International Journal of Scientific Research in Mathematical and Statistical Sciences
Vol.**11**, Issue.**2**, pp.**27-31**, April **2024**
E-ISSN: 2348-4519
Available online at: www.isroset.org

Research Article

Remarks on strong $\pi^* - \mathcal{H}$ −Open Set in GTS via Hereditary Class

R. Shankar[1]

[1]Dept. of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai, Tamil Nadu, India

Corresponding Author: shankarr@dgvaishnavcollege.edu.in

Received: 21/Feb/2024; **Accepted**: 23/Mar/2024; **Published**: 30/Apr/2024

Abstract — The aim of this paper is to introduce and study the properties of strong $\pi^* - \mathcal{H}$ −open sets. Also, we establish some of their properties and establish some characterization in a generalized topological space (X, λ) with a hereditary class $\mathcal{H}$**.** We discuss various characterization $\pi^* - \mathcal{H}$ −open set are given and properties of such sets are discussed. Finally we investigate various definitions using $\pi^* - \mathcal{H}$ −open sets in a generalized topological space (X, λ) with a hereditary class $\mathcal{H}$**.**

Keywords — Hereditary class, pre−$\mathcal{H}$ −open set, semi−$\mathcal{H}$ −open, $\alpha - \mathcal{H}$ −open set, $\sigma^* - \mathcal{H}$ −open set, weakly semi− $\mathcal{H}$ −open set and strong $\beta - \mathcal{H}$ −open set.

2010 Mathematics subject classification: 54A05, 54A10.

1. Introduction

Let X be a nonempty set and $\wp(X)$ denotes the power set of X. Then the collection $\lambda \subseteq \wp(X)$ is called a generalized topology (GT) on X [1] if $\emptyset \in \lambda$ and arbitrary union of members of λ belongs to λ. A set X, with a GT on it is said to be generalized topological space (GTS) (X, λ)**.** Members of λ are called λ −open sets. A subset $E \subset X$ is said to be λ −closed sets if $X - E$ is λ −open set. For each $E \subset X$, we denote by $int_\lambda(E)$ [2] the union of all λ −open sets contained in E and by $cl_\lambda(E)$ [2] the intersection of all λ-closed sets containing E. A GTS (X, λ) is said to be a quasi topological space [4] (QTS) if $K, L \in \lambda$ implies $K \cap L \in \lambda$. A hereditary class (HC) $\mathcal{H}$ is a nonempty family of subset of X such that $L \subset M$, $M \in \mathcal{H}$ implies $L \in \mathcal{H}$ [2]. Hereditary class $\mathcal{H}$ is said to be λ −codense [2] if $\lambda \cap \mathcal{H} = \{\emptyset\}$ [2]. For each $E \subset X$, a subset $E^*(\mathcal{H})$ is defined by $E^* = \{x \in X | N \cap E \notin \mathcal{H}$ for every $N \in \lambda$ such that $x \in N\}$ [3]. If $cl^*_\lambda(E) = E \cup E^*$ for each $E \subset X$, with respect to λ and a HC $\mathcal{H}$ of subsets of X, then $\lambda^* = \{E \subset X / cl^*_\lambda(X - E) = X - E\}$ [3] is a GT. Members of λ^* are called λ^* −open sets and its complement is called a λ^* −closed set. We denote the interior of E in (X, λ^*) is $int^*_\lambda(E)$. A subset $E \subset X$ is said to be λ^* −dense (resp. λ −dense) if $cl^*_\lambda(E) = X$ (resp. $cl_\lambda(E) = X$).

2. Related Work

Lemma. 2.1. [3] Let (X, λ) be a GTS with a HC $\mathcal{H}$ and $S, T \subset X$. Then the following holds.

(i) $S \subset T$ implies $S^* \subset T^*$.
(ii) $(S^*)^* = S^*$ *for every* $S \subset X$.
(iii) $S \subset T \subset X$ implies $cl^*_\lambda(S) \subset cl^*_\lambda(T)$.
(iv) $(E \cup E^*)^* \subset E^*$ *for every* $E \subset X$.
(v) $(S \cup T)^* = S^* \cup T^*$.
(vi) $\lambda \subset \lambda^*$.
(vii) F is λ^* − closed if and only if $F^* \subset F$.
(viii) $\beta = \{N - K : N \in \lambda, K \in \mathcal{H}\}$ *is a base for* λ^*.

Lemma 2.2. [3] Let (X, λ) be a GTS, $\mathcal{H}$ be a λ −codense hereditary class of subsets of X and $E \subset X$. If $E \subset E^*$, then $E^* = cl_\lambda(E) = cl^*_\lambda(E) = cl_\lambda(E^*)$.

Lemma 2.3. [3] Let (X, λ) be a generalized topological space with a hereditary class $\mathcal{H}$ and $E \subset X$. If E is a λ −semi closed set, then $int_\lambda(E) = int_\lambda(cl_\lambda(E))$.

Lemma 2.4. [3] Let (X, λ) be a GTS, $\mathcal{H}$ be a λ −codense hereditary class of subsets of X and $E \subset X$. Then the following hold

(i) $int_\lambda(E) = int^*_\lambda(E)$, for every λ^* −closed set E.
(ii) $cl_\lambda(E) = cl^*_\lambda(E)$, for every λ^* −open set E.

Lemma 2.5. [5] Let (X, λ) be a GTS with a HC $\mathcal{H}$ and $E \subset X$. Then the following holds.

(i) If $N \in \lambda$, then $N \cap cl_{\lambda}^{*}(E) \subset cl_{\lambda}^{*}(N \cap E)$.
(ii) If $N \in \lambda$, then $N \cap E^{*} \subset (N \cap E)^{*}$.

3. Research Design

Definition 3.1. A subset E ⊂ X of a GTS (X, λ) with a HC $\mathcal{H}$ is said to be

1. $\lambda^{*}-$Dense [3] in itself if $E \subset E^{*}$.
2. $\lambda^{*}-$Perfect [3] if $E = E^{*}$.
3. $\lambda^{*}-$Closed [3] if $E^{*} \subset E$.
4. $\mathcal{H}-$open [3] if $E \subset int_{\lambda}(E^{*})$
5. $\sigma - \mathcal{H}-$open [3] if $E \subset cl_{\lambda}^{*}\big(int_{\lambda}(E)\big)$.
6. $\pi - \mathcal{H}-$open [3] if $E \subset int_{\lambda}\big(cl_{\lambda}^{*}(E)\big)$.
7. $\alpha - \mathcal{H}-$open [3] if $E \subset int_{\lambda}\Big(cl_{\lambda}^{*}\big(int_{\lambda}(E)\big)\Big)$.
8. $\beta - \mathcal{H}-$open [3] if $E \subset cl_{\lambda}\Big(int_{\lambda}\big(cl_{\lambda}^{*}(E)\big)\Big)$.
9. $S - \mathcal{H}-$set [7] if $int_{\lambda}(E) = cl_{\lambda}^{*}\big(int_{\lambda}(E)\big)$.
10. Weaklysemi$-\mathcal{H}-$open[10] if $E \subset cl_{\lambda}^{*}\Big(int_{\lambda}\big(cl_{\lambda}(E)\big)\Big)$.
11. Almost strong$-\mathcal{H}-$open [10] if $E \subset cl_{\lambda}^{*}\big(int_{\lambda}(E^{*})\big)$.
12. Almost$-\mathcal{H}-$open [10] if $E \subset cl_{\lambda}\big(int_{\lambda}(E^{*})\big)$.
13. $\sigma^{*} - \mathcal{H}-$open [13] if $E \subset cl_{\lambda}\big(int_{\lambda}^{*}(E)\big)$.
14. $\pi^{*} - \mathcal{H}-$open [13] if $E \subset int_{\lambda}^{*}\big(cl_{\lambda}(E)\big)$.
15. $\beta^{*} - \mathcal{H}-$open [13] if $E \subset cl_{\lambda}\Big(int_{\lambda}^{*}\big(cl_{\lambda}(E)\big)\Big)$.
16. $\alpha^{*} - \mathcal{H}-$open [16] if $int_{\lambda}\Big(cl_{\lambda}^{*}\big(int_{\lambda}(E)\big)\Big) = int_{\lambda}(E)$.
17. Strong $\beta - \mathcal{H}-$open [19] if $E \subset cl_{\lambda}^{*}(int_{\lambda}\big(cl_{\lambda}^{*}(E)\big)$.

4. Materials and Method

The complement of a strong β$-\mathcal{H}-$ open (resp. $\sigma - \mathcal{H}-$open, $\sigma^{*} - \mathcal{H}-$open, $\pi - \mathcal{H}-$open, $\alpha - \mathcal{H}-$open) set is said to a strong β$-\mathcal{H}-$closed (resp. $\sigma - \mathcal{H} -$ Closed, $\sigma^{*} - \mathcal{H}-$closed, $\pi - \mathcal{H}-$closed, $\alpha - \mathcal{H}-$closed) set.

5. Results and Discussion

$$\begin{tikzcd}
 & \text{Strong } \pi^{*} - \mathcal{H} -\text{open set} \arrow[dr] & \\
\pi - \mathcal{H} -\text{open set} \arrow[ur] \arrow[dr] & & \pi^{*} - \mathcal{H} -\text{open set} \\
 & \lambda - \pi -\text{open set.} \arrow[ur] &
\end{tikzcd}$$

5.1 Strong $\pi^{*} - \mathcal{H}-$open Sets

Definition 5.1. A subset E ⊂ X of a GTS (X, λ) with a HC $\mathcal{H}$ is said to be a strong $\pi^{*} - \mathcal{H}-$open set if $E \subset int_{\lambda}^{*}\big(cl_{\lambda}^{*}(E)\big)$. The family of all strong $\pi^{*} - \mathcal{H}-$open set is denoted by $S\pi^{*}\mathcal{H}(X, \lambda)$. If its compliment is strong $\pi^{*} - \mathcal{H}-$closed set.

Theorem 5.2. Let (X, λ) be a GTS, $\mathcal{H}$ be a $\lambda-$codense hereditary class of subsets of X and $E \subset X$. Then every strong $\pi^{*} - \mathcal{H}-$open set is a strong $\beta - \mathcal{H}-$open set.

Proof. Let E be a strong $\pi^{*} - \mathcal{H}-$open set. Then $E \subset int_{\lambda}^{*}\big(cl_{\lambda}^{*}(E)\big)$. By Lemma 1.1, $E \subset int_{\lambda}^{*}\big(cl_{\lambda}^{*}(E)\big) = int_{\lambda}\big(cl_{\lambda}^{*}(E)\big) \subset int_{\lambda}\big(cl_{\lambda}^{*}(E)\big) \cup \Big(int_{\lambda}\big(cl_{\lambda}^{*}(E)\big)\Big)^{*} \subset cl_{\lambda}^{*}\Big(int_{\lambda}\big(cl_{\lambda}^{*}(E)\big)\Big)$. Then, E is a strong $\beta - \mathcal{H}-$open set.

The following Examples 5.3 shows that converses of Theorem 5.2 are not true.

Example 5.3. Let $X = \{s_1, s_2, s_3, s_4\}, \lambda = \{\emptyset, \{s_1\}, \{s_2\}, \{s_1, s_2\}, X\}$ and $\mathcal{H} = \{\emptyset\}$. If $E = \{s_1, s_3\}$, then $E^{*} = \{s_1, s_3, s_4\}$. Now $cl_{\lambda}^{*}\Big(int_{\lambda}\big(cl_{\lambda}^{*}(E)\big)\Big) = cl_{\lambda}^{*}\Big(int_{\lambda}\big(cl_{\lambda}^{*}(\{s_1, s_3\})\big)\Big) = cl_{\lambda}^{*}\big(int_{\lambda}(\{s_1, s_3\} \cup \{s_1, s_3, s_4\})\big) = cl_{\lambda}^{*}\big(int_{\lambda}(\{s_1, s_3, s_4\})\big) = cl_{\lambda}^{*}(\{s_1\}) = \{s_1\} \cup \{s_1, s_3, s_4\} = \{s_1, s_3, s_4\} \supset E$. Therefore E is a strong $\beta - \mathcal{H}-$open set. But $int_{\lambda}^{*}\big(cl_{\lambda}^{*}(E)\big) = int_{\lambda}^{*}(E \cup E^{*}) = int_{\lambda}(\{s_1, s_3, s_4\}) = X - cl_{\lambda}^{*}(X - \{s_1, s_3, s_4\}) = X - cl_{\lambda}^{*}(\{s_2\}) = X - (\{s_2\} \cup \{s_2, s_3, s_4\}) = X - \{s_2, s_3, s_4\} = \{s_1\} \not\supset E$. Therefore, E is not strong $\pi^{*} - \mathcal{H}-$open set.

Theorem 5.4. Let (X, λ) be a GTS, $\mathcal{H}$ be a $\lambda-$codense hereditary class of subsets of X and $E \subset X$. Then every strong $\pi^{*} - \mathcal{H}-$open set is a weakly semi $-\mathcal{H}-$open set.

Proof. Let E be a strong $\pi^{*} - \mathcal{H}-$open set, Then $E \subset int_{\lambda}^{*}\big(cl_{\lambda}^{*}(E)\big)$. By Lemma 1.1, $E \subset int_{\lambda}^{*}\big(cl_{\lambda}^{*}(E)\big) = int_{\lambda}\big(cl_{\lambda}^{*}(E)\big) \subset int_{\lambda}\big(cl_{\lambda}(E)\big) \subset cl_{\lambda}^{*}\Big(int_{\lambda}\big(cl_{\lambda}(E)\big)\Big)$. Therefore, E is a weakly semi$-\mathcal{H}-$open set.

The following Examples 5.5 shows that converses of Theorem 5.4 are not true.

Example 5.5. Let $X = \{s_1, s_2, s_3, s_4\}, \lambda = \{\emptyset, \{s_1\}, \{s_2\}, \{s_1, s_2\}, X\}$ and $\mathcal{H} = \{\emptyset\}$. If $E = \{s_1, s_4\}$, then $E^{*} = \{s_1, s_3, s_4\}$. Now $cl_{\lambda}^{*}\Big(int_{\lambda}\big(cl_{\lambda}(E)\big)\Big) = cl_{\lambda}^{*}\Big(int_{\lambda}\big(cl_{\lambda}(\{s_1, s_4\})\big)\Big) = cl_{\lambda}^{*}(int_{\lambda}(\{s_1, s_3, s_4\}) = cl_{\lambda}^{*}(\{s_1\}) = \{s_1\} \cup \{s_1, s_3, s_4\} = \{s_1, s_3, s_4\} \supset E$. Therefore E is a weakly semi$-\mathcal{H}-$open set. But $int_{\lambda}^{*}\big(cl_{\lambda}^{*}(E)\big) = int_{\lambda}^{*}(E \cup E^{*}) = int_{\lambda}^{*}(\{s_1, s_3, s_4\}) = X - cl_{\lambda}^{*}(X - \{s_1, s_3, s_4\}) = X - cl_{\lambda}^{*}(\{s_2\}) = X - (\{s_2\} \cup \{s_2, s_3, s_4\}) = X - \{s_2, s_3, s_4\} = \{s_1\} \not\supset E$. Therefore, E is not a strong $\pi^{*} - \mathcal{H}-$open set.

Theorem 5.6. Let (X, λ) be a generalized topological space with a hereditary class $\mathcal{H}$ and $E \subset X$. If E is a $\pi - \mathcal{H}$ −open set, then E is a strong $\pi^* - \mathcal{H}$ −open set.

Proof. Let E be a $\pi - \mathcal{H}$ −open set. Then $E \subset int_\lambda(cl_\lambda^*(E))$ and since λ^* finer than λ so $E \subset int_\lambda^*(cl_\lambda^*(E))$. Therefore, E is a strong $\pi^* - \mathcal{H}$ −open set.

The following Example 5.7 shows that the converse of Theorem 5.6 is not true.

Example 5.7. $X = \{s_1, s_2, s_3, s_4\}, \lambda = \{\emptyset, \{s_1\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}, X\}$ and $\mathcal{H} = \{\emptyset, \{s_1\}, \{s_4\}, \{s_1, s_4\}\}$. If $E = \{s_3, s_4\}$, then $E^* = \{s_2, s_3, s_4\}$. Now, $int_\lambda^*(cl_\lambda^*(E)) = int_\lambda^*(cl_\lambda^*(\{s_3, s_4\})) = int_\lambda^*(\{s_2, s_3, s_4\}) = X - cl_\lambda^*(X - \{s_2, s_3, s_4\}) = X - cl_\lambda^*(\{s_1\}) = X - (\{s_1\} \cup \emptyset) = X - \{s_1\} = \{s_2, s_3, s_4\} \supset E$. Therefore E is a strong $\pi^* - \mathcal{H}$ −open set. But $int_\lambda(cl_\lambda^*(E)) = int_\lambda(E \cup E^*) = int_\lambda(\{s_2, s_3, s_4\}) = \{s_2, s_3\} \not\supset E$. Therefore, E is not a $\pi - \mathcal{H}$ −open set.

Theorem 5.8. Let (X, λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$. Then every $\mathcal{H}$ −open set is a strong $\pi^* - \mathcal{H}$ −open set.

Proof. Let E is an $\mathcal{H}$ −open set, then $E \subset int_\lambda(E^*) \subset int_\lambda(E \cup E^*)$=$int_\lambda(cl_\lambda^*(E)) \subset int_\lambda^*(cl_\lambda^*(E))$. Hence E is a strong $\pi^* - \mathcal{H}$ −open set.

Theorem 5.9. Let (X, λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$. If E is both $\pi^* - \mathcal{H}$ −open set and λ −closed set, then E is a strong $\pi^* - \mathcal{H}$ −open set.

Proof. Let E is $\pi^* - \mathcal{H}$ −open set, the $E \subset int_\lambda^*(cl_\lambda(E))$. Also since E is closed set, $cl_\lambda(E) = E$ which implies that $E \subset int_\lambda^*(cl_\lambda(E)) = int_\lambda^*(E) \subset int_\lambda^*(E \cup E^*) = int_\lambda^*(cl_\lambda^*(E))$. It follows that E is a strong $\pi^* - \mathcal{H}$ −open set.

Theorem 5.10. Let (X, λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$ be a λ^* −perfect. If E is strong $\pi^* - \mathcal{H}$ −open set, then E is an almost strong$-\mathcal{H}$ −open set.

Proof. Let E is strong $\pi^* - \mathcal{H}$ −open set, $E \subset int_\lambda^*(cl_\lambda^*(E)) = int_\lambda(cl_\lambda^*(E)) \subset cl_\lambda^*\left(int_\lambda(cl_\lambda^*(E))\right) = cl_\lambda^*(int_\lambda(E^*))$. This is implies that E is an almost strong$-\mathcal{H}$ −open set.

Theorem 5.11. Let (X, λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$ such that every open set is $\lambda^* -$ closed, then every strong $\beta - \mathcal{H}$ −open set is a strong $\pi^* - \mathcal{H}$ −open set.

Proof. Let E is a strong $\beta - \mathcal{H}$ −open set, then $E \subset cl_\lambda^*\left(int_\lambda(cl_\lambda^*(E))\right)$. Since $int_\lambda(cl_\lambda^*(E))$ is open, by lemma $int_\lambda(cl_\lambda^*(E)) = cl_\lambda^*\left(int_\lambda(cl_\lambda^*(E))\right)$. so $E \subset cl_\lambda^*\left(int_\lambda(cl_\lambda^*(E))\right) = int_\lambda(cl_\lambda^*(E)) \subset int_\lambda^*(cl_\lambda^*(E))$. It shows that E is strong $\pi^* - \mathcal{H}$ −open set.

Theorem 5.12. Let (X, λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$. If E is both $\sigma - \mathcal{H}$ −open and $s - \mathcal{H}$ −set, then E is a strong $\pi^* - \mathcal{H}$ −open set.

Proof. Let E is a $\sigma - \mathcal{H}$ −open set. Then $E \subset cl_\lambda^*(int_\lambda(E))$. Since, E is a $s - \mathcal{H}$ −set, then $int_\lambda(E) = cl_\lambda^*(int_\lambda(E))$. Now $E \subset cl_\lambda^*(int_\lambda(E)) \subset int_\lambda(E) \subset int_\lambda(cl_\lambda^*(E)) \subset int_\lambda^*(cl_\lambda^*(E))$. It shows that strong $\pi^* - \mathcal{H}$ −open set.

Theorem 5.13. Let (X, λ) be a GTS with a hereditary class $\mathcal{H}$ and $E \subset X$. If E is strong $\pi^* - \mathcal{H}$ −closed if and only if $cl_\lambda^*(int_\lambda^*(E)) \subset E$.

Proof. Let E is a strong $\pi^* - \mathcal{H}$ −closed if and only if $(X - E)$ is a strong $\pi^* - \mathcal{H}$ −open if and only if $(X - E) \subset int_\lambda^*(cl_\lambda^*(X - E)) = X - cl_\lambda^*(int_\lambda^*(E))$ if and only if $cl_\lambda^*(int_\lambda^*(E)) \subset E$.

Theorem 5.14. Let (X, λ) be a GTS with a hereditary class $\mathcal{H}$ and $E, F \subset X$. Then E is a strong $\pi^* - \mathcal{H}$ −closed if and only if there exists an E is a strong $\pi^* - \mathcal{H}$ −closed F such that $int_\lambda^* \subset F \subset E$.

Proof. Let E be a strong $\pi^* - \mathcal{H}$ −closed, then $cl_\lambda^*(int_\lambda^*(E)) \subset E$. let $B = cl_\lambda^*(int_\lambda^*(E))$ be a λ^* −closet set that is B is strong $\pi^* - \mathcal{H}$ −closed set. $int_\lambda^*(E) \subset cl_\lambda^*(int_\lambda^*(E)) = F \subset E$. Conversely, if B is an is strong $\pi^* - \mathcal{H}$ −closed set such that $int_\lambda^*(E) \subset F \subset E$, then $int_\lambda^*(E) = int_\lambda^*(F)$. On the other hand, $cl_\lambda^*(int_\lambda^*(F)) \subset F$ and hence $E \supset F \supset cl_\lambda^*(int_\lambda^*(F))$= $cl_\lambda^*(int_\lambda^*(E))$. Thus $E \supset cl_\lambda^*(int_\lambda^*(E))$. Hence E is a strong $\pi^* - \mathcal{H}$ −closed

Theorem 5.15. Let (X, λ) be a QTS with a hereditary class $\mathcal{H}$ and $E, F \subset X$. If E is strong $\pi^* - \mathcal{H}$ −open set and F is λ −open set, then $E \cap F$ is strong $\pi^* - \mathcal{H}$ −open set.

Proof. Let E is a strong $\pi^* - \mathcal{H}$ −open set and F is λ −open set. Then we have $E \subset int_\lambda^*(cl_\lambda^*(E))$ and $F = int_\lambda(F)$. Now $E \cap F \subset int_\lambda^*(cl_\lambda^*(E)) \cap int_\lambda(F) \subset int_\lambda^*(cl_\lambda^*(E)) \cap int_\lambda^*(F) = int_\lambda^*(cl_\lambda^*(E) \cap F) = int_\lambda^*((E^* \cup E) \cap F) = int_\lambda^*((E^* \cap F) \cup (E \cap F)) \subset int_\lambda^*((E \cap F)^* \cup (E \cap F)) = int_\lambda^*(cl_\lambda^*(E \cap F))$. Hence $E \cap F$ is strong $\pi^* - \mathcal{H}$ −open set.

Theorem 5.16. Let (X, λ) be a GTS with a hereditary class $\mathcal{H}$ and $E, F \subset X$. If E is a strong $\pi^* - \mathcal{H}$ −open set and F is λ −pre open set, then $E \cup F$ is $\pi^* - \mathcal{H}$ −open set.

Proof. Let E is a strong $\pi^* - \mathcal{H}$ −open set and F is λ −pre open set. Then we have $E \subset int_\lambda^*(cl_\lambda^*(E))$ and $F \subset int_\lambda(cl_\lambda(F))$. Now $E \cup F \subset int_\lambda^*(cl_\lambda^*(E)) \cup int_\lambda(cl_\lambda(F)) \subset int_\lambda^*(cl_\lambda^*(E)) \cup int_\lambda^*(cl_\lambda(F)) \subset int_\lambda^*(cl_\lambda(E)) \cup int_\lambda^*(cl_\lambda(F)) \subset int_\lambda^*(cl_\lambda(E \cup F))$. Hence $E \cup F$ is a $\pi^* - \mathcal{H}$ −open set.

Theorem 5.17. Let (X, λ) be a generalized topological space with a λ −codense hereditary class $\mathcal{H}$ and $E \subset X$. Then every strong $\pi^* - \mathcal{H}$ −open set is a $\pi^* - \mathcal{H}$ −open set.

Proof. Let E be a strong $\pi^* - \mathcal{H}$ −open set. Then $E \subset int_\lambda^*(cl_\lambda^*(E))$. Since λ^* finer than λ, $E \subset int_\lambda^*(cl_\lambda^*(E)) \subset int_\lambda^*(cl_\lambda(E))$. Therefore, E is a $\pi^* - \mathcal{H}$ −open set.

The following Examples 5.18 shows that converses of Theorem 5.17 are not true.

Example 5.18. Let $X = \{s_1, s_2, s_3, s_4\}$, $\lambda = \{\emptyset, \{s_3\}, \{s_1, s_2, s_3\}, X\}$ and $\mathcal{H} = \{\emptyset, \{s_1\}\}$. If $E = \{s_1\}$, then $E^* = \emptyset$. Now, $int_\lambda^*(cl_\lambda(E)) = int_\lambda^*(cl_\lambda(\{s_1\})) = int_\lambda^*(\{s_1, s_2, s_4\}) = X - cl_\lambda^*(X - \{s_1, s_2, s_4\}) = X - cl_\lambda^*(\{s_3\}) = X - (\{s_3\} \cup \{s_3\}) = X - \{s_3\} = \{s_1, s_2, s_4\} \supset E$. Therefore E is a $\pi^* - \mathcal{H}$ −open set. But $int_\lambda^*(cl_\lambda^*(E)) = int_\lambda^*(E \cup E^*) = int_\lambda^*(\{s_1\} \cup \emptyset) = int_\lambda^*(\{s_1\}) = X - cl_\lambda^*(X - \{s_1\}) = X - cl_\lambda^*(\{s_2, s_3, s_4\}) = X - (\{s_2, s_3, s_4\} \cup X) = X - X = \emptyset \not\supset E$. Hence E is not a strong $\pi^* - \mathcal{H}$ −open set.

Theorem 5.19. Let (X, λ) be a generalized topological space with a hereditary class $\mathcal{H}$ and $E \subset X$. If E is a strong $\pi^* - \mathcal{H}$ −closed set if and only if $cl_\lambda^*(int_\lambda^*(E)) \subset E$.

Proof. If E is a strong $\pi^* - \mathcal{H}$ −closed set if and only if $X - E$ is a strong $\pi^* - \mathcal{H}$ −open set if and only if $X - E \subset int_\lambda^*(cl_\lambda^*(X - E)) = int_\lambda^*(X - int_\lambda^*(E)) = X - cl_\lambda^*(int_\lambda^*(E))$ if and only if $cl_\lambda^*(int_\lambda^*(E)) \subset E$.

Theorem 5.20. Let (X, λ) be a generalized topological space with a hereditary class $\mathcal{H}$ and $E \subset X$ be a λ −pre open set. If E is a λ −semi closed set, then E is a strong $\pi^* - \mathcal{H}$ −open set.

Proof. Let E be a λ −pre open. Then $E \subset int_\lambda(cl_\lambda(E))$. Since, E is a λ −semi closed set, then $int_\lambda(E) = int_\lambda(cl_\lambda(E))$ and so $E = int_\lambda(E) \subset int_\lambda^*(cl_\lambda^*(E))$. Therefore, E is a strong $\pi^* - \mathcal{H}$ −open set.

Theorem 5.21. Let (X, λ) be a generalized topological space with a λ −codense hereditary class $\mathcal{H}$ and $E \subset X$ be a λ^* −perfect set. Then E is a strong $\pi^* - \mathcal{H}$ −open set if and only if E is a $\mathcal{H}$ −open set.

Proof. Let E be a strong $\pi^* - \mathcal{H}$ −open set, then $E \subset int_\lambda^*(cl_\lambda^*(E))$. Since E is a λ^* −perfect set, then $cl_\lambda^*(E) = E^*$. Now by Lemma 2.3, $E \subset int_\lambda^*(cl_\lambda^*(E)) = int_\lambda(cl_\lambda^*(E)) = int_\lambda(E^*)$. Hence E is a $\mathcal{H}$ −open set. Converse is clear.

Theorem 5.22. Let (X, λ) be a generalized topological space with a hereditary class $\mathcal{H}$ and $E, F \subset X$. If E is a strong $\pi^* - \mathcal{H}$ −open set and F is a weakly semi−$\mathcal{H}$ −open set, then $E \cup F$ is a $\beta^* - \mathcal{H}$ −open set.

Proof. Let E be a strong $\pi^* - \mathcal{H}$ −open set and F be a weakly semi−$\mathcal{H}$ −open set. Then $E \subset int_\lambda^*(cl_\lambda^*(E))$ and $F \subset cl_\lambda^*\left(int_\lambda(cl_\lambda(F))\right)$.Now, $E \cup F \subset int_\lambda^*(cl_\lambda^*(E)) \cup cl_\lambda^*\left(int_\lambda(cl_\lambda(F))\right)$ by Theorem 5.3 $\subset cl_\lambda\left(int_\lambda^*(cl_\lambda(E))\right) \cup cl_\lambda\left(int_\lambda^*(cl_\lambda(F))\right) = cl_\lambda\left(int_\lambda^*(cl_\lambda(E)) \cup int_\lambda^*(cl_\lambda(F))\right) \subset cl_\lambda\left(int_\lambda^*(cl_\lambda(E \cup F))\right)$. Hence $E \cup F$ is a $\beta^* - \mathcal{H}$ −open set.

Theorem 5.23. Let (X, λ) be a generalized topological space with λ −codense hereditary class $\mathcal{H}$ and $E \subset X$. Then E is a strong $\pi^* - \mathcal{H}$ −closed set if and only if E is a $\pi - \mathcal{H}$ −closed set.

Proof. Suppose E be a strong $\pi^* - \mathcal{H}$ −closed set. Then $E \supset cl_\lambda^*(int_\lambda^*(E))$. By lemma 1.17, $E \supset cl_\lambda^*(int_\lambda^*(E)) = cl_\lambda^*(int_\lambda(E))$ and so $cl_\lambda^*(int_\lambda(E)) \subset E$. Therefore, E is a $\pi - \mathcal{H}$ −closed set. Conversely, let E be a $\pi - \mathcal{H}$ −closed set. Then $E \supset cl_\lambda^*(int_\lambda(E))$. Since λ^* finer than λ, $E \supset cl_\lambda^*(int_\lambda(E)) \supset cl_\lambda^*(int_\lambda^*(E))$ and so $E \subset cl_\lambda^*(int_\lambda^*(E))$. Therefore, E is a strong $\pi^* - \mathcal{H}$ −closed set.

6. Conclusion and Future scope

As stated in the abstract, we have defined the strong $\pi^* - \mathcal{H}$ −open set in a Generalized Topological space (X, λ) with a hereditary class $\mathcal{H}$. Moreover, many counter examples were investigated, we discuss its properties and give various characterization of this sets. In future, researchers doing research in this filed can study about the properties and characterizations of these sets. Moreover, one can define the functions which preserve this sets. Further, one can study about the compactness, covering and separation properties of these sets.

Conflict of Interest
There is no Conflict of Interest due to this Article.

Funding statement
The Research leading to this Article is funded by Author's own resources

Author contributions
The entire Article is contributed by the author and nobody else has any role in its preparation

Acknowledgements
The author sincerely thank the referees for their valuable suggestions and comments.

References

[1] A. Csaszar, "Generalized Open Sets", Acta Math. Hungar, vol. 75 issue 1-2, pp 65-87, 1997.
[2] A.Csaszar, "Generalized topology, generalized continuity", Acta Math. Hungar, vol. 96, pp 351-357, 2002.
[3] A.Csaszar, "Modifications of generalized topologies via hereditary classes", Acta Math. Hungar, vol. 115, pp 29 – 36, 2007.
[4] A.Csaszar, "Remarks on quasi-topologies", Acta Math. Hungar, vol.119 issue 1-2, pp 197 – 200, 2007.
[5] V.Renukadevi and K. Karuppayi. "On modifications of generalized topologies via hereditary classes", J. Adv. Res. Pure. Math., vol. 2, pp 14-20, 2010.
[6] V.Renukadevi and Sheena Scaria. "On hereditary classes in generalized topological spaces", J. Adv. Res. Pure. Math., vol. 3 issue 2, pp 21-30, 2011.
[7] K.Karuppayi, "Some subsets of GTS with hereditary classes", Journal of Advanced Studies in Topology, vol. 5, issue 1, pp 25-33, 2013.
[8] V.Pankajam and D.Sivaraj, "Some separation axioms in GTS", Bolo. Soc. Paran. Mat, vol. 31, issue 1, pp 29-42, 2013.
[9] K.Karuppayi, "A Note on δ-H-sets in GTS with hereditary classes", International Journal of Mathematical Archive, vol. 5, issue 1, pp 226-229, 2014.
[10] K.Karuppayi, "A Note on RH-open sets in GTS with hereditary classes", International Journal of Mathematical Archive, vol. 5 issue 1, pp 312-316, 2014.
[11] V.Renukadevi and P.Vimaladevi, "Note on generalized topological space with hereditary classes", Bolo. Soc. Paran. Mat, vol. 32, issue 1, pp 89-97, 2014.
[12] V.Renukadevi and D.Sivaraj, "Weak separation axioms in GTS", KYUNGPOOK Math, J, vol. 54, pp 387-399, 2014.
[13] K.Karuppayi, "On decomposition of continuity and complete continuity with hereditary class", J.Adv. Stud. Topology, vol. 5, pp 18-26, 2014.
[14] A.P.Ponraj and D.Sivaraj, "Semi^*-H-open sets", International Journal pure and applied mathematics, vol. 117, issue 17, pp 1-6, 2017.
[15] R.Shankar, A.P. Ponraj and D. Sivaraj, "Weakly semi− H −open sets and almost strong H−κ−open sets", International Journal of Pure and Applied Mathematics, vol. 117 issue 17, pp 7-13, 2017.
[16] A.P.Ponraj, R. Shankar and D.Sivaraj, "H_R-closed set in generalized topological space", International Journal of Pure and Applied Mathematics, vol. 119 issue 15, pp 2153-2158, 2018.
[17] R. Shankar, A.P.Ponraj and D.Sivaraj, "A_HR-sets in generalized topological space", International Journal of Pure and Applied Mathematics, vol. 119, issue 15, pp 2165-2170, 2018.
[18] P. Montagantirud and S. Phonrakkhet, "Generalized quotient topologies and hereditary classes", Acta Math. Hungar, vol. 161, pp 1-15, 2020.
[19] R. Shankar and D. Sivaraj, "Strong β-H-Open sets in Generalized Topological Space", International Journal of Mathematics Trends and Technology, vol. 67 issue 7, pp 183-186, 2021.
[20] R. Shankar and D. Sivaraj, "Semi-H-regular set and H_AB-sets in generalized topological space", In the Proceedings of the AIP Conference 2516, pp 420002, 2022.

AUTHORS PROFILE

Dr. R Shankar was born in Chennai, Tamil Nadu, India in 1981. He received his B.sc and M.Sc in Mathematics from A.M. Jain College and Pachiyappa's College affiliated to university of Madras, He received his M.phil from PRIST University, Thanjavur. He completed State Eligibility Test (SET) in 2017 and did research work in Meenakshi Academy of Higher education and research (Meenakshi University) Chennai, received his P.hD in 2023. His area of interest Topology and Graph Theory. He published more than 10 research papers in various International and National Journals.

