} = 0$$

$$\Rightarrow \rho^3 + a_1\rho^2 + a_2\rho + a_3 = 0,$$

where

$$a_1 = \beta(S^*+I^*) + \delta - (\gamma+\alpha+d),$$

$$a_2 = (\beta I^*+d)(\gamma+\alpha+d) + \beta(\delta+d)(S^*+I^*) + (\delta+d)(\gamma+\alpha+2d) - d\beta S^*,$$

$$a_3 = (\beta I^*+d)(\gamma+\alpha+d)(\delta+d) - d\beta S^*(\delta+d) - \beta I^*\gamma d.$$

Clearly

$a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$ if $R_0 > 1$ and by the Routh Hurwitz Criteria , the endemic equilibrium is locally asymptotically stable for $R_0 > 1$.

V. EXAMPLE

We take the parameters of the system, $d = 2.33$, $K = 1.4$, $b = 2.9$, $\delta = 1$, $\alpha = 0.28$, $\beta = 1.2$, $\gamma = 0.49$. Then $E_0 = (1.7424,0,0)$ and $R_0 = 0.5 < 1$. Hence the diseases free equilibrium is asymptotically stable.

Now, we take parameters of the system as $d = 0.16$, $K = 1.4$, $b = 2.9$, $\delta = 1$, $\alpha = 0.5$, $\beta = 1.2$, $\gamma = 0.18$. Then $E_1 = (0.7,5.764,0.894)$ and $R_0 = 4.905 > 1$. Hence the endemic equilibrium point E_1 is locally asymptotically stable.

Remark If $\delta = 0$, then we gets one of the models of Hethcote [5].

VI. CONCLUSION

In this paper, we see that the basic reproduction number plays an important role to control the disease. If $R_0 < 1$ then there is a disease free equilibrium which is locally stable, that is the disease dies out. But when $R_0 > 1$ then the diseases persist and the endemic equilibrium is stable.

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