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Independent Roman Domination Number of Graphs

D.K. Thakkar[1], S.M. Badiyani[2*]

[1]Department of Mathematics, Saurashtra University Campus, Rajkot, India
[2]Department of Mathematics, Saurashtra University Campus, Rajkot, India

[*]*Corresponding Author: sankycolors@gmail.com*

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Abstract — In this manuscript we consider Independent Roman Dominating Functions for graphs. We characterize minimal Independent Roman Dominating Functions. We observed the change in the Independent Roman Domination Number of a graph when a vertex is removed from the graph. We prove a necessary and sufficient condition under which the Independent Roman Domination Number of a graph increases or decreases. We have defined a new class of graphs called Independent Roman graphs. A necessary and sufficient condition is given under which a graph is an Independent Roman graph.

Keywords—Independent Roman Dominating Function, Independent Roman Domination Number, minimal Independent Roman Dominating Function, minimum Independent Roman Dominating Function, Independent Roman graph.

I. INTRODUCTION

The concept of Independent Roman Domination was introduced in [4] by Ernie J. Cockayne, T.W.Haynes, and others. Later on this concept was studied by M. Adabi et al. in [6]. Here we further study the concept of Independent Roman Dominating Functions. We consider the operation of vertex removal and observe its effect on the Independent Roman Domination Number of the graph. In particular we established conditions under which this number increases or decreases. A set $S \subseteq V(G)$ is an Independent Dominating set if S is an Independent and Dominating set both. We also introduce a new class of graphs called Independent roman graphs and prove the characterization for a graph to be an Independent Roman graph. Also a necessary and sufficient condition is given for the minimal Independent Roman Dominating Functions. We will see some interesting results about Independent Roman Domination Number of graphs. Some of the examples are given in this paper.

The paper contains five sections in which Section I contains the introduction of Independent Roman Domination in graphs. Section II contains preliminaries and notations. In Sections III characterization of a minimal Independent Roman Dominating Function has been given. In Section IV the operation of vertex removal from the graph is considered and characterizations for change in the Independent Roman Domination Number have been proved. In Section V concluding remarks indicates future directions and further scope of results.

II. PRELIMINARIES AND NOTATIONS

In this paper we consider only those graphs which are simple, finite and undirected. If G is a graph, $V(G)$ will denote the vertex set of graph G and $E(G)$ will denote the edge set of graph G. If G is a graph and $v \in V(G)$ then $G - v$ will denote the subgraph obtained by removing the vertex v from G. The Independent Roman Domination Number of the graph G is denoted as $i_R(G)$. If $f: V(G) \to \{0,1,2\}$ is a function then we write,

$$\boldsymbol{V_2(f)} = \{v \in V(G) \,/\, f(v) = 2\}$$
$$\boldsymbol{V_1(f)} = \{v \in V(G) \,/\, f(v) = 1\}$$
$$\boldsymbol{V_0(f)} = \{v \in V(G) \,/\, f(v) = 0\}$$

Obviously the above sets are mutually disjoint and their union is the vertex set V(G). The weight of this function $f = \sum_{v \in V(G)} f(v)$. This number is denoted as $w(f)$. We will also use the following notations:

$$S^0 = \{v \in V(G) \,/\, i_R(G - v) = i_R(G)\}$$
$$S^+ = \{v \in V(G) \,/\, i_R(G - v) > i_R(G)\}$$
$$S^- = \{v \in V(G) \,/\, i_R(G - v) < i_R(G)\}$$

If $S \subset V(G)$ and $v \in S$, then the external private neighborhood of v with respect to the set

S = $\{ w \in V(G-S) \,/\, N(w) \cap S = \{v\}\}$. It is denoted as $expn[v;S]$.

Definition 2.1 [8]: Let G be a graph. A function $f: V(G) \to \{0,1,2\}$ is called a Roman Dominating Function if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$.

Definition 2.2 [4]: Let G be a graph. A Roman Dominating Function $f: V(G) \to \{0,1,2\}$ is called an *Independent Roman Dominating Function* if $V_1(f) \cup V_2(f)$ is an Independent set.

Definition 2.3 [4]: An Independent Roman Dominating Function with minimum weight is called a *minimum Independent Roman Dominating Function*.

Definition 2.4 [4]: The weight of a minimum Independent Roman Dominating Function is called the *Independent Roman Domination Number* of the graph. It is denoted as $i_R(G)$.

Definition 2.5 [4]: Let G be a graph. A function $f: V(G) \to \{0,1,2\}$ is called a *minimal Independent Roman Dominating Function* if:
(i) f is an Independent Roman Dominating Function.
(ii) Whenever $g: V(G) \to \{0,1,2\}$ and $g < f$ then g is not an Independent Roman Dominating Function.

Definition 2.6 [9]: The cardinality of a minimum Independent Dominating set is called an *Independent Domination Number* of the graph. It is denoted as $i(G)$.

We introduce the following concept.

Definition 2.7: A graph G is said to be an *Independent Roman graph* if $i_R(G) = 2 \cdot i(G)$.

III. Minimal Independent Roman Dominating Functions

Now we characterize the minimal Independent Roman Dominating Functions.

Theorem 3.1: Let G be a graph and f be an Independent Roman Dominating Function on G then f is a minimal Independent Roman Dominating Function if and only if for every $v \in V_2(f)$, $expn[v;V_2(f)] \neq \emptyset$.
Proof: Suppose that f is a minimal Independent Roman Dominating Function and $v \in V_2(f)$.
Suppose there is no vertex from $V_0(f)$ which is adjacent to v then v is an isolated vertex. Now define f' on $V(G)$ as follows:
$f'(v) = 1$ and
$f'(w) = f(w); \forall\, w \neq v$
Then $f' < f$ and f is an Independent Roman Dominating Function on G and $w(f') < w(f)$; which contradicts the minimality of f.
Suppose there is a vertex x in $V_0(f)$ such that x is adjacent to v and $x \notin expn[v;V_2(f)]$.
Then x is adjacent to some other vertex $v' \in V_2(f)$. Now define $g: V(G) \to \{0,1,2\}$ as follows:
$g(v) = 1$, and
$g(w) = f(w); \forall\, w \neq v$
Then g is an Independent Roman Dominating Function on G and $g < f$; which contradicts the minimality of f.
Therefore there is a vertex y such that $y \in expn[v;V_2(f)]$. Thus $expn[v;V_2(f)] \neq \emptyset$.
Conversely suppose $expn[v;V_2(f)] \neq \emptyset$; $\forall\, v \in V_2(f)$.
Suppose f is not a minimal Independent Roman Dominating Function. Then there is a function $g: V(G) \to \{0,1,2\}$ such that g is an Independent Roman Dominating Function and $g < f$ then for some $v \in V(G)$ we have $g(v) < f(v)$.
Case-1: Suppose $f(v) = 2$ then we must have $g(v) = 0$ or $g(v) = 1$.
Suppose $g(v) = 1$.Now there is a vertex x such that $f(x) = 0$ and $x \in expn[v;V_2(f)]$ then $g(x) = 0$ and x is adjacent to v with $g(v) = 1$.
If there is a vertex v' such that $g(v') = 2$ and x is adjacent to v' then $f(v') = 2$ which implies that $x \notin expn[v;V_2(f)]$. Therefore there is no vertex v' such that $v' \neq v$, $g(v') = 2$ and x is adjacent to v'. Thus g is not an Independent Roman Dominating Function; which is a contradiction.
Suppose $g(v) = 0$. Now v must be adjacent to some vertex w for which $g(w) = 2$ then $f(w) = 2, f(v) = 2$ and v and w are adjacent vertices; which contradicts the fact that f is an Independent Roman Dominating Function.
Case-2: Suppose $f(v) = 1$ then we must have $g(v) = 0$.
Again there is a vertex w such that $g(w) = 2$ and w is adjacent to v then $f(w) = 2$ with $f(v) = 1$ and v and w are adjacent vertices which again contradicts the fact that f is an Independent Roman Dominating Function.
Thus it follows that there is no function g such that $g < f$ and g is an Independent Roman Dominating Function on G. Therefore f is a minimal Independent Roman Dominating Function.

Proposition 3.2: Let G be a graph and v be an isolated vertex of G. If f is a minimum Independent Roman Dominating Function on G then $f(v) = 1$.
Proof: If $f(v) = 0$ then there is no vertex u adjacent to v such that $f(u) = 2$. Therefore $f(v) = 0$ is not possible. If $f(v) = 2$ then define $g: V(G) \to \{0,1,2\}$ as follows:
$g(v) = 1$, and
$g(w) = f(w); \forall\, w \neq v$
Then g is an Independent Roman Dominating Function on G with $w(g) < w(f)$; this is a contradiction.
Therefore $f(v) = 1$.

In [2] we have proved the following theorem.

Theorem 3.3: Let $f: V(G) \to \{0,1,2\}$ be a Roman Dominating Function. Then f is a minimal Roman Domination Function if and only if each of the following two conditions is satisfied:

1) If $v \in V(G)$ and $f(v) = 2$ then there is a vertex x such that $f(x) = 0$, x is adjacent to v and x is not adjacent to any other vertex w for which $f(w) = 2$.
2) If $f(w) = 1$ then w is not adjacent to any vertex x for which $f(x) = 2$.

We have the following corollary of **theorem 3.1.**

Corollary 3.4 If G is a graph and $f: V(G) \to \{0,1,2\}$ is a minimal Independent Roman Dominating Function then f is also a minimal Roman Dominating Function.

Proof: The result follows from the fact that every Independent Roman Dominating Function is a Roman Dominating Function and **theorem 3.3**.

IV. Effects oF vertex Removal on Independent Roman Domination Number

Now we state and prove the necessary & sufficient under which the Independent Roman Domination Number increases when a vertex is removed from the graph.

Theorem 3.5: Let G be a graph and $v \in V(G)$ then $i_R(G-v) > i_R(G)$ if and only if the following conditions are satisfied:

i) v is not an isolated vertex of G.
ii) $f(v) = 2$ for every minimum Independent Roman Dominating Function f on G.
iii) There is no Independent Roman Dominating Function g on $G - v$ such that $w(g) \leq i_R(G)$ and $V_2(g)$is a subset of $V(G) - N[v]$.

Proof: Suppose $iR(G-v) > iR(G)$.

i) Suppose v is an isolated vertex in G. Let f be any minimum Independent Roman Dominating Function on G then $f(v) = 1$ by the **proposition 3.2.** Let g equal to the restriction of f on $G - v$. Then g is an Independendnet Roman Dominating Function on $G - v$. Therefore $i_R(G-v) \leq w(g) < w(f) = i_R(G)$; this is a contradiction. Thus v cannot be an isolated vertex of G.

ii) Suppose $f(v) = 1$ for some minimum Independent Roman Dominating Function f on G. Here again consider the restriction g of f on $G - v$. Then g is an Independent Roman Dominating Function on $G - v$. Therefore $i_R(G - v) \leq w(g) < w(f) = i_R(G)$; this is again a contradiction.

Suppose $f(v) = 0$ for some minimum Independent Roman Dominating Function f on G. Here also consider the restriction g of f on $G - v$, then g is an Independent Roman Dominating Function on $G - v$. Therefore $i_R(G-v) \leq w(g) \leq w(f) = i_R(G)$; this is a contradiction.

Thus we conclude that for any minimum Independent Roman Dominating Function f on $G, f(v) = 2$.

iii) Suppose there is an Independent Roman Dominating Function g on $G - v$ with $w(g) \leq i_R(G)$ and $V_2(g)$is a subset of $V(G) - N[v]$. Then $i_R(G-v) \leq w(g) \leq i_R(G)$; which is a contradiction. Therefore no such function exists.Thus conditions i), ii) and iii) are satisfied if $i_R(G) < i_R(G-v)$.

Conversely suppose the conditions i), ii) and iii) are satisfied.

First suppose $i_R(G-v) = i_R(G)$. Let g be a minimum Independent Roman Dominating Function on $G - v$. Suppose v is adjacent to some vertex x for which $g(x) = 2$. Now define f on $V(G)$ as follows:

$f(v) = 0$ and

$f(w) = g(w); \forall\, w \neq v$

Then f is a minimum Independent Roman Dominating Function on G and $w(f) = w(g)$.

But $f(v) = 0$ which contradicts condition (ii). Therefore v is not adjacent to any vertex x for which $g(x) = 2$. Therefore $V_2(g)$ is a subset of $V(G) - N[v]$, Thus we have an Independent Roman Dominating Function g on $G - v$ such that $w(g) \leq i_R(G)$; which contradicts condition (iii).

Thus $i_R(G-v) = i_R(G)$ is not possible. Suppose $i_R(G-v) < i_R(G)$. Let g be a minimum Independent Roman Dominating Function on $G - v$. If v is adjacent to some vertex x for which $g(x) = 2$ then as observed above it will imply that$i_R(G) \leq i_R(G-v)$; which is a contradiction.

Therefore $V_2(g)$ is a subset of $V(G) - N[v]$ and g is an Independent Roman Dominating Function on $G - v$ with $w(g) \leq iR(G)$. This is again contradiction.

Thus $i_R(G-v) < i_R(G)$ is also not possible.

Hence $i_R(G-v) > i_R(G)$.

Now we state and prove the necessary & sufficient under which the Independent Roman Domination Number decreases when a vertex is removed from the graph.

Theorem 3.6: Let G be a graph and $v \in V(G)$ then $i_R(G-v) < i_R(G)$ if and only if there is a minimum Independent Roman Dominating Function f on G such that one of the following conditions is satisfied:

i) $f(v) = 1$
ii) $f(v) = 0$ and $expn[u; V_2(f)] = \{v\}$; for some vertex $u \in V_2(f)$.

Proof: Suppose $i_R(G-v) < i_R(G)$.

Let g be a minimum Independent Roman Dominating Function on $G - v$. Suppose $N(v) \cap V_2(g) \neq \emptyset$. Now define f on $V(G)$ as follows:

$f(v) = 0,$ and

$g(w) = f(w); \forall\, w \neq v$

Then f is an Independent Roman Dominating Function on G and $i_R(G) \leq w(f) = w(g) = i_R(G-v)$.
i.e.$i_R(G) \leq i_R(G-v)$; this is a contradiction.
Thus $N(v) \cap V_2(g) = \emptyset$.
Case 1: $N(v) \cap V_1(g) = \emptyset$.
Now define $f: V(G) \to \{0,1,2\}$ as follows:
$f(v) = 1$, and
$f(w) = g(w); \forall\, w \neq v$
Then f is an Independent Roman Dominating Function on G and $w(f) = w(g) + 1$ and therefore f is a minimum Independent Roman Dominating Function on G as $i_R(G) > i_R(G-v)$. Thus f is a minimum Independent Roman Dominating Function with $f(v) = 1$.

Case 2: $N(v) \cap V_1(g) \neq \emptyset$.
Define $f: V(G) \to \{0,1,2\}$ as follows:
$f(v) = 0$
$f(u) = 2$; Where $u \in V_1(g)$ such that u is adjacent to v.
$f(w) = g(w); \forall\, w \neq \{u, v\}$
Then f is a minimum Independent Roman Dominating Function on G with $f(v) = 0$ and $v \in expn[u; V_2(f)]$. Let x is a vertex such that $x \neq v$ and $x \in expn[u; V_2(f)]$. Then $f(x) = 0$. Note that $g(x) = 0$. Since g is an Independent Roman Dominating Function x is adjacent to y for some $y \in V_2(g)$. Since $V_2(g) \subset V_2(f)$ this implies that x is adjacent to $y \in V_2(f)$ and $y \neq u$. This contradicts the fact that $x \in expn[u; V_2(f)]$. Thus there is no x such that $x \neq v$ and $x \in expn[u; V_2(f)]$. Therefore $expn[u; V_2(f)] = \{v\}$.
Conversely suppose any one of the two conditions is satisfied. First suppose that condition (i) is satisfied.
Then there is a minimum Independent Roman Dominating Function$f: V(G) \to \{0,1,2\}$ such that $f(v) = 1$.
Define $g: V(G-v) \to \{0,1,2\}$ as follows:
$g(w) = f(w); w \in V(G-v)$
Then g is an Independent Roman Dominating Function on $G - v$ such that $w(g) < w(f)$.
Therefore $i_R(G-v) \leq w(g) < w(f) = i_R(G)$.
i.e.$i_R(G-v) < i_R(G)$.
Now suppose condition (ii) satisfied therefore there is a minimum Independent Roman Dominating Function $f: V(G) \to \{0,1,2\}$ such that $f(u) = 0$ and for some $u \in V_2(f)$; $expn[u; V_2(f)] = \{v\}$.
Now define $g: V(G-v) \to \{0,1,2\}$ as follows:
$g(u) = 1$, and
$g(w) = f(w); \forall\, w \neq u$
Then g is an Independent Roman Dominating Function on $G - v$ and $w(g) < w(f)$.
Also $i_R(G-v) \leq w(g) < w(f) = i_R(G)$.
i.e.$i_R(G-v) < i_R(G)$.

Corollary 3.7: Let G be a graph and $v \in V(G)$. If $i_R(G-v) < i_R(G)$ then $i_R(G-v) = i_R(G) - 1$.
Proof: Let g be a Minimum Independent Roman Dominating Function of $G - v$ then from the proof of the first part of **theorem 3.6** there is a function $f: V(G) \to \{0,1,2\}$ which is a minimum Independent Roman Dominating Function on G and $w(f) = w(g) + 1$.
Therefore $i_R(G) = w(f) = w(g) + 1 = i_R(G-v) + 1$.
i.e.$i_R(G-v) = i_R(G) - 1$.

Corollary 3.8: Let G be a graph and $u, v \in V(G)$ such that $i_R(G-u) > i_R(G)$ and $i_R(G-v) < i_R(G)$ then u and v cannot be adjacent vertices.
Proof: Since $i_R(G-v) < i_R(G)$ there is a minimum Independent Roman Dominating Function f on G such that one of the two following conditions are satisfied:
i) $f(v) = 1$
ii) $f(v) = 0$ and there is some vertex $w \in V_2(f)$ such that $expn[w; V_2(f)] = \{v\}$.
First suppose that $f(v) = 1$.Also $f(u) = 2$ by **theorem 3.1** and therefore $u, v \in V_2(f) \cup V_1(f)$; which is and independent set. Therefore u and v are non-adjacent vertices.
Suppose ii) holds. Claim: $u \neq w$.
Suppose $u = w$ then $expn[u; V_2(f)] = \{v\}$. Now consider the subgraph $G - u$.
Define $h: V(G-u) \to \{0,1,2\}$ as follows:
$h(v) = 1$; Where v is not adjacent to any vertex of $V_1(f)$.
$h(v) = 0$ with $h(x) = 2$; Where v is adjacent to some vertex x of $V_1(f)$.
$h(y) = f(y)$; for other vertices of $V(G-u)$
Then h is an Independent Roman Dominating Function on $G - u$.
Also $i_R(G-u) \leq w(h) < w(f) = i_R(G)$.
i.e.$i_R(G-u) < i_R(G)$; this is a contradiction.
Thus $u \neq w$. Since $expn[w; V_2(f)] = \{v\}$, v cannot be adjacent to u.

Corollary 3.9: Let G be a graph and $u \in V(G)$ with $i_R(G-u) > i_R(G)$ then for every minimum Independent Roman Dominating Function f on G, $expn[u; V_2(f)]$ contains at least two vertices.
Proof: By **theorem 3.1** we have $expn[u; V_2(f)] \neq \emptyset$. Suppose $expn[u; V_2(f)]$ contains only one vertex say v. Now define $h: V(G-u) \to \{0,1,2\}$ as follows:
$h(v) = 1$; Where v is not adjacent to any vertex of $V_1(f)$.
$h(v) = 0$ with $h(x) = 2$; Where v is adjacent to some vertex x of $V_1(f)$.
$h(y) = f(y)$; for all other vertices of $V(G-u)$.
Then h is an Independent Roman Dominating Function on $G - u$ and $i_R(G-u) \leq w(h) < w(f) = i_R(G)$.
i.e. $i_R(G-u) < i_R(G)$.
This is a contradiction. Thus $expn[u; V_2(f)]$ contains at least two vertices.

Remark 3.10: Let G be a graph and $u \in V(G)$ such that $i_R(G-u) > i_R(G)$ then by the above **corollary 3.9** $expn[u; V_2(f)]$ contains at least two vertices say x_1 and x_2 for every minimum Independent Roman Dominating function f on G. Since x_i is adjacent to u

for $i = 1,2$ we have $i_R(G - x_i) \nless i_R(G)$. Also $x_i \notin V_2(f)$ for this particular function f so we have $i_R(G - x_i) \ngtr i_R(G)$ .Therefore $i_R(G - x_i) = i_R(G)$ for $i = 1,2$.
Thus every vertex of $u \in S^+$ gives rises to two distinct vertices x_1 and x_2 in S^0. Moreover if u_1 and u_2 are distinct vertices of S^+ then the corresponding vertices in S^0 are all distinct. Thus $|S^0| \geq 2|S^+|$.

Proposition 3.11: Let G be a graph and h v be a pendant vertex of G and its neighbor u is also pendant vertex of G then $i_R(G - v) < i_R(G)$ and $i_R(G - u) < i_R(G)$.
Proof: We may observe that there is a minimum Independent Roman Dominating Function f on G such that $f(v) = 0$ and $f(u) = 2$ and also there is a minimum Independent Roman Dominating Function g on G such that $g(v) = 2$ and $g(u) = 0$.
If $f(v) = 0$ then u is an isolated vertex in $G - v$. Define $f': V(G - v) \to \{0,1,2\}$ as follows:
$f'(u) = 1$, and
$f'(w) = f(w); \forall\, w \neq u$
Then f' is an Independent Roman Dominating Function on $G - v$ and $w(f') < w(f)$.
Thus $i_R(G - v) \leq w(f') < w(f) = i_R(G)$.
i.e.$i_R(G - v) < i_R(G)$.
On the other hand if $g(u) = 0$ then v is an isolated vertex in $G - u$ and $g(v) = 2$. Define $g': V(G - u) \to \{0,1,2\}$ as follows:
$g'(v) = 1$, and
$g'(w) = g(w); \forall\, w \neq v$
Then g' is an Independent Roman Dominating Function on $G - u$ and $w(g') < w(g)$.
Thus $i_R(G - u) \leq w(g') < w(g) = i_R(G)$.
i.e. $i_R(G - u) < i_R(G)$.

Corollary 3.12: Let G be a graph and consider u and v are adjacent vertices. Suppose there is a minimum Independent Roman Dominating Function f on G such that $f(v) = 2$ and $expn[u; V_2(f)] = \{v\}$ then $i_R(G - v) < i_R(G)$ and $i_R(G - u) < i_R(G)$.
Proof: Suppose the condition is satisfied then by the **theorem 3.6** $i_R(G - v) < i_R(G)$.
Now define $g: V(G - u) \to \{0,1,2\}$ as follows:
$g(v) = 1$; Where v is not adjacent to any vertex of $V_1(f)$.
$g(v) = 0$ with $g(x) = 2$; Where v is adjacent to some vertex of $V_1(f)$.
$g(y) = f(y)$; for all other vertices of $V(G - u)$
Then g is an Independent Roman Dominating Function on $G - u$ and $i_R(G - u) \leq w(g) < w(f) = i_R(G)$.
i.e. $i_R(G - u) < i_R(G)$.

The following theorem provides a characterization of Independent Roman graphs.

Theorem 3.13: A graph is an Independent Roman graph if and only if there is a minimum Independent Roman Dominating Function $f: V(G) \to \{0,1,2\}$ such that $V_1(f) = \emptyset$.
Proof: Suppose G is an Independent Roman graph. Let S be a minimum Independent Dominating set of G then $|S| = i(G)$.
Define $f: V(G) \to \{0,1,2\}$ as follows:
$f(x) = 2$; if $x \in S$
$f(x) = 0$; if $x \notin S$
Then f is an Independent Roman Dominating Function. Now $w(f) = 2 \cdot |S| = 2 \cdot i(G) = i_R(G)$.
Therefore f is a minimum Independnet Roman Dominating Function on G. Note that $V_1(f) = \emptyset$.
Conversely suppose there is a minimum Independent Roman Dominating Function f such that $V_1(f) = \emptyset$.
Consider $S = \{v \in V(G) / f(v) = 2\}$.
Suppose $x \in V(G - S)$ then $f(x) = 0$. Since f is a minimum Independent Roman Dominating Function x is adjacent to some vertex y for which $f(y) = 2$ which means that $y \in S$.
Thus we have proved that each vertex of $V(G - S)$ is adjacent to some vertex of S. Thus S is an Independent Dominating set. We assert that S is a minimum Independent Dominating set of G.
Suppose S_1 is an Independent Dominating set of G such that $|S_1| < |S|$.
Define $h: V(G) \to \{0,1,2\}$ as follows:
$h(x) = 2$; if $x \in S_1$
$h(x) = 0$; if $x \notin S_1$
Then h is an Independent Roman Dominating Function on G. Now we have,
$i_R(G) \leq w(h) = 2 \cdot |S_1| < 2 \cdot |S| = w(f) = i_R(G)$.
i.e.$i_R(G) < i_R(G)$; this is a contradiction. Therefore S is a minimum Independent Dominating set of G.
Therefore $2 \cdot i(G) = 2 \cdot |S| = w(f) = i_R(G)$.
Thus G is an Independent Roman graph.
Example 3.14: Consider the cycle graph $G = C_5$ with the vertex set $\{v_1, v_2, v_3, v_4, v_5\}$.

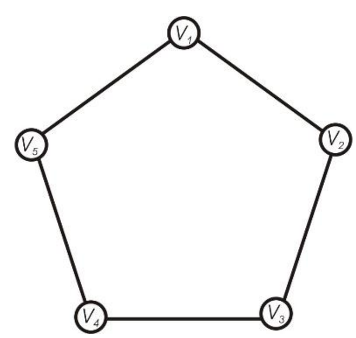

Figure 1 (G)

Let $f: V(G) \to \{0,1,2\}$ be any function such that
$f(v_1) = 0, f(v_2) = 2, f(v_3) = 0,$
$f(v_4) = 2$ and $f(v_5) = 0$

Clearly Independent Roman Domination Number of the graph G is **4**; whereas the Independent Domination Number of G is **2**. Thus we have $i_R(G) = 2 \cdot i(G)$.
Therefore $G = C_5$ is an Independent Roman graph.

Note that the cycle graph $G = C_4$ is not an Independent Roman graph.

Example 3.15: Consider the graph G with the vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$.

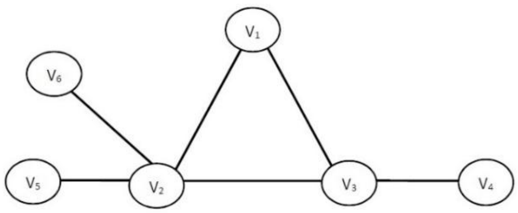

Figure 2(G)

Let $f: V(G) \to \{0,1,2\}$ be any function such that
$f(v_1) = 0, f(v_2) = 2, f(v_3) = 0, f(v_4) = 1,\ f(v_5) = 0$ and $f(v_6) = 0$
Then f is a minimum Independent Roman Dominating Function and $i_R(G) = 3$.
Now consider the graph $G - v_4$.
Define $g: V(G - v_4) \to \{0,1,2\}$ as follows:
$g(v_1) = 0, g(v_2) = 2, g(v_3) = 0, g(v_5) = 0$ and $g(v_6) = 0$.
Then g is a minimum Independent Roman Dominating Function of the graph $G - v_4$ and $i_R(G - v_4) = 2$. Thus we have $i_R(G) > i_R(G - v_4)$.
Now consider the graph $G - v_2$.
Define $h: V(G - v_2) \to \{0,1,2\}$ as follows:
$h(v_1) = 0, h(v_3) = 2, h(v_4) = 0,$
$h(v_5) = 1$ and $h(v_6) = 1$
Then h is a minimum Independent Roman Dominating Function of the graph $G - v_2$ and $i_R(G - v_2) = 4$. Thus we have $i_R(G) < i_R(G - v_2)$.
Note that v_2 and v_4 are the non-adjacent vertices.

V. CONCLUDING REMARKS

It may be interesting to find necessary and sufficient conditions under which the Independent Roman Domination Number increases, decreases or does not change when an edge is removed from the graph. It may be also interesting to find the upper bound on the number of edges whose removal changes the Independent Roman Domination Number of the graph.

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AUTHORS PROFILE

Dr.D.K.Thakkar is in the Department of Mathematics of Saurashtra University, Rajkot. His areas of interest are Graph Theory, Topology and Discrete Mathematics. He has published over 75 research papers in various journals**.**

Mr S.M.Badiyani is a young researcher student who is working in the area of Domination in graphs. He is working in Department of Mathematics, Saurashtra University, Rajkot. He likes to work in challenging environment.