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Bayes Estimation of Parameter in Inverse Maxwell Distribution under Weighted Quadratic Loss Function

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Abstract— The inverse Maxwell distribution is the probability distribution of the reciprocal of the Maxwell random variable. This distribution has applications in reliability and life testing. In this paper, Bayes estimator of the parameter in the inverse Maxwell distribution is derived employing Jeffreys non-informative prior under weighted quadratic loss function. The Bayes estimator in this case is the minimax estimator. Efficiency of the Bayes estimator is compared with the maximum likelihood estimator. Application of the inverse Maxwell distribution to a real life situation is also studied.

Keywords*— Bayes estimator, Inverse Maxwell distribution, Jeffreys prior, Maximum likelihood estimator.*

I. INTRODUCTION

Maxwell distribution was introduced during 19th century by [6] for studying on the motions and collisions of perfectly elastic spheres. After a decade again described by [1]. The probability distributions of reciprocal of random variable have also attracted the attention of researchers for their applications in various fields. Recently, [8] proposed the inverse Maxwell distribution as a lifetime distribution. They discussed estimation of parameter in the inverse Maxwell distribution applying the methods of moments and the maximum likelihood based on uncensored data as well as type-II censored data. [5] showed that the maximum likelihood estimator based on the uncensored sample is the uniformly minimum variance unbiased and the minimum variance bound estimator. They discussed the asymptotic distribution of the maximum likelihood estimator. They determined the confidence limits based on small and large samples.

In the situations where the prior information is difficult to obtain or the experiment is performed for the first time, the Bayesian estimation is discussed employing non-informative prior. No attempts to study the Bayesian estimation of parameter of the inverse Maxwell distribution under weighted squared error loss function in the literature. In this paper Bayesian estimation of the parameter in the inverse Maxwell distribution is considered under weighted quadratic loss function employing Jeffreys non-informative prior. This study is useful for researchers and practitioners in reliability theory and life testing, where inverse Maxwell distribution is widely used.

The inverse Maxwell distribution and its properties are presented in Section 2. Posterior distribution of the parameter and the Bayes estimator are derived in Section 3. Simulation studies are carried out to compare the accuracy of the Bayes estimator with the maximum likelihood estimator. Results are discussed in Section 4. Section 5 deals with the application of the inverse Maxwell distribution to a real life example. Finally the results are briefly summarised in Section 6.

II. INVERSE MAXWELL DISTRIBUTION

A random variable X is said to be distributed according to an inverse Maxwell distribution with parameter θ if its probability density function is given by

$$f(x|\theta)=\begin{cases}\dfrac{4}{\sqrt{\pi}}\dfrac{\exp(-1/\theta x^2)}{\theta^{3/2}x^4}, & \text{for } x>0, \quad \theta>0\\ 0, & \text{otherwise}\end{cases}$$

Hereafter, this distribution will be referred as IM (θ). The probability distribution of Z=1/X is the Maxwell distribution with the probability density function

$$h(z|\theta)=\begin{cases}\dfrac{4}{\sqrt{\pi}}\dfrac{x^2\exp(-x^2/\theta)}{\theta^{3/2}}, & \text{for } x>0, \quad \theta>0\\ 0, & \text{otherwise}\end{cases}$$

As mentioned in [5], $Y=1/X^2$ is a Gamma (3/2, θ) random variable.

The likelihood function of θ for a random sample $\underset{\sim}{x}=(x_1, x_2, ..., x_n)$ of n observations drawn from the IM (θ) distribution given by

$$L(\theta\,|\,\underset{\sim}{x})=\left(\frac{4}{\sqrt{\pi}}\right)^{n}\left(\frac{1}{\theta}\right)^{3n/2}\prod_{i=1}^{n}\frac{1}{x_i^4}\exp\left\{-\frac{1}{\theta}\sum_{i=1}^{n}\frac{1}{x_i^2}\right\} \qquad (1)$$

It is straight forward, as noted in [5] that the family of IM (θ) distribution belongs to one-parameter exponential family of distributions and thereby $T(\underset{\sim}{X})=\sum_{i=1}^{n}\frac{1}{X_i^2}$ is the complete minimal sufficient statistic. The probability distribution of $T(\underset{\sim}{X})$ is the gamma (3n/2, θ) distribution.

[8] derived the maximum likelihood estimator of θ as $\hat{\theta}_{ML}(\underset{\sim}{X})=\frac{2}{3n}T(\underset{\sim}{X})$. [5] showed that $\hat{\theta}_{ML}(\underset{\sim}{X})$ is the uniformly minimum variance unbiased as well as the minimum variance bound estimator.

III. BAYES ESTIMATION OF θ

Recently, [5] derived Fisher's information about θ for given $\underset{\sim}{x}$ as

$$I(\theta)=\frac{3n}{2\theta^2}.$$

Here, Jeffreys non-informative prior for θ is given by

$$g(\theta)\propto\sqrt{\frac{3n}{2}}\frac{1}{\theta}. \qquad (2)$$

The probability density function of θ for given sample $\underset{\sim}{x}$ and $T(\underset{\sim}{x})=t$ can be obtained from (1) and (2) as

$$g(\theta\,|\,\underset{\sim}{x})=\frac{t^{3n/2}}{\Gamma\left(\frac{3n}{2}\right)}\left(\frac{1}{\theta}\right)^{\frac{3n}{2}+1}e^{-t/\theta}\ ,\ \theta>0 \qquad (3)$$

Thus, the posterior distribution of θ for given sample $\underset{\sim}{X}=\underset{\sim}{x}$ is the $IG\left(\frac{3n}{2},\ T(\underset{\sim}{X})\right)$ distribution.

The weighted, quadratic loss function is defined, in general, as ([3])

$$L(\theta,\hat{\theta}(\underset{\sim}{x}))=w(\theta)(\theta-\hat{\theta}(\underset{\sim}{x}))^2 .$$

Suppose that the weight is proportional to 1/θ for each θ. Then, the Bayes estimate

$$\hat{\theta}_B(\underset{\sim}{x})=E\left(\theta\,|\,\underset{\sim}{x}\right)$$

$$=\frac{2T(\underset{\sim}{x})}{(3n+2)}$$

The risk for corresponding to the weighted quadratic loss function is given by

$$R(\theta,\hat{\theta}(\underset{\sim}{x}))=\frac{1}{\theta^2}\left\{\theta^2-2\theta E(\hat{\theta})+E(\hat{\theta})^2\right\}$$

$$=\frac{1}{\theta^2}\left\{\theta^2-\frac{6n}{3n+2}\theta^2+\frac{3n}{2}\left(\frac{3n}{2}+1\right)\theta^2\right\}$$

$$=1-\frac{6n}{3n+2}+\frac{3n(3n+2)}{4}$$

After simplifications,

$$R(\theta,\hat{\theta}(\underset{\sim}{x}))=\frac{9n^2(3n+4)+8}{4(3n+2)}$$

It may be noted that the risk function is a constant for all θ and for given values of θ. Hence, $\hat{\theta}_B$ is a minimax estimator ([4], pp. 249-250) and [7], pp.288).

IV. SIMULATION STUDIES

Simulation studies are carried out to compare the quality of the Bayes estimator, $\hat{\theta}_B$ with the maximum likelihood estimator $\hat{\theta}_{ML}$. Random samples are generated from the IM(θ) distribution for each θ = 0.25, 0.50, 1.00, 2.00, 5.00 and 10.00. The sample sizes are taken as n=10, 25, 50, 100 and 200. The values of $\hat{\theta}_{ML}$ are calculated for each sample and are presented in Table 1.

A set of 10,000 observations is obtained from the posterior distribution of θ for each sample and the values of Bayes estimates are computed. The values are displayed in Table 1. It can be observed in general from the Table 1 that the Bayes estimates and the maximum likelihood estimates become very close to the true value of θ when the sample size increases. There are deviations among these two types of estimates. Bayes estimates are relatively close to the true value than the maximum likelihood estimates in each case.

Table 1. Bayes Estimates and Maximum Likelihood Estimates

Sample Size	Parameter	$\hat{\theta}_{ML}$	$\hat{\theta}_{B}$
n=10	θ=0.25	0.18392	0.19685
	θ=0.50	0.41005	0.43982
	θ=1	0.86897	0.93314
	θ=2	1.77294	1.89801
	θ=5	4.16630	4.46673
	θ=10	8.48586	9.09199
n=25	θ=0.25	0.21863	0.22468
	θ=0.50	0.43267	0.44534
	θ=1	0.93599	0.96241
	θ=2	2.10071	2.04682
	θ=5	4.71102	4.84091
	θ=10	9.76377	9.83127
n=50	θ=0.25	0.24023	0.24331
	θ=0.50	0.48272	0.48926
	θ=1	1.02895	1.04508
	θ=2	1.96827	1.99435
	θ=5	4.91864	4.98325
	θ=10	9.76985	9.93228
n=100	θ=0.25	0.24972	0.25123
	θ=0.50	0.49500	0.49853
	θ=1	1.14813	1.15564
	θ=2	1.91973	1.93299
	θ=5	4.97914	4.99454
	θ=10	9.79782	9.80198
n=200	θ=0.25	0.25634	0.25700
	θ=0.50	0.50170	0.50341
	θ=1	1.03118	1.02736
	θ=2	2.04187	2.03535
	θ=5	5.07772	5.09416
	θ=10	9.93856	9.97476

V. REAL LIFE EXAMPLE

[2] obtained the percentage of silica is a randomly selected 22 chrondites meteors as

20.77	22.56	22.71	22.99	26.39	27.08
27.32	27.33	27.57	27.81	28.69	29.36
30.25	31.89	32.88	33.23	33.28	33.40
33.52	33.83	33.95	34.82		

Since Rayleigh and gamma distributions are two other competing distributions due to their distributional properties, fitness of the inverse Maxwell, Rayleigh and gamma distributions is studied applying the Kolmogorov-Smirnov test. Estimates of the parameters of the inverse Maxwell, Rayleigh and the gamma distributions are computed using their respective maximum likelihood estimators. The value of the test statistic is computed as 0.31005, 0.35267 and 0.40998 corresponding to the inverse Maxwell, Rayleigh and gamma distributions. Theoretical value of the test statistic at 1% level of significance is 0.33666. Hence, it may be decided that the inverse Maxwell distribution provides relatively better to the above data.
The Bayes estimate of θ is calculated based on this data is 0.00082. The maximum likelihood estimate of θ is calculated as 0.00084 from the above data.

VI. SUMMARY

The inverse Maxwell distribution is the probability distribution of the reciprocal of a random variable distributed according to the Maxwell distribution. The inverse Maxwell distribution has applications in reliability and life testing. Bayes estimator is calculated and the risk function is derived, it is constant. Therefore the Bayes estimator is also called minimax estimator. Through simulation studies, Bayes estimator values are very close to the original parameter values as compared to maximum likelihood estimator values. Appropriateness of the inverse Maxwell distribution in a real life situation is discussed and is shown that it provides relatively better fit than the Rayleigh and gamma distributions.

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