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Mixture of Exponential and Weighted Exponential Distribution: Properties and Applications

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Abstract-In this paper, a new model entitled as "Mixture of Exponential and Weighted Exponential Distribution (MEWED): Properties and Applications" is obtained. Some statistical properties of the new model are unfolded. The method of maximum likelihood is used to estimate the parameters of the proposed model. The usefulness of the proposed model is illustrated by using real life data sets.

Keywords- Exponential Distribution, Weighted Exponential Distribution, Maximum Likelihood Estimation and Real Life Data.

I. INTRODUCTION

Mixture distributions are used to generate the new models which are more flexible in modeling a variety of data sets. Pearson [1] developed the idea of mixture distributions by considering two normal distributions and estimates its parameters. Several other researchers have studied in detail the properties and characteristics of mixture distributions, such as Jiang et al. [2] proposed the Inverse Weibull mixture model while Adnan et al. [3] studied Laplace Mixture distribution. Kamaruzzaman et al. [4] fit the two component mixture normal distribution. Mendenhall and Hader [5] estimated the parameters of mixed exponentially distributed failure time distributions. Marco et al. [6] obtained the estimates of mixture Pareto distribution using MLE technique. Sankaran and Nair [7] studied the finite mixture of Pareto distribution using Bayesian approach. Recently, Satsayamon Suksaengrakcharoen and Winai Bodhisuwan [8] studied the mixture Generalized Gamma distribution. Roy et al. [9-11] proposed and studied various mixtures of various standard distributions.

II. MATERIALS AND METHODS

The pdf and cdf of exponential distribution is given as

$$f_1(x)=\lambda e^{-\lambda x} \quad ; x>0,\ \lambda>0. \qquad (1)$$

$$F_1(x)=1-e^{-\lambda x}.$$

Let $X \geq 0$ be a r.v. having pdf $f(x)$ and $w(x)\geq 0$ be the weight function. Then the weighted density function $f_w(x)$ is given by:

$$f_w(x)=\frac{w(x)f(x)}{E\left(w(x)\right)} \quad , \quad \text{where } E\left(w(x)\right)=\mu_w<\infty. \qquad (2)$$

By choosing $w(x)=x^{\omega}$ and using (3), we get the pdf of weighted exponential distribution (WED) given as:

$$f_w(x)=\frac{(\lambda^{\omega+1}x^{\omega}e^{-\lambda x})}{\Gamma(\omega+1)} \quad ; x>0,\ \lambda>0,\ \omega>0. \qquad (3)$$

While the corresponding cdf is given as:

$$F_w(x)=\frac{\gamma(\omega+1,\lambda x)}{\Gamma(\omega+1)}.$$

A mixture distribution is obtained by mixing two or more distributions. Since, in this article, we consider the mixture of two distributions. Thus, the pdf of mixture distribution is given by: $f(x) = pf_1(x) + (1-p) f_2(x)$

Where p is the mixing parameter, $f_1(x)$ is the pdf of exponential distribution and $f_2(x)$ is the pdf of WED.
The pdf of MEWED is obtained by mixing (1) and (3) given as:

$$f(x) = p(\lambda e^{-\lambda x}) + (1-p)\frac{(\lambda^{\omega+1} x^{\omega} e^{-\lambda x})}{\Gamma(\omega+1)} \; ; x > 0, \lambda > 0, \omega > 0 . \qquad (4)$$

Where $\lambda > 0$ is the rate parameter and $\omega > 0$ is the weight parameter of the distribution.
The cumulative distribution function corresponding to (4) is given as

$$F(x) = p(1-e^{-\lambda x}) + (1-p)\frac{\gamma(\omega+1, \lambda x)}{\Gamma(\omega+1)}$$

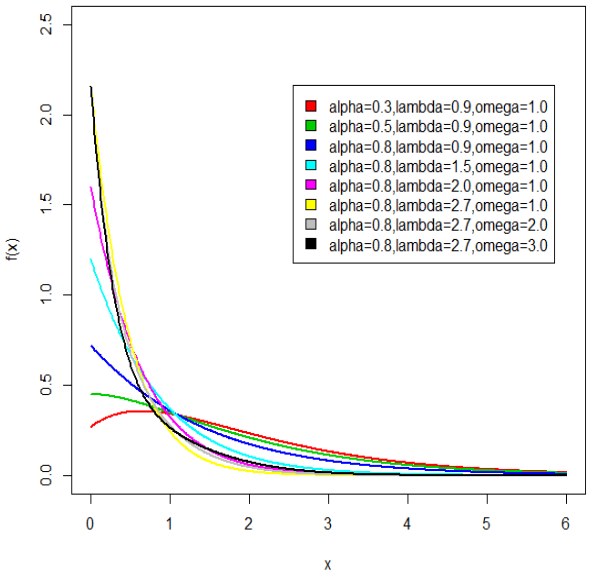

Figure 1: Probability distribution function

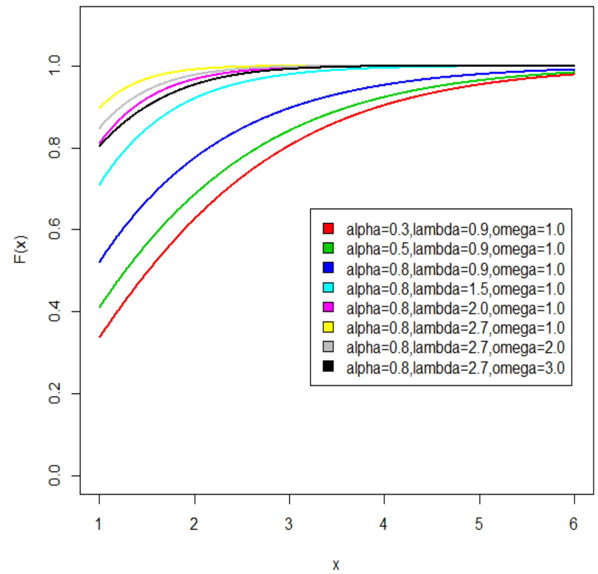

Figure 2: cumulative distribution function

III. RELATIONSHIP WITH OTHER DISTRIBUTIONS

i. If $p = 0$ in the expression (4), it reduces to the WED with pdf given as:

$$f(x) = \frac{\lambda^{\omega+1} x^{\omega} e^{-\lambda x}}{\Gamma(\omega+1)} .$$

ii. If $p = 0$ and $\omega = 1$ in the expression (4), we get length biased exponential distribution (LBED) with pdf given below:

$$f(x) = \lambda^2 x e^{-\lambda x} .$$

iii. If $p = 0$ and $\omega = 2$ in the expression (4), it reduces to the ABED with pdf as follows: $f(x) = \dfrac{\lambda^3 x^2 e^{-\lambda x}}{2}$.

iv. If $p = 1$ and $\omega = 0$ in the expression (4), it reduces to the exponential distribution (ED) with the following pdf:

$$f(x) = \lambda e^{-\lambda x} .$$

v. If $p = 1$ and $\lambda = 1$ in the expression (4), it reduces to the Standard exponential distribution (SED) with pdf as follows:

$$f(x) = e^{-x} .$$

IV. RELIABILITY ANALYSIS

This section is devoted to obtain the expression for reliability function, hazard function, reverse hazard function, cumulative hazard function and odds function.

i. The expression for reliability function of MEWED is given as:

$$R(x)=1-\left\{p(1-e^{-\lambda x})+(1-p)\frac{\gamma(\omega+1,\lambda x)}{\Gamma(\omega+1)}\right\}.$$

ii. The expression for hazard function of MEWED is given by:

$$h(x)=\frac{p(\lambda e^{-\lambda x})+(1-p)\dfrac{(\lambda^{\omega+1}x^{\omega}e^{-\lambda x})}{\Gamma(\omega+1)}}{1-\left\{p(1-e^{-\lambda x})+(1-p)\dfrac{\gamma(\omega+1,\lambda x)}{\Gamma(\omega+1)}\right\}}.$$

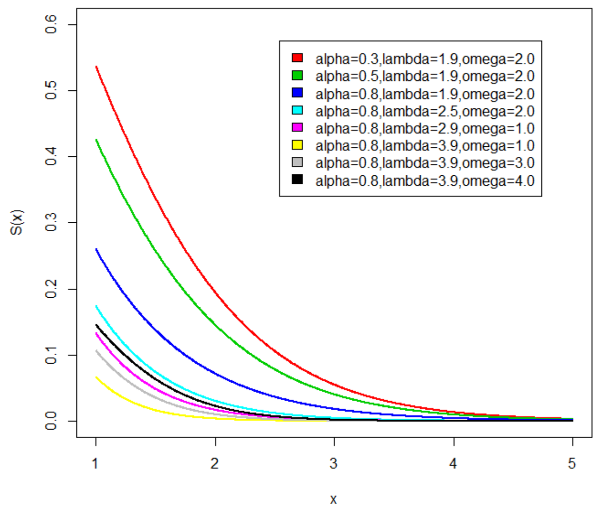

Figure 3: Reliability function

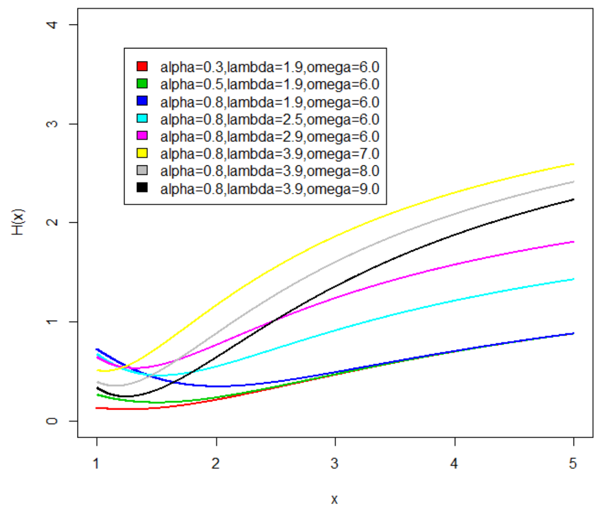

Figure 4: Hazard function

iii**.** The expression for reverse hazard function is given by:

$$\phi(x)=\frac{p(\lambda e^{-\lambda x})+(1-p)\dfrac{(\lambda^{\omega+1}x^{\omega}e^{-\lambda x})}{\Gamma(\omega+1)}}{p(1-e^{-\lambda x})+(1-p)\dfrac{\gamma(\omega+1,\lambda x)}{\Gamma(\omega+1)}}.$$

iv**.** The expression for cumulative hazard function is given by:

$$\Lambda(x)=\log\left(\frac{1}{1-\left(p(1-e^{-\lambda x})+(1-p)\dfrac{\gamma(\omega+1,\lambda x)}{\Gamma(\omega+1)}\right)}\right).$$

v**.** The expression for odds function is given by:

$$O(x)=\frac{p(1-e^{-\lambda x})+(1-p)\dfrac{\gamma(\omega+1,\lambda x)}{\Gamma(\omega+1)}}{1-\left\{p(1-e^{-\lambda x})+(1-p)\dfrac{\gamma(\omega+1,\lambda x)}{\Gamma(\omega+1)}\right\}}.$$

V. STATISTICAL PROPERTIES

This section is devoted to study some statistical properties of the MEWED.

5.1. Moments

The r^{th} moment about origin of a random variable X having MEWED is given by:

$$\mu_r' = E(x^r) = \int_0^\infty x^r f(x)dx$$

$$= \frac{p\Gamma(r+1)}{\lambda^r} + \frac{(1-p)\Gamma(\omega+r+1)}{\lambda^r\Gamma(\omega+1)}. \qquad (5)$$

The mean and variance of MEWED is obtained by using (5) given as

$$\text{Mean} = \frac{p+(1-p)(\omega+1)}{\lambda}. \qquad (6)$$

$$\text{Variance} = \frac{2p+(1-p)(\omega+1)(\omega+2)-\{p+(1-p)(\omega+1)\}^2}{\lambda^2}. \qquad (7)$$

The co-efficient of variation of MEWED is given by

$$C.V = \frac{\sqrt{[2p+(1-p)(\omega+1)(\omega+2)]-[p+(1-p)(\omega+1)]^2}}{p+(1-p)(\omega+1)}.$$

Also, the measure of skewness and kurtosis are given by

$$\beta_1 = \frac{\left[\left\{\frac{6p+(1-p)(\omega+1)(\omega+2)(\omega+3)}{\lambda^3}\right\} - 3\left\{\frac{p+(1-p)(\omega+1)}{\lambda}\right\} \left\{\frac{2p+(1-p)(\omega+1)(\omega+2)}{\lambda^2}\right\} + 2\left\{\frac{p+(1-p)(\omega+1)}{\lambda}\right\}^3\right]^2}{\left[\left\{\frac{2p+(1-p)(\omega+1)(\omega+2)}{\lambda^2}\right\} - \left\{\frac{p+(1-p)(\omega+1)}{\lambda}\right\}\right]^3}$$

$$\beta_2 = \frac{\left[\left\{\frac{24p+(1-p)(\omega+1)(\omega+2)(\omega+3)(\omega+4)}{\lambda^4}\right\} - 4\left\{\frac{p+(1-p)(\omega+1)}{\lambda}\right\}\left\{\frac{6p+(1-p)(\omega+1)(\omega+2)(\omega+3)}{\lambda^3}\right\} + 6\left\{\frac{p+(1-p)(\omega+1)}{\lambda}\right\}^2\left\{\frac{2p+(1-p)(\omega+1)(\omega+2)}{\lambda^2}\right\} - 3\left\{\frac{p+(1-p)(\omega+1)}{\lambda}\right\}^4\right]}{\left[\left\{\frac{2p+(1-p)(\omega+1)(\omega+2)}{\lambda^2}\right\} - \left\{\frac{p+(1-p)(\omega+1)}{\lambda}\right\}^2\right]^2}$$

5.2. Moment Generating Function (MGF)

The mgf of a random variable X denoted by $M_x(t)$ is given by:

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$$

$$= \int_0^\infty e^{tx}\left(p\lambda e^{-\lambda x} + \frac{(1-p)\lambda^{\omega+1}x^\omega e^{-\lambda x}}{\Gamma(\omega+1)}\right)dx$$

After solving the above integral, we get

$$M_x(t) = \frac{p\lambda(\lambda - t)^{\omega} + (1-p)\lambda^{\omega+1}}{(\lambda - t)^{\omega+1}}.$$

5.3. Characteristic Function

The characteristic function denoted by $(\phi_x(t))$ is given by:

$$\phi_x(t) = E(e^{itx}) = \int_0^{\infty} e^{itx} f(x)dx.$$

$$= \int_0^{\infty} e^{itx}\left(p\lambda e^{-\lambda x} + \frac{(1-p)\lambda^{\omega+1}x^{\omega}e^{-\lambda x}}{\Gamma(\omega+1)}\right)dx.$$

After solving the above integral, we get

$$\phi_x(t) = \frac{p\lambda(\lambda - it)^{\omega} + (1-p)\lambda^{\omega+1}}{(\lambda - it)^{\omega+1}}.$$

VI. INFORMATION MEASURES

This section is devoted to obtain the expression for Renyi entropy and beta entropy.

6.1. The Renyi entropy [12] of order ρ is given as

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\int_0^{\infty} f(x)^{\rho} dx\right\}.$$

Where $\rho > 0$ and $\rho \neq 1$.

$$I_R(\rho) = \frac{1}{1-\rho}\log\int_0^{\infty} \lambda^{\rho} e^{-\rho\lambda x}\sum_{n=0}^{\rho}\binom{\rho}{n}p^{\rho-n}\frac{(1-p)^n\lambda^{\omega n}x^{\omega n}}{\Gamma(\omega+1)^n}dx.$$

After solving the above integral, we get

$$I_R(\rho) = \frac{1}{1-\rho}\log\left[\sum_{n=o}^{\rho}\binom{\rho}{n}p^{\rho-n}\frac{(1-p)^n\lambda^{\rho-1}\Gamma(\omega n+1)}{\rho^{\omega n+1}\Gamma(\omega+1)^n}\right].$$

6.2. The beta or q-entropy introduced by Havrda and Charvat [13] as one parameter generalization of Shannon entropy (1948) is given by:

$$I_H(q) = \frac{1}{q-1}\left\{1 - \int_0^{\infty} f(x)^q dx\right\}.$$

Where $q > 0$ and $q \neq 1$.

$$I_H(q) = \frac{1}{q-1}\left[1 - \left\{\lambda^q \sum_{n=o}^{q}\binom{q}{n}p^{q-n}\frac{(1-p)^n\lambda^{\omega n}}{\Gamma(\omega+1)^n}\int_0^{\infty} e^{-q\lambda x}x^{\omega n}dx\right\}\right]$$

$$= \frac{1}{q-1}\left[1 - \sum_{n=o}^{q}\binom{q}{n}p^{q-n}\frac{(1-p)^n\lambda^{q-1}\Gamma(\omega n+1)}{q^{\omega n+1}\Gamma(\omega+1)^n}\right].$$

VII. ORDER STATISTICS

Suppose n samples are taken from the mixture of exponential and weighted exponential distribution and $X_{(1)}, X_{(2)}, X_{(3)}, ..., X_{(n)}$ be the corresponding order statistics. Then the pdf of r^{th} order statistics is given as :

$$f_r(x)=\frac{n!}{(r-1)!(n-r)!}\left[p(1-e^{-\lambda x})+(1-p)\frac{\gamma(\omega+1,\lambda x)}{\Gamma(\omega+1)}\right]^{r-1}\left[1-p(1-e^{-\lambda x})-(1-p)\frac{\gamma(\omega+1,\lambda x)}{\Gamma(\omega+1)}\right]^{n-r}$$
$$\left[p\lambda e^{-\lambda x}+\frac{(1-p)\lambda^{\omega+1}x^{\omega}e^{-\lambda x}}{\Gamma(\omega+1)}\right]. \tag{8}$$

By using (8), the pdf of first and nth order statistics are given as:

$$f_1(x)=n\left[1-p(1-e^{-\lambda x})-(1-p)\frac{\gamma(\omega+1,\lambda x)}{\Gamma(\omega+1)}\right]^{n-1}\left[p\lambda e^{-\lambda x}+\frac{(1-p)\lambda^{\omega+1}x^{\omega}e^{-\lambda x}}{\Gamma(\omega+1)}\right].$$

$$f_n(x)=n\left[p(1-e^{-\lambda x})+(1-p)\frac{\gamma(\omega+1,\lambda x)}{\Gamma(\omega+1)}\right]^{n-1}\left[p\lambda e^{-\lambda x}+\frac{(1-p)\lambda^{\omega+1}x^{\omega}e^{-\lambda x}}{\Gamma(\omega+1)}\right].$$

VIII. CHARACTERIZATION OF MEWED

Theorem 1: Suppose $X_1, X_2,..., X_n$ are n positive iid random samples taken from the MEWED with sample mean $\bar{x}_n$ and sample variance s_n^2 , then

$$\underset{n\to\infty}{lt}\,E\left[\frac{s_n^2}{\bar{x}_n^2}\right]=\lim_{n\to\infty}E\left[\frac{s_n}{\bar{x}_n}\right]^2\to\left(\frac{\sigma}{\mu}\right)^2$$

Proof: Since

$$E(\bar{x}_n)=\mu,\ \mathrm{var}(\bar{x}_n)=\sigma^2/n$$

Also, $E(\bar{x}_n^2)=\mathrm{var}(\bar{x}_n)+[E(\bar{x}_n)]^2$ (9)

Using (6) and (7) in (9), we obtain

$$E(\bar{x}_n^2)=\frac{2p+(1-p)(\omega+1)(\omega+2)-(1-n)\{p+(1-p)(\omega+1)\}^2}{n\lambda^2} \tag{10}$$

Also, we have $E(s_n^2)=\sigma^2$

$$=\frac{2p+(1-p)(\omega+1)(\omega+2)-\{p+(1-p)(\omega+1)\}^2}{\lambda^2}. \tag{11}$$

Now, $E(s_n^2)=E\left[\frac{s_n^2}{\bar{x}_n^2}\bar{x}_n^2\right].$

$$\Rightarrow\frac{E[s_n^2]}{E[\bar{x}_n^2]}=E\left[\frac{s_n^2}{\bar{x}_n^2}\right]. \tag{12}$$

Using (10) and (11) in (12), we get

$$E\left[\frac{s_n^2}{\bar{x}_n^2}\right]=\frac{2p+(1-p)(\omega+1)(\omega+2)-\{p+(1-p)(\omega+1)\}^2}{\frac{1}{n}\{2p+(1-p)(\omega+1)(\omega+2)\}-\left(\frac{1}{n}-1\right)\{p+(1-p)(\omega+1)\}^2}.$$

$$\therefore\underset{n\to\infty}{lt}\,E\left[\frac{s_n^2}{\bar{x}_n^2}\right]=\frac{2p+(1-p)(\omega+1)(\omega+2)-\{p+(1-p)(\omega+1)\}^2}{\{p+(1-p)(\omega+1)\}^2}$$

$$=\left(\frac{\sigma}{\mu}\right)^2.$$

Hence the result.

IX. MAXIMUM LIKELIHOOD ESTIMATION

Suppose $x_1, x_2, ..., x_n$ be the n sample observations taken from the MEWED, then the log-likelihood function can be expressed as

$$l=\sum_{i=1}^{n}\log\left[p\lambda e^{-\lambda x_i}+\frac{(1-p)\lambda^{\omega+1}x_i^{\omega}e^{-\lambda x_i}}{\Gamma(\omega+1)}\right]. \tag{13}$$

The estimates of the unknown parameters are obtained by differentiating partially (13) with respective parameters and equating to zero, we get

$$\frac{\partial l}{\partial\lambda}=\sum_{i=1}^{n}\left[\frac{\frac{\partial}{\partial\lambda}\left\{p\lambda e^{-\lambda x_i}+\frac{(1-p)\lambda^{(\omega+1)}x_i^{\omega}e^{-\lambda x_i}}{\Gamma(\omega+1)}\right\}}{p\lambda e^{-\lambda x_i}+\frac{(1-p)\lambda^{(\omega+1)}x_i^{\omega}e^{-\lambda x_i}}{\Gamma(\omega+1)}}\right]=0$$

$$\frac{\partial l}{\partial\omega}=\sum_{i=1}^{n}\left[\frac{\frac{\partial}{\partial\omega}\left\{p\lambda e^{-\lambda x_i}+\frac{(1-p)\lambda^{(\omega+1)}x_i^{\omega}e^{-\lambda x_i}}{\Gamma(\omega+1)}\right\}}{p\lambda e^{-\lambda x_i}+\frac{(1-p)\lambda^{(\omega+1)}x_i^{\omega}e^{-\lambda x_i}}{\Gamma(\omega+1)}}\right]=0$$

$$\frac{\partial l}{\partial p}=\sum_{i=1}^{n}\left[\frac{\frac{\partial}{\partial p}\left\{p\lambda e^{-\lambda x_i}+\frac{(1-p)\lambda^{(\omega+1)}x_i^{\omega}e^{-\lambda x_i}}{\Gamma(\omega+1)}\right\}}{p\lambda e^{-\lambda x_i}+\frac{(1-p)\lambda^{(\omega+1)}x_i^{\omega}e^{-\lambda x_i}}{\Gamma(\omega+1)}}\right]=0$$

The above system of equations is non-linear and can't be solved analytically. In order to overcome this hindrance, Newton-Raphson procedure has been implemented for obtaining the estimate of the parameters.

X. REAL LIFE APPLICATIONS

In order to check the flexibility of the proposed model, two data sets have been analysed and compared with its sub-models by using R software.

The data set I used by Tahir et al. [14] consists of the failure times of 84 aircraft wind shields. The second data set is previously used by Ghitnay et al. [15] which comprises the waiting times (in minutes) of 100 bank costumers before service.

By using the above data sets, we have calculated the estimates of the unknown parameters. We have also obtained -2log-likelihood, AIC and BIC to compare proposed model with its sub-models and the results are shown in the tables below:

Table 1: MLEs of the model parameters using data set I, the resulting SEs in braces and criteria for model comparison:

Model	**Estimate, standard error in parentheses**			**-2logl**	**AIC**	**BIC**
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\omega}$			

MEWED	0.04986912 (0.02674411)	2.23821798 (0.37053845)	4.98821077 (0.96720316)	255.6965	261.6965	268.989
WED	-	1.375701 (0.2169170)	2.525402 (0.5172271)	276.7907	280.7907	285.6523
ED	-	0.3902253 (0.0423256)	-	329.9754	331.9754	334.4062
ABED	-	1.170675 (0.07331048)	-	277.9556	280.9556	282.3864

Table 2: MLEs of the model parameters using data set II, the resulting SEs in braces and criteria for model comparison:

Model	**Estimate, standard error in parentheses**			**-2logl**	**AIC**	**BIC**
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\omega}$			
MEWED	0.0194790 (0.005186)	0.2306874 (0.0738858)	1.3164784 (1.0455389)	635.7819	641.7819	649.5974
WED	-	0.2813555 (0.04125455)	1.7790716 (0.3718379)	641.2448	645.2448	650.4551
ED	-	0.1012468 (0.01012369)	-	658.0418	660.0418	662.647
ABED	-	0.3037361 (0.01753602)	-	645.1408	647.1408	649.746

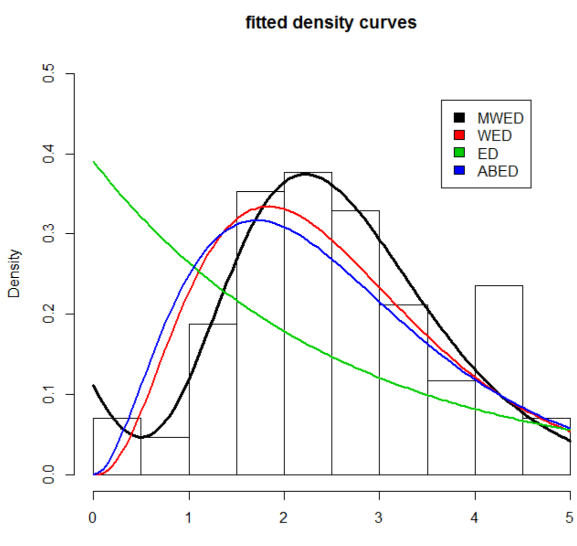

Figure 5 Plots for the histogram and estimated densities using data set I

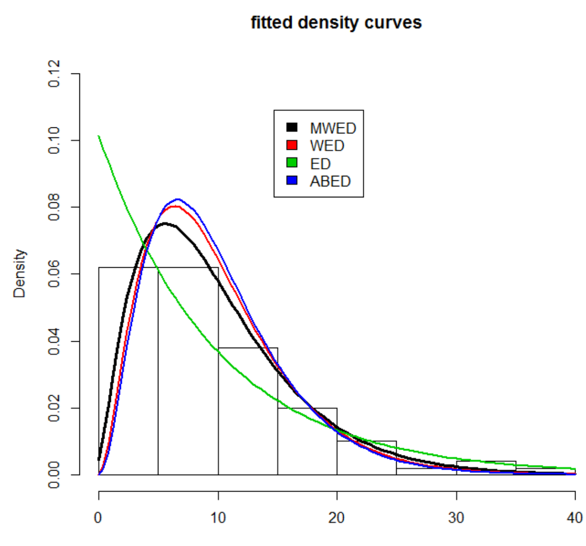

Figure 6 Plots for the histogram and estimated densities using data set II

XI. CONCLUSION

We introduce a new three-parameter probability distribution known as "Mixture of Exponential and Weighted Exponential Distribution: Properties and Applications" and study its various properties. The Renyi and q-entropies are also derived. We estimate the parameters of the new model using maximum likelihood method of estimation. To check the superiority of the proposed mixture model over some other models, two real life data sets are used and the results are obtained through R-software. An application of the proposed model to real data sets shows that the fit of the MEWED is superior to the fits using WED, ABED and ED.

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