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Research Article

Discovering The Properties, Applications and Factual Manifestations of The Fibonacci Series

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***Abstract*—** The Fibonacci series is a sequence of numbers that has fascinated mathematicians, scientists, and artists for centuries due to its intriguing properties and wide-ranging applications across various fields. This research paper aims to delve into the origins, mathematical properties, and practical applications of the Fibonacci series. Beginning with an overview of its historical background, the paper progresses to examine its mathematical properties, recurrence relation, and closed-form expression. Furthermore, it explores diverse applications of the Fibonacci series in mathematics, computer science, nature, and art, highlighting its significance and versatility in real-world contexts. By consolidating information from various disciplines, this paper presents a comprehensive understanding of the Fibonacci series and its enduring relevance in modern society.

***Keywords*—** Fibonacci, Golden Ratio, Encryption, Golden Spiral

1. Introduction

The Fibonacci series is a sequence of numbers in which each number is the sum of the two preceding ones. This sequence was first introduced to the Western world by Leonardo of Pisa, also known as Fibonacci, in his book Liber Abaci in 1202[1]. The series begins with 0 and 1, and each subsequent number is the sum of the two preceding ones: 0, 1, 1, 2, 3, 5, 8, 13, 21, and so on. This research paper provides a comprehensive overview of the Fibonacci series, encompassing its historical significance, mathematical properties, and practical applications. Through interdisciplinary exploration, it highlights the enduring relevance and profound impact of this sequence in various fields, underscoring its status as a fundamental concept in mathematics and beyond [6].

In this paper will go on by studying the properties of Fibonacci Series, features related to Fibonacci series such as golden ratio, golden rectangle, golden spiral etc. and further we will delve into the application of Fibonacci Series in the field of computer science by using the Fibonacci Numbers for *encryption*. Further we examine the manifestation of Fibonacci Sequence and its properties in the field of Biology, Architecture, Art, Music.

2. Historical Background

The Fibonacci sequence finds its roots in ancient Indian mathematics, where it appeared in Sanskrit prosody as early as 200 BC. However, it gained prominence in the Western world after Fibonacci's seminal work in Liber Abaci.

Over the course of his life, Fibonacci wrote several books, including Liber Abaci, which publicized the advantages of the Hindu numerals and discussed various mathematical problems, a book on geometry, which included trigonometry and proofs, a book on flowers, and a book on number theory, which brought him much recognition as an extremely talented mathematician. By far the most well-known of his works is Liber Abaci, which means “book of calculating” or “book of computation.” [12].

In spite of his influential contributions to the field of European mathematics, Fibonacci is not most remembered for any of these reasons, but rather for a single sequence of numbers that provided the solution to a problem included in Liber Abaci. Like most of the problems in the book, Fibonacci did not invent this problem himself, but his solution to it has forever immortalized him in the mathematical world [4].

The problem, dealing with the regeneration of rabbits, calculated the number of rabbits after a year if there is only one pair the first month. The problem states that it takes one month for a rabbit pair to mature, and the pair will then produce one pair of rabbits each month following. Fibonacci’s solution stated that in the first month there would be only one pair; the second month there would be one adult pair and one baby pair; the third month there would be two adult pairs and one baby pair; and so forth [12]. When the total number of rabbits for each month is listed, one after the other, it generates the sequence of numbers for which Fibonacci is most famous: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377…

Since then, mathematicians, scientists, and scholars have continued to study and unravel the mysteries of this sequence, leading to significant discoveries and applications. As a major part of its application, Fibonacci series will also be applied for the purpose end-to-end encryption.

3. Mathematical Properties

A closer inspection of the numbers making up the Fibonacci sequence brings to light all sorts of fascinating patterns and mathematical properties. Fibonacci himself makes no mention of these patterns in his book, but the following patterns are a few that have been brought to light over years of examination of the numbers in the sequence.

- Any two consecutive Fibonacci numbers are relatively prime, having no factors in common with each other [5]

  For example:

$$5,8,13,21,34$$
$$5=1\cdot 5$$
$$8= 2\cdot 2\cdot 2$$
$$13= 1\cdot 13$$
$$21= 3\cdot 7$$
$$34= 2\cdot 17$$

- Summing together any ten consecutive Fibonacci numbers will always result in a number which is divisible by eleven [12].

$$1+ 1+ 2+ 3+ 5+ 8+ 13+ 21+ 34+ 55=143$$
$$143/11= 13$$
$$89+ 144+ 233+ 377+ 610+ 987+ 1{,}597+ 2{,}584+ 4{,}181+ 6{,}675= 17{,}567$$
$$17{,}567/11=1{,}597$$

- Following tradition, F_n will be used to represent the n-th Fibonacci number in the sequence.

n	F_n
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
11	89
12	144
13	233
14	377
15	610

- Every third Fibonacci number is divisible by two, or F_3. Every fourth Fibonacci number is divisible by three, or F_4. Every fifth Fibonacci number is divisible by five, or F_5. Every sixth Fibonacci number is divisible by eight, or F_8, and the pattern continues. In general, every nth Fibonacci number is divisible by the nth number in the Fibonacci sequence, or F_{mn} is divisible by F_n [5].

- Fibonacci numbers in composite-number positions are always composite numbers, with the exception of the fourth Fibonacci number. In other words, if n is not a prime, the nth Fibonacci number will not be a prime [12].

$$F_6 = 8$$
$$F_9 = 34$$
$$F_{16} = 987$$

- Multiplying any Fibonacci number by two and subtracting the next number in the sequence will result in the answer being the number two places before the original [5].

$$\ldots 3,5,8,13,21,34,55,89,144,233\ldots$$
$$2\cdot F_6\text{-}F_7= (2\cdot 8)\text{-}13=16\text{-}13= 3 =F_4$$
$$2\cdot F_{11}\text{-}F_{12}= (2\cdot 89)\text{-} 144= 178\text{-}144= 34 =F_9$$
$$2\cdot F_n\text{-}F_n\text{-}1=F_n\text{-}2$$

- Summing consecutive odd-positioned Fibonacci numbers, starting with the first odd-positioned number, will result in a number that is the next Fibonacci number in the sequence after the last term in the sum[12].

$$F_1+F_3=1+ 2= 3=F_4$$
$$F_1+F_3+F_5= 1+ 2+ 5= 8=F_6$$
$$F_1+F_3+F_5+F_7=1+2+ 5+ 13= 21=F_8$$

- A similar pattern emerges when summing consecutive, even-positioned Fibonacci numbers beginning with, only this time, the result is a number that is one less than the Fibonacci number following the last even number in the sum [12].

$$F_2+F_4=1+ 3= 4=F_5\text{-}1$$
$$F_2+F_4+F_6= 1+ 3+ 8= 12=F_7\text{-}1$$
$$F_2+F_4+F_6+F_8=1+3+ 8+ 21= 33=F_9\text{-}1$$

- When the squares of two consecutive Fibonacci numbers are added, the sum is also a Fibonacci number [12].

$$F_3{}^2+ F_4{}^2 = 2^2+3^2 = 13 = F_7$$
$$F_6{}^2+ F_7{}^2=8^2+13^2 = 233 = F_{13}$$
$$F_{13}{}^2+F_{14}{}^2=223^2 + 377^2 =196{,}418 = F_{27}$$
$$F_n{}^2+F_{n+1}{}^2= F_{2n+1}$$

- Summing any number of consecutive Fibonacci numbers will result in a number that is one less than the Fibonacci number two places beyond the last one added.

$$1,1,2,3,5,8,13$$

$F_1 + F_2 + F_3 + F_4 + F_5 = 1 + 1 + 2 + 3 + 5 = 12 = 13 - 1$
$= F_7 - 1$

This gives a general formula for a simple way to find the sum of any number of Fibonacci numbers [12].

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

3.1 Recurrence Relation and Closed-Form Expression

One of the defining characteristics of the Fibonacci series is its recurrence relation, which defines each term in the sequence in terms of its predecessors. Additionally, the Fibonacci sequence can be expressed using a closed-form formula, providing a concise way to compute its terms without recursion. This section explores both the recurrence relation and the closed-form expression of the Fibonacci series, offering insights into their computational efficiency and mathematical elegance.

Discovering the value of a Fibonacci number, given its location in the sequence, can be very time consuming and tedious, particularly if it has a later placement in the sequence. Finding the fifth Fibonacci number is not difficult. Finding the fiftieth is much more cumbersome, as the process involves finding and summing the previous forty-nine terms. In 1843, the French mathematician Jacques-Philippe-Marie Binet discovered a formula which could find any Fibonacci number without having to find any of the previous numbers in the sequence. This formula finds the n-th Fibonacci number using a number called the golden ratio, $\frac{1+\sqrt{5}}{2}$, and its inverse [12].

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^n - \left(-\frac{1}{\phi}\right)^n\right] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Because the Fibonacci sequence is a linear, homogeneous, recurrence relation of the second degree, the above formula can be derived as follows:

Recurrence relation: $f_n = f_{n-1} + f_{n-2}$
Initial conditions: $f_0 = 0, f_1 = 1$
Assume that $f_n = r^n$ is a solution

Then $r^n = r^{n-1} + r^{n-2}$
$\Rightarrow r^2 = r + 1 \Rightarrow r^2 - r - 1 = 0$

Using the quadratic formula to solve this equation results in

$r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$

$$f_n = \alpha_1 r_1^n + \alpha_2 r_2^n \Rightarrow f_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\alpha_1 = -\alpha_2 \text{ and } \alpha_2 = -\frac{1}{\sqrt{5}}$$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\frac{1-\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{1+\sqrt{5}} = \frac{1-5}{2+2\sqrt{5}} = -\frac{2}{1+\sqrt{5}} = -\frac{1}{\phi}$$

$$\therefore f_n = \frac{1}{\sqrt{5}}\left[\phi^n - \left(-\frac{1}{\phi}\right)^n\right]$$

[14]

The more the Fibonacci sequence is studied, the more fascinating and intriguing patterns begin to surface. As various mathematical operations are performed on the numbers, all sorts of relationships between the numbers come to light. This is one of the many reasons this string of numbers has captivated the mathematical world for centuries.

3.2 Golden Ratio

The Fibonacci numbers also have a geometric manifestation in the form of the golden ratio. The golden ratio can be found by partitioning a line segment in such a way that the longer portion (L) is to the shorter portion (S) as the entire line segment is to the longer portion. This relationship is generally expressed by the formula

$\frac{L}{S} = \frac{L+S}{L}$

To find the numerical value for the golden ratio, let $x = \frac{L}{S}$. Then $x = 1 + \frac{1}{x}$.

Finally, solving for using the quadratic equation gives the numerical value for the golden ratio, which is often denoted by the Greek letter phi.

$$\phi = \frac{L}{S} = x = \frac{1+\sqrt{5}}{2} = 1.6180339887 \ldots$$

Interestingly, after the reciprocal of phi is simplified, it turns out to be only one less than phi.

$$\frac{1}{\phi} = \frac{S}{L} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}+1}{2} - 1 = \phi - 1$$
$$= 0.6180339887 \ldots$$

This reveals a very unique relationship between phi and its reciprocal. $\phi - \frac{1}{\phi} = 1$, but it is also true that $\phi . \frac{1}{\phi} = 1$. Phi and the reciprocal of phi are the only two numbers whose difference and product are both equal to one [13].

As it turns out, phi can also be calculated using Fibonacci numbers. Dividing a Fibonacci number by the preceding

Fibonacci number will result in a number that approaches phi. The larger the numbers used, the closer the result will be to the actual value of phi.

$$\frac{F_8}{F_7} = \frac{21}{13} = 1.6153846153 \ldots$$
$$\frac{F_{14}}{F_{13}} = \frac{377}{233} = 1.6180257511 \ldots$$
$$\frac{F_{20}}{F_{19}} = \frac{6{,}765}{4{,}181} = 1.6180339632 \ldots$$

This can be shown to be true in general by taking the limit as approaches infinity of any Fibonacci number divided by the preceding one.

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{5}}\left[\phi^{n+1} - \left(-\frac{1}{\phi}\right)^{n+1}\right]}{\frac{1}{\sqrt{5}}\left[\phi^{n} - \left(-\frac{1}{\phi}\right)^{n}\right]}$$
$$= \lim_{n\to\infty} \frac{\phi^{n+1} - \left(-\frac{1}{\phi}\right)^{n+1}}{\phi^{n} - \left(-\frac{1}{\phi}\right)^{n}}$$
$$= \lim_{n\to\infty} \frac{\phi\left[\phi^{n} - \frac{1}{\phi}\left(-\frac{1}{\phi}\right)^{n+1}\right]}{\phi^{n} - \left(-\frac{1}{\phi}\right)^{n}}$$
$$= \phi \lim_{n\to\infty} \frac{\phi^{n} + \left(-\frac{1}{\phi}\right)^{n+2}}{\phi^{n} - \left(-\frac{1}{\phi}\right)^{n}} \cdot \frac{\frac{1}{\phi^n}}{\frac{1}{\phi^n}}$$
$$= \phi \lim_{n\to\infty} \frac{1 + \frac{(-1)^{n+2}}{\phi^{2n+2}}}{1 + \frac{(-1)^{n}}{\phi^{2n}}}$$
$$= \phi \cdot \frac{1+0}{1+0} = \phi \cdot 1 = \phi$$
$$\therefore \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$$

Conversely, if a Fibonacci number is divided by the following Fibonacci number, the result will be close to the reciprocal of phi. Again, the larger the two numbers used, the closer the result will be to the reciprocal of phi [12].

$$\frac{F_7}{F_8} = \frac{13}{21} = 0.6190476190 \ldots$$
$$\frac{F_{13}}{F_{14}} = \frac{233}{377} = 0.6180371353 \ldots$$
$$\frac{F_{19}}{F_{20}} = \frac{4{,}181}{6{,}765} = 0.6180339985 \ldots$$

Fibonacci numbers become even more closely linked to the golden ratio when powers of phi are considered. First, ϕ^2 is written in terms of ϕ, which after simplification yields $\phi^2 = \phi + 1$. Each successive power of phi can then be written in terms of factors of previous powers of phi. The result of each power is a multiple of ϕ plus a constant. It turns out that the coefficient of phi and the constant are consecutive Fibonacci numbers in sequential order [12].

$$\phi^3 = \phi \cdot \phi^2 = \phi(\phi+1) = \phi^2 + \phi = (\phi+1) + \phi = 2\phi + 1$$
$$\phi^4 = \phi^2 \cdot \phi^2 = 3\phi + 2$$
$$\phi^5 = \phi^3 \cdot \phi^2 = 5\phi + 3$$
$$\phi^6 = \phi^3 \cdot \phi^3 = 8\phi + 5$$
$$\phi^7 = \phi^4 \cdot \phi^3 = 13\phi + 8$$

3.3 Golden Rectangle

Throughout the course of history, there is a rectangle whose proportions are found most pleasing to the eye. It is neither too fat nor too skinny, neither too long nor too short. People will subconsciously choose this rectangle over another one with different proportions. This rectangle, considered the most perfectly shaped rectangle, is known as the golden rectangle [5]. This rectangle is one in which the ratio of length to width is the golden ratio and follows the formula $\frac{w}{l} = \frac{l}{w+l}$.

In the late 1800s, Gustav Fechner, a German psychologist, invested a good deal of time into researching the subject. He measured thousands of common rectangles, from playing cards and books to windows and writing pads, and he ultimately found that in most of them, the ratio of length to width was close to phi. Fechner also conducted a study in which he asked a large number of people to choose the rectangle out of a group of rectangles was the most pleasing to the eye. His findings showed that the largest percentage of people preferred the rectangle with a ratio of 21:34. These numbers are consecutive Fibonacci numbers, and their ratio approaches the reciprocal of phi. The rectangle most preferred by people was a golden rectangle [12].

3.4 Golden Angle

The golden angle is the angle which divides a complete circle of 360° into central angle portions corresponding to the golden ratio. This golden angle, represented by the symbol, is found when 360° is multiplied by the reciprocal of phi, and that result is then subtracted from 360°

$$\psi = 360° - (360°)\left(\frac{1}{\phi}\right) = 137.5077640501 \ldots° \approx 137.5°$$

This golden angle is approached when 360°, multiplied by the ratio of two consecutive Fibonacci numbers, is subtracted from 360°. As with the golden ratio, this approximation of the golden angle becomes more accurate as the Fibonacci numbers used grow larger [2].

$$360° - (360°)\left(\frac{3}{5}\right) = 144°$$
$$360° - (360°)\left(\frac{13}{21}\right) = 137.1428571428 \ldots°$$
$$360° - (360°)\left(\frac{34}{55}\right) = 137.\bar{45}°$$

3.5 Golden Spiral

The golden spiral, also known as a logarithmic spiral, is a spiral whose form remains the same, even as it continues to grow in size. When the length increases, the radius also increases proportionally, so the actual shape of the spiral is unchanged. This spiral is also referred to as an equiangular spiral because its curve intersects each radius vector from the center of the spiral at the same constant angle [8]. This golden spiral's construction can be approximated using both a golden rectangle and Fibonacci squares. To construct it with a golden rectangle, a golden rectangle is divided up by cutting off successive squares. For example, a rectangle of length 89 and width 55, which is composed of two Fibonacci numbers and is very close to a golden rectangle, can be sectioned into a square with a side length of 55 and a rectangle with side lengths of 55 and 34. The new rectangle is divided into a square with a side length of 34 and a rectangle with side lengths of 34 and 21. This new rectangle is divided into a square with a side length of 21 and a rectangle with side lengths of 21 and 13. This pattern continues, and when quarter-circle arcs are drawn between opposing corners of each square, they form a spiral, as shown in the figure 1[12].

Alternatively, the spiral can also be approximated using squares with side lengths of the sequential Fibonacci numbers. It begins with a square of length 1. Another square of length 1 is attached to that. A square of length 2 is attached to the sides of the previous two squares where it fits, as 1 + 1 = 2. Then a square of length 3 is attached to the squares of lengths 1 and 2, where 1 + 2 = 3. A square of length 5 is attached to the squares of lengths 2 and 3, a square of length 8 is attached to the squares of lengths 3 and 5, and so forth. Quarter-circle arcs are then drawn to sequentially connect the opposing corners of the squares, as can be seen in the figure 1 [5].

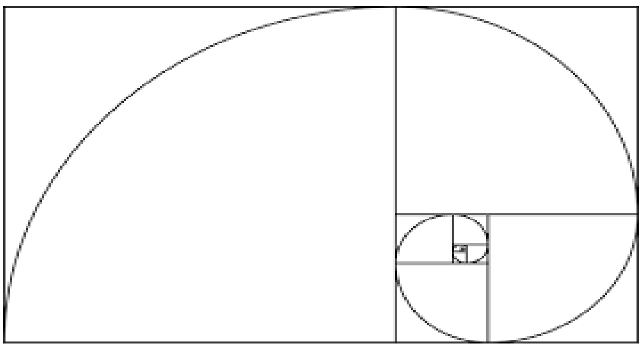

Figure 1. Golden Spiral

4. Procedure

Fibonacci Series is very well known for being the sequence where numbers become much varied and distinct as we move ahead in the sequence. So given this fact, we will be using the Fibonacci Series for the generation of key for the purpose of end-to-end encryption. In order to do so let us first understand the concept of Fibonacci in brief.

4.1 Encryption

Encryption can be defined as the process of transforming the message into some scrambled or unreadable form so that it can prevent the unauthorized access of the sensitive information. The purpose of encryption is to keep information, to be shared between two parties, secure from third person. Each encryption algorithm is based on two general principles namely substitution and transposition. In substitution principle, each element in the plaintext is substituted or mapped to another element.

As, internet is highly vulnerable to various attacks, sending sensitive information over the Internet may be dangerous. One of the ways to protect the sensitive Information is using the cryptographic techniques. Encryption is the process of transforming the information into unreadable form. Here Fibonacci term haven been used to randomly generate the secret key for encryption and decryption purpose. Using Fibonacci numbers and generating random keys provide significant security to shared information [10].

4.2 Program Code

Now we are coding a program using C language which generates an encryption key using Fibonacci number.

main.c Save Run

```c
#include <stdio.h>
#include <string.h>

int fibonacci(int n) {
    if (n <= 1)
        return n;
    return fibonacci(n - 1) + fibonacci(n - 2);
}

void encrypt(char *sentence, int shift) {
    int len = strlen(sentence);
    for (int i = 0; i < len; i++) {
        if (sentence[i] >= 'A' && sentence[i] <= 'Z') {
            sentence[i] = 'A' + (sentence[i] - 'A' + shift) % 26;
        } else if (sentence[i] >= 'a' && sentence[i] <= 'z') {
            sentence[i] = 'a' + (sentence[i] - 'a' + shift) % 26;
        }
    }
}

int main() {
    char sentence[100];
    int shift;

    printf("Enter a sentence: ");
    fgets(sentence, sizeof(sentence), stdin);
```

Figure 2. Coding-I

```c
    sentence[strcspn(sentence, "\n")] = '\0'; // Remove newline character

    printf("Enter the shift (Fibonacci number): ");
    scanf("%d", &shift);

    int fib = fibonacci(shift);

    encrypt(sentence, fib);

    printf("Encrypted sentence: %s\n", sentence);

    return 0;
}
```

Figure 3. Coding-II

4.3 Methodology of the Program

This code implements a Caesar cipher encryption program with a twist. Instead of using a fixed shift value, it uses the Fibonacci sequence to determine the shift amount.

1. Header Files:
- stdio.h: This header file provides input/output functions like printf and scanf.
- string.h: This header file provides string manipulation functions *like strlen and strcspn.*

2. Fibonacci Function:
- fibonacci(int n): This function calculates the nth Fibonacci number recursively. It checks if n is less than or equal to 1. If so, it returns n. Otherwise, it returns the sum of the (n-1)th and (n-2)th Fibonacci numbers.

3. Encrypt Function:
- encrypt(char *sentence, int shift): This function encrypts the given sentence using a Caesar cipher with the specified shift value. It iterates through each character in the sentence and checks if it's an uppercase or lowercase letter. If it is, it calculates the new character by adding the shift value to its ASCII code and taking the modulo 26 to wrap around the alphabet.

4. Main Function:
- This function first prompts the user to enter a sentence and stores it in the sentence array.
- It then prompts the user to enter a shift value, which should be a Fibonacci number.
- It calculates the actual Fibonacci number corresponding to the entered shift value using the Fibonacci function.
- It calls the encrypt function to encrypt the sentence using the calculated Fibonacci shift.
- Finally, it prints the encrypted sentence.

5. Results

This program takes a word as input and encrypts it using a key based on the Fibonacci series. It shifts each letter in the word by a position determined by the Fibonacci number at that position.

5.1 Output of the Program

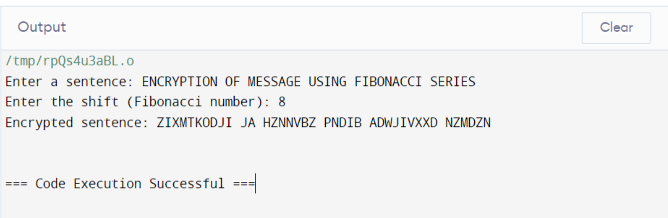

Figure 4. Output-I

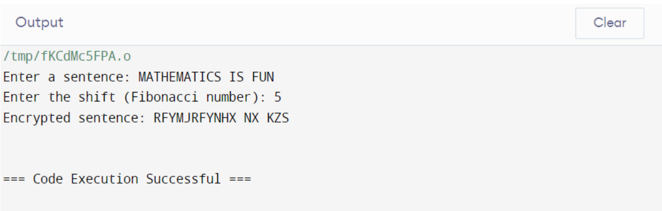

Figure 5. Output-II

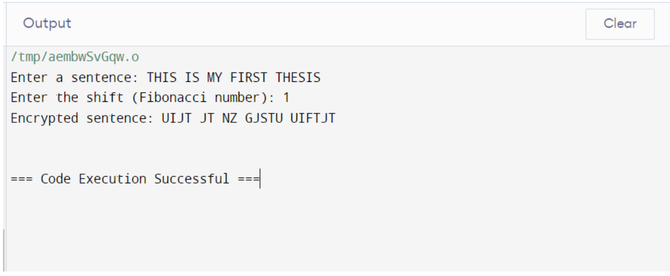

Figure 6. Output-III

6. Manifestations of The Fibonacci Series Around Us

The Fibonacci sequence finds diverse applications across various disciplines, underscoring its relevance and utility in real-world scenarios. From mathematics and computer science to biology, finance, music and art, the Fibonacci series serves as a foundational concept that facilitates problem-solving and innovation. This section delves into specific examples of how the Fibonacci sequence is utilized in different fields, demonstrating its versatility and impact.

6.1 Biology

One place where Fibonacci numbers consistently appear is in the leaf arrangement on plants, a field of study known as phyllotaxis. As leaves go up a plant stem, they follow a spiral arrangement. Starting at one leaf, let x be the number of turns of the spiral before a leaf is reached that is directly above the first leaf. Let y be the number of leaves encountered along the spiral between the first leaf and the last leaf in this arrangement, not counting the first. This ratio of x/y is known as the divergence of the plant [4].

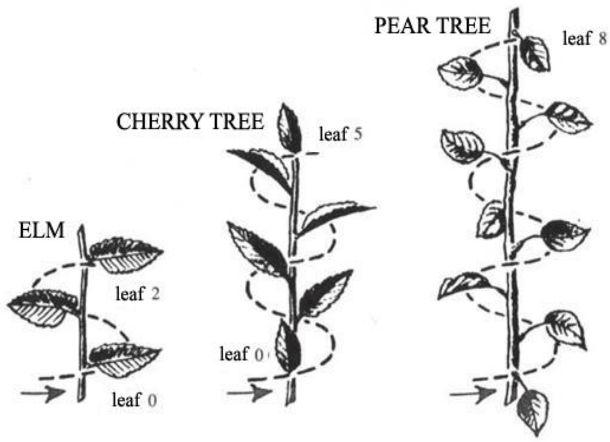

Figure 7. Leave arrangement as pe Fibonacci Numbers

In this phyllotactic ratio, the numerator and denominator are very often Fibonacci numbers. For example, leaves are generated after about 3/8 of a revolution for poplar, willow, and pear trees, 1/3 for beech and hazel, 2/5 for oak, cherry, and apple, 1/2 for elm and lime, and 5/13 for almond. Other phyllotactic ratios include 3/5, 5/13, and 8/13 [2].

Figure 8. Leave arrangement as per Fibonacci Numbers

Figure 9. Primrose

Figure 10. Larkspur

Figure11. Buttercup

Figure 12. Delphiniums

Figure 13. Daisy

Figure 14. Chicory

The Fibonacci numbers are present in the leaf or petal arrangement of most plants. It has been speculated that the reason these numbers are often present in such arrangements could be to maximize the amount of light received or the space allotted for each leaf or petal on the plant. A stem growing upwards will generate leaves, which branch out at regular angular intervals, spiraling up the stalk. If the leaves on a stem all grew with angular intervals that were multiples of 360°, then they would be growing, one directly above the other. The top few leaves would then block the lower leaves and prevent them from receiving as much sunlight and moisture [2].

A pineapple is covered in hexagonally shaped scales, known as bracts. These bracts form spirals in three different directions, each passing through opposing sides of the hexagon. Five spirals rise gradually in one direction, eight spirals rise at a medium rate in a second direction, and thirteen spirals rise steeply in the third direction, giving three consecutive Fibonacci numbers for the three different sets [12].

Figure 15. Pineapple

Figure 16. Pinecone

Some pinecones have three gradual and five steep spirals, while others have eight gradual and thirteen steep spirals. There is one set of spirals going steeply in one direction and another set of spirals going gradually in the other direction, and the number of spirals in each set is a Fibonacci number [5].

A similar Fibonacci spiraling tendency also surfaces when examining the centers of flowers, the spines of various types of cacti, and the leaves on certain succulents. In this case, one set of spirals can be found going in a clockwise direction, and a second set is found going in a counterclockwise direction. The number of spirals going clockwise and the number of spirals going counterclockwise are consecutive Fibonacci numbers.

Figure. 17 Sunflower's Centre

This is most clearly shown in the sunflower. The seeds at the center of the flower head spiral clockwise and counterclockwise. While the numbers of spiral sets depend on the age and development of the sunflower, they are always Fibonacci numbers. The two numbers can vary from 13 and 21, to 34 and 55, to 89 and 144 [12].

Also, when a budding rose is viewed from above, we see that the petals are unfolding in a spiraling pattern. If the angles between any two successive petals are measured, it is found that the angles are about 137.5°, the golden angle [7].

The golden spiral, found in pinecones, flower seed heads, and pineapples, can be found in countless other places in nature as well. The curl of a growing fern follows the pattern of a logarithmic spiral, starting out tightly furled, but loosening as it grows. This same spiral can be traced in ocean waves curling forward upon themselves before crashing on the shore. The spiral form within a galaxy conforms to a golden spiral as well, as does the spiraling shape of a storm [5].

Figure 18. Fern

Figure 19. Storm

Figure 20. Ocean Waves

Figure 21. Galaxy

Also, in fruits a similar pattern is observed. If we cut a banana we will see three rings, if we cut the apple there is a five-pointed star, a date star with 8 edges.

Figure 22. Banana

Figure 23. Apple

Fibonacci numbers also manifest themselves in the animal kingdom. This sequence of numbers, which first made its appearance in a problem about the regeneration of rabbits, also shows up in the regeneration of other living creatures. The numbers can be discovered by an inspection of the family tree of the male bee. There are three types of bees living in a bee hive: the queen, who produces eggs; the male bees, who do no work; and the female bees, who do all the work [12]. The female bees develop from fertilized eggs, meaning they have both mothers and fathers. The male bees, on the other hand, develop from unfertilized eggs, meaning they have only mothers but no fathers. They do, however, have grandfathers, as each female bee has a father [5]. So, one male bee has one mother, two grandparents, three great-grandparents, five great-great-grandparents, and eight great great-great-grandparents. The number of bees in each preceding generation is a Fibonacci number [7].

One of the most intriguing appearances of the Fibonacci sequence in the animal kingdom is in the spiral which indicates animal growth. One of the best examples of the golden spiral can be found in the shell of the chambered nautilus. This spiral can be found in many other places throughout the animal kingdom such as parrot beaks, elephant tusks, the tail of a seahorse, and the horns of bighorn sheep. Other manifestations of the spiral include spider webs, cat claws, the growth patterns of many seashells, and an insect's path as it approaches a light source. All these spirals possess the basic characteristics of the golden spiral, as they all increase in size while still maintaining the same shape, and most exhibit the Fibonacci proportions in their spirals [5].

Figure 24. Shell of Chambered Nautilus

Figure 25. Bighorn Sheep

Figure 26. Elephant's Tusk

Figure 27. Seahorse

Humans exhibit Fibonacci characteristics too. Every human has two hands, each one of these has five fingers and each finger has three parts which are separated by two knuckles. All of these numbers fit into the sequence. Moreover, the lengths of bones in a hand are in Fibonacci numbers. The cochlea of the inner ear and the fingerprints of humans forms a Golden Spiral.

Figure 28. Human Hand

Figure 29. Fingerprint

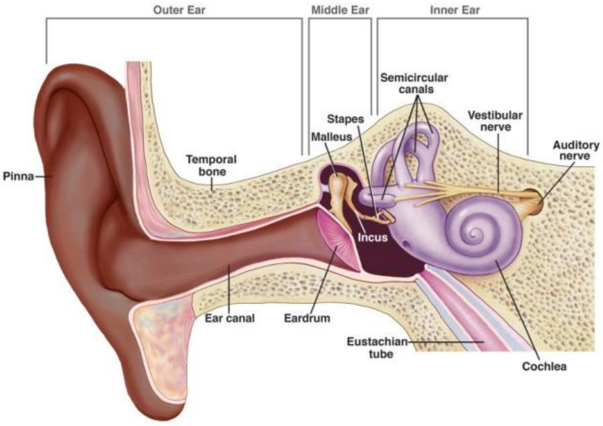

Figure 30. Cochlea

These examples illustrate only a few instances where Fibonacci numbers are found in nature. The manifestations of the Fibonacci numbers and the golden ratio are seemingly endless. When one begins looking for these occurrences, they suddenly can be found everywhere. Fibonacci numbers and the golden ratio also play a part in nature exhibited by pentagons. These pentagons appear in many places in nature, often manifested as a star-like shape. The center design of a sand dollar bears the shape of a pentagon, as do the shapes of many starfish, the seed placement in the cross section of an apple, and the forms of many flowers. Snowflakes are constructed according to the golden ratio. Pine needles often grow in groups of 2, 3, or 5. The number of segments in most plant pods is a Fibonacci number [5]. These manifestations occur far too often to be pure chance or coincidence. Instead, they indicate the mathematical nature of a world formed with order and precision.

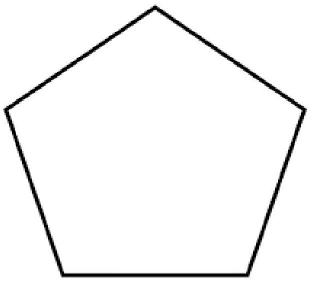

Figure 31. Pentagon

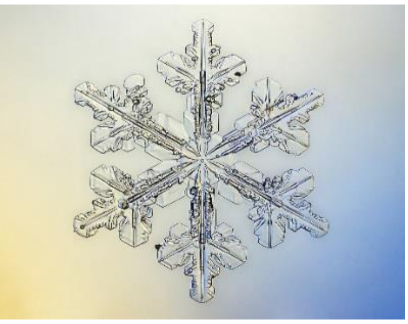

Figure 32. Snowflakes

6.2 Architecture

One of the earliest examples can be found in the Great Pyramid at Giza. Let be the base of a triangle which goes from the midpoint of a side of the pyramid to the center of the square base. Let be the diagonal up the side of the pyramid from the same midpoint of the side to the very top of the pyramid. For the Great Pyramid, the approximate lengths of and are 612.01 feet and approximately 377.9 feet, respectively

$$\frac{a}{b} = \frac{612.01}{377.9} = 1.62$$

which is very close to the golden ratio [9].

Figure 33. Pyramid of Giza

Whether this indicates that the ancient Egyptians knew about the golden ratio, or simply that they chose those dimensions because they were visually appealing is a point of great debate.

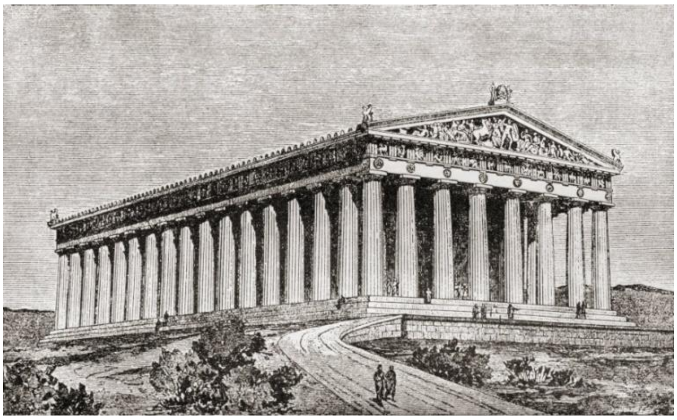

Figure 34. Parthenon

Another well-known example of the golden ratio in architecture is the Parthenon of ancient Greece. Located on the Acropolis in Athens, the Parthenon was built as a temple to house the statue of the Greek goddess Athena. The dimensions of the front of the building fit into a golden rectangle, and the structure of the building lends itself to being partitioned off into all sorts of golden rectangles. Much of the ornamentation involves the golden ratio in its measurements. Exactly how much of this was intentional on the part of the ancient architects remains uncertain [12]. The designs of many buildings built during the Renaissance involve Fibonacci numbers or the golden ratio. For example, the Cathedral in Florence involves the Fibonacci numbers 55, 89, and 144, as well as 17, which is half of 34, and 72, which is half of 144. The strongest example can be found in the windows, which have proportions of 89 and 55

$$\frac{89}{55} = 1.6181818 \ldots,$$

which is very close to the golden ratio [12].

Figure 35. Florence Cathedral

6.3 Art

The golden ratio also figures quite prominently in works of art, both in sculptures and in paintings. In the case of the statue Apollo Belvedere, the measurements from his feet to his navel and from his navel to the top of his head form the golden ratio, as do the measurements from his navel to his shoulders and from his shoulders to the top of his head. The entire figure of the statue Aphrodite of Melos is divided into the golden ratio by her navel [12]. Again, how much of this was intentional by the sculptors is uncertain. The golden ratio can be found in art everywhere, from the Middle Ages paintings of Madonna to ancient Chinese bowls to Syrian floor mosaics to Indian statues of Buddha [5] Leonardo da Vinci, a man of science as well as a brilliant painter, utilized the golden ratio in the majority of his work. In his well-known sketch of the Vitruvian man, the ratio of the side of the square which corresponds to the man's arm span and height to the radius of the circle which contains his outstretched arms and legs is the golden ratio [12]. In the Mona Lisa, a golden rectangle can be used to enclose the space from the top of her head to the top of her bodice. Dividing this rectangle into a square result in a square that precisely encloses her head, with her left eye at the center [3].

Figure 36. Mona Lisa

Figure 37. Madonna

Figure 38. Apollo Belvedere

Figure 39. Buddha

Figure 40. Chinese Bowls

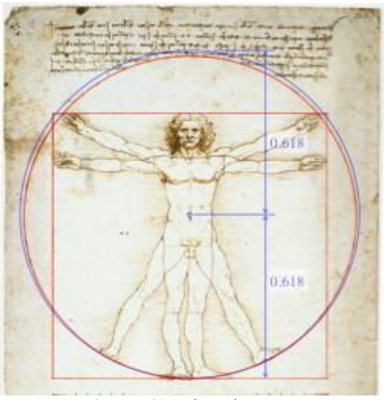

Figure 41. Vitruvian Man

Figure 42. Syrian Mosaic

These are but a few of countless examples of how the golden ratio, Fibonacci numbers, and golden rectangles are involved in the construction and architecture of buildings, as well as in the structure of sculptures and paintings, both ancient and modern.

6.4 Music

Fibonacci numbers are clearly illustrated when looking at the keyboard of a piano. The first six numbers in the Fibonacci sequence can be found by looking at just one octave of keys. Each octave is composed of 13 keys, 8 of which are white and 5 of which are black, and the black keys are partitioned into groups of 2 and 3 [5]. The violins made by Antonio Stradivarius are the most sought after of all violins, and today they can cost several million dollars. The proportions and components of these instruments have been carefully studied by those who wish to replicate them, and it turns out the violin is divided into proportions of 2, 3, 5, 8, and 13 [12]. The true relationship between music and Fibonacci numbers can only be found when the actual musical compositions are examined. In many of Chopin's preludes, the climax of the music is located very near the place where the golden ratio would divide the length of the piece. This is especially true of his Prelude No. 1 in C major, which has 34 measures. The climax of this piece occurs in measure 21, and the ratio of the two comes close to the golden ratio, as

$$\frac{34}{21} = 1.619 \ldots$$

Something similar happens in his Prelude No. 9 in E major. This piece contains 48 beats, and the climax occurs on beat 29. The ratio of these two numbers also comes close to the golden ratio, as $\frac{48}{29} = 1.655 \ldots$. This occurs in a number of preludes, although there are also many in which it does not happen [12]. The first movement of Beethoven's Fifth Symphony is divided into the golden ratio by the opening five measures, which repeat 372 measures later, and again after 228 measures. There are 377 measures before the middle repetition and 233 measures after the middle repetition, providing a ratio of $\frac{377}{233} = 1.618\ldots$ [5].

Out of Mozart's seventeen piano sonatas employing what is known as the sonata-allegro form, six are exactly divided into the golden ratio, and eight are very close. Since a total of 82% of his sonatas are divisible by the golden section, it would seem that the use of the golden ratio was very important to Mozart in his compositions [12]. These composers, along with Haydn, Wagner, and Bartok, represent only a few of the musicians, both modern and classical, who have compositions that are divided into the golden ratio [12]. The use of the golden ratio seems more intentional in music than in art.

Figure 43. Keyboard

Figure 44. Violin

7. Conclusion and Future Scope

In conclusion, the Fibonacci series stands as a testament to the beauty and ubiquity of mathematical patterns in nature and human endeavors. Through centuries of study and exploration, this sequence has captured the imagination of scholars and practitioners alike, inspiring countless discoveries and applications. By unraveling its historical origins, mathematical properties, and practical implications, this paper seeks to deepen our understanding of the Fibonacci series and its enduring legacy in the modern world.

The program for encryption which we coded using Fibonacci Numbers uses the Fibonacci sequence to introduce a dynamic and unpredictable shift value, making the encryption more secure than a simple Caesar cipher with a fixed shift.
The Fibonacci sequence is famously associated with technical analysis in financial markets, where Fibonacci retracement levels are used to predict price movements. Future advancements may refine these techniques or apply Fibonacci concepts to new areas within finance and investment strategies
Fibonacci coding is a technique used in data compression, and future developments in data science and information technology may lead to further applications of Fibonacci-based encoding methods for efficient data storage and transmission.

Overall, the future scope of the Fibonacci series is broad and multidisciplinary, with opportunities for exploration and innovation in diverse areas ranging from mathematics and computer science to art, biology, finance, and education.

Data Availability
The availability of data comes from various research papers, each offering valuable insights and contributing to our understanding of the subject. Through an extensive review of literature and various online sources, we have gleaned much knowledge and identified potential avenues for future exploration. These insights serve as a foundation for uncovering new possibilities and addressing the boundaries that emerge from current research.

Conflict of Interest
The authors declare that they do not have any conflict of interest.

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Authors' Contributions
Isha Patil researched literature and conceived the study. Pooja Sharma was involved in protocol development, gaining ethical approval, and data analysis. Garima Singh finalized the draft of the manuscript. All authors reviewed and edited the manuscript and approved the final version of the manuscript.

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References

[1] A. Grigas. "*The Fibonacci Sequence Its History, Significance, and Manifestations in Nature*", Liberty University, 2003
[2] Adam, J. A. " *Mathematics in nature: Modeling patterns in the natural world*. " Princeton, NJ: Princeton University Press pp.217-271, 2003
[3] Atalay, B., Math and the Mona Lisa: "*The art and science of Leonardo da Vinci*. " New York, NY: Smithsonian Books, pp. 176-177,2006
[4] Devlin, K., "*The man of numbers: Fibonacci's arithmetic revolution*. " New York, NY: Walker, pp.143-146 , 2011
[5] Garland, T. H., "*Fascinating Fibonaccis: Mystery and magic in numbers*. " Palo Alto, CA: Dale Seymour pp. 9-69, 1987
[6] Gosai, Ishita, "*Fibonacci Sequence and It's Applications*", International Journal of Research and Analytical Reviews, Vol.6,Issue.2,pp.241-247, 2019
[7] Hemenway, P., "*Divine proportion: Phi in art, nature, and science*. ",2019 New York, NY: Sterling Publishing, pp.135,2005
[8] Huntley, H. E.,"*The divine proportion: A study in mathematical beauty*. " New York, NY: Dover Publications, pp.168, 1970
[9] Livio, Mario, "*The golden ratio: The story of phi, the world's most astonishing number.* " New York, NY: Broadway Books, pp. 56-57, 2002
[10] Mohd Vasim Ahamad, "*An Improved Playfair Encryption Technique Using Fibonacci Series Generated Secret Key.* ",2018
[11] Orhani, Senad, "*Fibonacci Numbers as a Natural Phenomenon*", International Journal of Scientific Research and Innovative Studies, Issue.1,pp.7-13,2022
[12] Posamentier, A. S., & Lehmann, I. "*The fabulous Fibonacci numbers.* " *Amherst, NY: Prometheus Books,* pp.19-296, 2007
[13] Posamentier, A. S., & Lehmann, I.. "*The glorious golden ratio*. " Amherst, NY: Prometheus Books pp.15,105, 2012
[14] Rosen, K. H., "*Discrete mathematics and its applications*. " New York, NY: McGraw-Hill pp.514-517, 2012
[15] S. Dubey, R. Jhaggar, R. Verma, D. Gaur, "*Encryption and Decryption of Data by Genetic Algorithm*", International Journal of Scientific Research in Computer Science and Engineering, Vol.5, Issue.3, pp.47-52, 2017
[16] Sinha, Sudipta, "*The Fibonacci Numbers and Its Amazing Applications*", International Journal of Engineering Science Invention, Vol.6, Issue.9, pp.7-14, 2017
[17] Vaishali Billore, Naresh Patel, "*Matrix Modelling of Extended Generalized Fibonacci Polynomial*", International Journal of Scientific Research in Mathematical and Statistical Sciences, Vol.9, Issue.6, pp.61-64, 2022

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