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# A Fixed Point Theorem for a Contractive Mapping in Dislocated Quasi Metric Space

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***Abstract-*** Aage and Salunke [1] proved the result on fixed point theorem in dislocated and dislocated quasi metric space. Dass and Gupta [2] given an extentionsion of Banach Contraction Principle through rational expression. In this paper, we establish a common fixed point theorem for continuous contractive mapping in dislocated quasi metric space which is the generalized result of Isufati. A [4] ,Mujeeb Ur Rahman and Muhammad Sarwar [11].

***Keywords-*** Dislocated Quasi metric space, common fixed point, continuous contractive mapping

**AMS Subject Classification-** 47H10, 54H25

## I. INTRODUCTION AND PRELIMINARIES

In1922, Banach proved fixed point theorem for contraction mapping in complete metric space. It is well known as a Banach fixed point theorem . In 1975 Dass and Gupta [2] , generalized Banach contraction principle in metric space. In 1977 Rhoades [7], gave a comparison of various definitions of contractive mappings. In 2005 Zeyada et al. [10] , given a generalization of fixed point theorem due to Hiltzler and Seda [3] , in dislocated quasi metric space . In 2008 Aage and Saluke [1] proved result on fixed point theorem in dislocated & dislocated quasi metric space. After this in 2010 Isufati [4] , established a fixed point theorem in dislocated quasi metric space , also in 2010 Kohliet al.[5] , in 2011 Shrivastava and Gupta [8] , Pagey and Nighojkar [6] and in 2014 Shrivastava et al. [9],Mujeeb Ur Rahman and Muhammad sarwar [11], worked on a common fixed point theorem in dislocated quasi metric space. In this paper , we establish a common fixed point theorem for continuous contractive mapping in dislocated quasi metric space which is the generalized result of Isufati, A. [4] and Mujeeb Ur Rahman and Muhammad sarwar [11].

**Definition 1.1** [3&10] Let X be a non-empty set and let d :X × X → [0,∞)

be a function satisfying the following conditions:

($d_1$) $d(x\,,x)=0$

($d_2$) $d(x\,,y)$=$d(y\,,x)$=0 implies$x=y$.

($d_3$) $d(x\,,y)$= $d(y\,,x)$for all $x\,,y$ ∈X

($d_4$)$d(x\,,y)\ \leq d(x\,,z)+d(z\,,y)$for all $x\,,y,z$ ∈X

If $d$ satisfies conditions only ($d_2$) and ($d_4$), then $d$ is called a dislocated quasi metric on X.

If $d$ satisfies conditions ($d_1$), ($d_2$) and ($d_4$) , then $d$ is called a quasi metric on X. If $d$ satisfies conditions ($d_2$), ($d_3$) and ($d_4$) , then $d$ is called a dislocated metric on X . If $d$ satisfies all the conditions ($d_1$), ($d_2$) ($d_3$) and ($d_4$) , then $d$ is called a metric on X.

**Definition 1.2** [10] A sequence $\{x_n\}$ in a *d-q*metric space (dislocated quasi metric space) (X, d) is called a Cauchy sequence if for given ∈ >0, there corresponds $n_0$∈N such that for all m, n ≥ $n_0$, implies $d(x_n,x_m) <$∈

**Definition 1.3** [10] A sequence in *d-q*metric space converges to a point $x$ if there exists $x$∈X such that $d(x_{n,}x)\rightarrow 0$ as n→∞ or $\lim_{n\to\infty} d(x_{n,}x)=0$

**Definition 1.4** [3] A dislocated quasi metric space (X, d) is a complete metric space if every Cauchy sequence in (X, d) is convergent sequence with respect to d.

**Definition 1.5 [10]** Let (X , d) and (Y , ρ) be any two dislocated quasi metric spaces and Let $T$: X → Y be a function then $T$ is a continuous function at $x_0$ ∈ X,if for each sequence $\{x_n\}$ which is convergent to $x_0$ in X, the sequence $\{T(x_n)\}$ is convergent to $\{T(x_0)\}$ in Y.

**Definition 1.6**[10] Let (X, d) be a d-metric space. A map $T$: X → X iscalled contraction mapping if there exists a number λ with 0 ≤ λ < 1 such that$d(Tx,Ty) \leq \lambda\, d(x, y$ .for all $x$ ,$y$ in X.

**Lemma 1.1**[10] Limits in a d-q metric space are unique.

**Theorem 1.1[1]** Let ( X , d) be a complete *d-q* metric space and suppose there exist non negative constants α , β , γ with α+β+γ< 1.Let $T$ : X→X be a continuous mapping satisfying condition

$d(Tx,Ty) \leq \alpha\, d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty)$ for all $x,y \in X$ .

Then $T$ has a unique fixed point **.**

**Theorem 1.2** [4] Let (X, d) be a *d-q* metric space and let

T :X→X be a continuous mapping satisfying the following condition

$$d(Tx,Ty)= \alpha\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta\, d(x,y) \forall x,y\in X,$$

andα> 0,β> 0 , α+β< 1 . Then T has a unique fixed point.

**Theorem 1.3** [9] Let (X, d) be a *d-q* metric space and

T :X→X be a continuous mapping satisfying the following condition

$$d(Tx,Ty)= \alpha\frac{d(y,Ty)[1+d(x,Tx)]}{(d(x,Ty))[1+d(x,Ty)]}+\beta\, d(x,y)+\gamma d(x,Ty)$$

$\forall x,y\in X$,

and α> 0,β> 0 ,γ> 0 , α+β+γ< 1 ; Then T has a unique fixed point.

**Theorem 1.5[**11**]** Let ( X , d) be a complete *d-q* metric space and let $T$ : X →X be a continuous self-mapping satisfying the condition

$$d(Tx,Ty) \leq \alpha\, d(x,y) + \beta\frac{d(x,Ty)d(y,Ty)}{d(x,y)+d(y,Ty)} + \gamma\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}$$

$$+ \mu\frac{d(x,Tx)d(x,Ty)}{1+d(x,y)} \text{for all } x,y\in X$$

andα , β , γ , μ ≥ 0 $with\ \alpha+\beta+\gamma+2\mu<1$ .

Then $T$ has a unique fixed point

## II. Main Result

**Theorem 2.1** Let(X, d) be a complete *d-q* metric space and T :X→X be a continuous mapping satisfying the following condition

$$d(Tx,Ty) \leq \alpha\frac{d(y,Ty)d(x,Tx)}{[1+d(x,Tx)][1+d(y,Ty)]} + \beta\frac{d(x,y)d(x,Tx)}{1+d(x,Tx)[1+d(y,Ty)]}$$

$$+ \gamma\frac{d(x,y)d(y,Ty)}{1+d(x,y)[1+d(y,Ty)]} \forall x,y\in X \quad (1)$$

andα> 0,β> 0 ,γ> 0 , α+β+γ< 1 ; Then T has a unique fixed point.

**Proof.**

Let $\{x_n\}$ bea sequence in *d-q* metric space (X, d) defined as under, For any $x_0 \in X$, we define $T(x_0) = x_1$ ,

$T(x_1) = x_2$ ,$T(x_2) = x_3$ ,…,$T(x_n) = x_{n+1}$…$\forall n \in N$ **(2)**

Consider n , n+1 ≥ $n_0$ where $n_0 \in N$

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n)$$

$$\leq \alpha\frac{d(x_n,Tx_n)d(x_{n-1},Tx_{n-1})}{[1+d(x_{n-1},Tx_{n-1}][1+d(x_n,Tx_n)]}$$

$$+\beta\frac{d(x_{n-1},x_n)d(x_{n-1},Tx_{n-1})}{1+d(x_{n-1},Tx_{n-1})[1+d(x_n,Tx_n)]}$$

$$+\gamma\frac{d(x_{n-1},x_n)d(x_n,Tx_n)}{1+d(x_{n-1},x_n)[1+d(x_n,Tx_n)]}$$

$$= \alpha\frac{d(x_n,x_{n+1})d(x_{n-1},x_n)}{[1+d(x_{n-1},x_n)][1+d(x_n,x_{n+1})]} + \beta\frac{d(x_{n-1},x_n)d(x_{n-1},x_n)}{1+d(x_{n-1},x_n)[1+d(x_n,x_{n+1})]}$$

$$+\gamma\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)[1+d(x_n,x_{n+1})]}$$

Since $1+d(x_{n-1},x_n) > d(x_{n-1},x_n)$

$$\Longrightarrow 1 > \frac{d(x_{n-1},x_n)}{1+d(x_{n-1},x_n)}$$

$$< \alpha\frac{d(x_n,x_{n+1})}{1+d(x_n,Tx_{n+1})} + \beta\, d\frac{d(x_{n-1},x_n)}{[1+d(x_n,x_{n+1})]} + \gamma\frac{d(x_n,x_{n+1})}{[1+d(x_n,x_{n+1})]}$$

This gives

$$d(x_n,x_{n+1}) < \alpha\, d(x_n,x_{n+1}) + \beta\, d(x_{n-1},x_n)$$

$$+ \gamma\, d(x_n,x_{n+1})$$

$$\Longrightarrow d(x_n,x_{n+1}) < \frac{\beta}{1-\alpha-\gamma} d(x_{n-1},x_n)$$

$$\Longrightarrow \delta = \frac{\beta}{1-\alpha-\gamma} \text{where} 0 < \delta < 1$$

Therefore we have

$$d(x_n,x_{n+1}) < \delta d(x_{n-1},x_n) ,$$

Similarly, we have

$$d(x_{n-1},x_n) < \delta d(x_{n-2},x_{n-1}),$$

$$d(x_{n-2},x_{n-1}) < \delta d(x_{n-3},x_{n-2}),$$

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$$d(x_2,x_1) < \delta\, d(x_1,x_0),$$

Finally, we have

$$d(x_n,x_{n+1}) < \delta^n d(x_1,x_0) ,$$

$\Rightarrow| d(x_n, x_{n+1}) | < \delta^n | d(x_1, x_0) |$

Since $0 < \delta < 1$ and letting $n \to\infty, \Rightarrow \delta^n \to 0$

Implies that

$| d(x_n, x_{n+1})| \to 0$ as $n \to\infty$

Hence**,** The sequence $\{x_n\}$ is Cauchy sequence in the complete dislocated quasi metric space (X,d) .
Thus the sequence $\{x_n\}$ is a convergent sequence in dislocated quasi metric space (X,d) to the point $x\in$x .
i.e.$\lim_{n\to\infty} x_n = x$ .
Since T : X→X is continuous then we have

$$T(x) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

i.e. $T(x) = x$
Thus T has a fixed point.

**For uniqueness :**

To prove T has unique fixed point we suppose $x$ and $y$ are any two common fixed point of T

i.e.$T(x) = x$ and $T(y) = y$

Consider

$$d(x,y) = d(Tx,Ty)$$

$$\le \alpha\frac{d(y,Ty)d(x,Tx)}{[1+d(x,Tx)][1+d(y,Ty)]} + \beta\frac{d(x,y)d(x,Tx)}{1+d(x,Tx)[1+d(y,Ty)]} + \gamma\frac{d(x,y)d(y,Ty)}{1+d(x,y)[1+d(y,Ty)]}$$

$$\le \alpha\frac{d(y,y)d(x,x)}{[1+d(x,x)][1+d(y,y)]} + \beta\frac{d(x,y)d(x,x)}{1+d(x,x)[1+d(y,y)]} + \gamma\frac{d(x,y)d(y,y)}{1+d(x,y)[1+d(y,y)]}$$

$d(x,y) \le 0$ [ ∵$x$ and $y$ are any two common fixed point of T ,i.e.$T(x) = x$ and $T(y) = y$ and $d(x,x) = 0$ & $d(y,y) =0$ ] but $d(x,y) \ge 0$

This implies that

$d(x,y) = 0$

i.e. $x = y$, this proves the uniqueness of fixed point of T in X

This completes the proof of theorem **2.1**

**Corollary 2.1** Let (X, d) be a complete *d-q* metric space ant T :X→X be a continuous mapping, satisfying the following condition

$$d(Tx,Ty) \le \beta\frac{d(x,y)d(x,Tx)}{1+d(x,Tx)[1+d(y,Ty)]} + \gamma\frac{d(x,y)d(y,Ty)}{1+d(x,y)[1+d(y,Ty)]}$$

$\forall x,y\in$X

and $\beta > 0$, $\gamma > 0$, $\beta+\gamma < 1$ ; Then T has a unique Fixed point.

**Proof :** The proof of the corollary 2.1 follows immediately by putting $\alpha = 0$ in Theorem 2.1

**Corollary2.2** Let(X, d) be a complete *d-q* metric space and T :X→X be a continuous mapping satisfying the following condition

$$d(Tx,Ty) \le \alpha\frac{d(y,Ty)d(x,Tx)}{[1+d(x,Tx)][1+d(y,Ty)]} + \gamma\frac{d(x,y)d(y,Ty)}{1+d(x,y)[1+d(y,Ty)]}$$

$\forall x,y\in$X

and $\alpha > 0, \gamma > 0$ , $\alpha+\gamma < 1$ ; Then T has a unique fixed point.

**Proof :**The proof of the corollary 2.2 follows immediately by putting $\beta = 0$ in Theorem 2.1

**Corollary2.3** Let(X, d) be a complete *d-q* metric space and T :X→X be a continuous mapping

Satisfying the following condition

$$d(Tx,Ty) \le \alpha\frac{d(y,Ty)d(x,Tx)}{[1+d(x,Tx)][1+d(y,Ty)]} + \beta\frac{d(x,y)d(x,Tx)}{1+d(x,Tx)[1+d(y,Ty)]}$$

$\forall x,y\in$X

and$\alpha > 0$, $\beta > 0$, $\alpha+\beta < 1$ ; Then T has a unique fixed point.

**Proof :**The proof of the corollary 2.3 follows immediately by putting $\gamma = 0$ in Theorem 2.1

**Corollary 2.4**Let (X, d) be a complete *d-q* metric space and T :X→X be a continuous

mapping also $T^n$ : X → X is acontinuous mapping,satisfying the following condition

$$d(T^n x,T^n y) \le \alpha\frac{d(y,T^n y)d(x,T^n x)}{[1+d(x,T^n x)][1+d(y,T^n y)]} + \beta\frac{d(x,y)d(x,T^n x)}{[1+d(x,T^n x)][1+d(y,T^n y)]} + \gamma\frac{d(x,y)d(y,T^n y)}{[1+d(x,y)][1+d(y,T^n y)]} \forall x,y\in X$$

where n is an integer such that n > 0 and $\alpha > 0$, $\beta > 0$ , $\gamma > 0$ , $\alpha+\beta+\gamma < 1$ ; Then T has a unique fixed point.

**Proof :**From theorem 2.1 , $T^n$ has a unique fixed point$x$ in complete *d-q* metric space X

Therefore $T^n x = x$

Now consider

 

$T^n(T\boldsymbol{x}) = T(T^n x) = Tx$

i.e. $Tx$ is a fixed point of $T^n$. But $x$ is a unique fixed point of $T^n$ and so $Tx = x$

Hence $x$ is a fixed point of T.

**For uniqueness :**For uniqueness of fixed point let $x \neq y$ be an another fixed point of T
Then
$d(y, x) = d(Ty, Tx)$

$$\leq \alpha\frac{d(x,Tx)d(y,Ty)}{[1+d(y,Ty))][1+d(x,Tx)]} + \beta\frac{d(y,x)d(y,Ty)}{[1+d(y,Ty)]][1+d(x,Tx)]}$$

$$+ \gamma\frac{d(y,x)d(x,Tx)}{[1+d(y,,x)][1+d(x,Tx)]} \quad \forall x,y \in X$$

$$= \alpha\frac{d(x,x)d(y,y)}{[1+d(y,y)][1+d(x,x)]} + \beta\frac{d(y,x)d(y,y)}{[1+d(y,y)]][1+d(x,x)]}$$

$$+ \gamma\frac{d(y,x)d(x,x)}{[1+d(y,,x)][1+d(x,x)]}$$

*i.e.* $d(y,x) \leq 0$ .But $d(y,x) \geq 0$

This gives us

$d(y,x) = 0$

Which is possible iff $x = y$ and so $x$ is a unique fixed point

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