International Journal of Scientific Research in Mathematical and Statistical Sciences
Volume-5, Issue-2, pp.46-56, April (2018)

Research Paper

E-ISSN: 2348-4519

A Comprehensive study of Various Queue Characteristics using Tri-Cum Biserial Queuing Model

Sachin Kumar Agrawal[1*], B.K. Singh [2]

[1*] Department of Mathematics, IFTM University, Moradabad, India
[2]Department of Mathematics, IFTM University, Moradabad, India

[*]*Corresponding Author: sachin269mit@gmail.com*

Available online at: www.isroset.org

Received 16/Mar/2018, Revised 25/Mar/2018, Accepted 10/Apr/2018, Online 30/Apr/2018

Abstract— In the present article, a queuing model having three servers connected in parallel in a tri-cum biserial way has been presented. Statistical tools, generating function technique and the law of calculus have been implemented to compute the various queuing characteristics such as Queue length, variance, average waiting time and probabilities. The arrival and service pattern is supposed to follow the Poisson law. Generic mathematical formulation of queuing model has been presented to ensure the implementation of the presented model in various problems occurred in real-time domain. The presented model is easy to understand and also deliver an important tool for the decision makers dealing with multi-tasking problems.

Keywords— Queue length, variance, average waiting time, probabilities.

I. INTRODUCTION

Operational research exemplifies a wide range of techniques that can improve the way we plan and organize the complex problems encountered in the daily life. Implementation of these techniques (i.e. Queuing theory) facilitates the person to become a better decision-maker. Queuing theory deals with the study of queues which occur in real-world situations and arise so long as arrival rate of any system is faster than the system can manage [1]. Queuing theory is applicable to any situation in daily life, for example, vehicle arriving at the petrol pump, a patient arriving at a doctor's clinic, customers arriving at the bank etc.

Numerous studies have been carried out in the past which dealt with the investigations of queuing model characteristics. Maggu [2] investigated the various attributes of phase type service queues with two servers in bi-series to study the queue model. Maggu & Das [3] presented an equivalent job-block theorem to deal the queuing problems in the production line. Singh, Man [4] examined the Steady-state behavior of serial queuing processes with impatient customers. Gupta *et al* [5] emphasized on the study of bi-serial and parallel channels associated with a common server to study the queuing model. Agrawal & Singh [7] investigated the various queue characteristics in which they considered tri-cum biserial queue model connected with a common server.

II. PRACTICAL ENACTMENT OF THE MODEL

Several situations can arise in which queuing theory will be helpful to take the best decision. For example, in a mall, several department/sections are there such as food corner, drinks and ice-cream corner and multiplex. Every department has it's own billing facility. The customers taking food may also have some ice-cream or drinks. Further, he/she may proceed to watch a movie. It is also possible that the customer entered in the mall to watch only movie or to have some food only. These activities may occur one by one or any two of them or even only one may also be possible. In such cases, the present queuing model can be effectively implemented to take the decision for hassle-free trip.

III. MATHEMATICAL DESCRIPTION OF THE MODEL

In the present work, a queue model consist of three servers (Sr_a, Sr_b and Sr_c) are connected in parallel in tri cum biserial way. Let Q_a, Q_b and Q_c are the queue lengths associated with servers Sr_a, Sr_b and Sr_c respectively. The number of customers (n_a) coming at mean arrival rate λ_a, after completion of service at server Sr_a, can avail the facility at server Sr_b or Sr_c (either of two or both) with the probabilities p_{ab}, p_{ac} and p_a such that $p_{ab} + p_{ac} + p_a = 1$. The same criterion will be applicable

to those customers who entered in servers Sr_b and Sr_c. The pictorial representation of the considered problem is demonstrated in Fig 1.

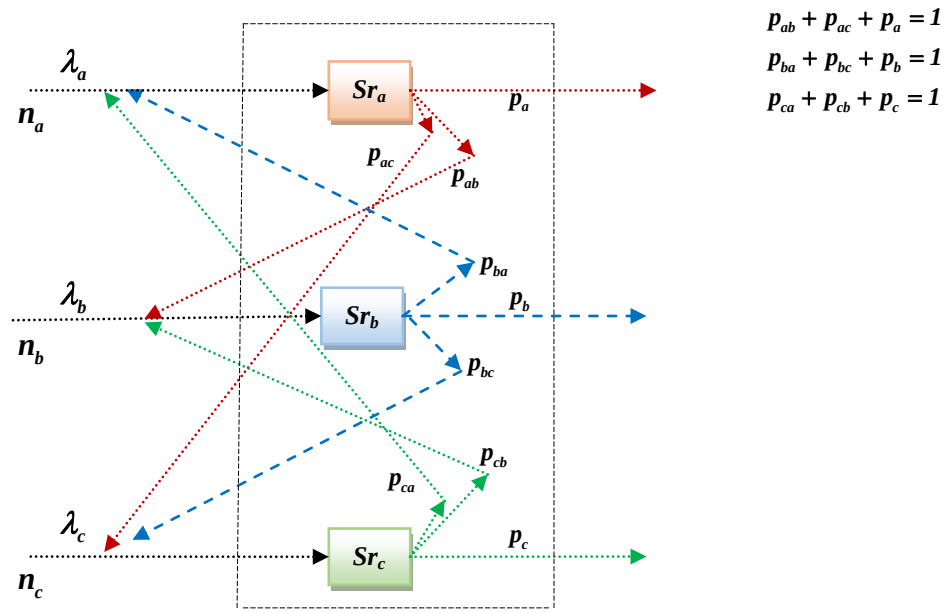

Fig 1: Tri-cum Biserial queuing network

Differential difference equation in steady (transient) state of the model is

$$\begin{aligned}(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_b+\mu_c)P_{n_a,n_b,n_c} &= \lambda_a P_{n_a-1,n_b,n_c}+\lambda_b P_{n_a,n_b-1,n_c}+\lambda_c P_{n_a,n_b,n_c-1}\\ &+\mu_a p_{ab}P_{n_a+1,n_b-1,n_c}+\mu_a p_{ac}P_{n_a+1,n_b,n_c-1}+\mu_a p_a P_{n_a+1,n_b,n_c}\\ &+\mu_b p_{ba}P_{n_a-1,n_b+1,n_c}+\mu_b p_{bc}P_{n_a,n_b+1,n_c-1}+\mu_b p_b P_{n_a,n_b+1,n_c}\\ &+\mu_c p_{cb}P_{n_a,n_b-1,n_c+1}+\mu_c p_{ca}P_{n_a-1,n_b,n_c+1}+\mu_c p_c P_{n_a,n_b,n_c+1}\end{aligned} \tag{1}$$

Considering $n_a = 0$; the Eq (1) can be given as

$$\begin{aligned}(\lambda_a+\lambda_b+\lambda_c+\mu_b+\mu_c)P_{0,n_b,n_c} &= \lambda_b P_{0,n_b-1,n_c}+\lambda_c P_{0,n_b,n_c-1}\\ &+\mu_a p_{ab}P_{1,n_b-1,n_c}+\mu_a p_{ac}P_{1,n_b,n_c-1}+\mu_a p_a P_{1,n_b,n_c}\\ &+\mu_b p_{bc}P_{0,n_b+1,n_c-1}+\mu_b p_b P_{0,n_b+1,n_c}\\ &+\mu_c p_{cb}P_{0,n_b-1,n_c+1}+\mu_c p_c P_{0,n_b,n_c+1}\end{aligned} \tag{2}$$

Considering $n_b = 0$; the Eq (1) can be given as

$$\begin{aligned}(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_c)P_{n_a,0,n_c} &= \lambda_a P_{n_a-1,0,n_c}+\lambda_c P_{n_a,0,n_c-1}\\ &+\mu_a p_{ac}P_{n_a+1,0,n_c-1}+\mu_a p_a P_{n_a+1,0,n_c}\\ &+\mu_b p_{ba}P_{n_a-1,1,n_c}+\mu_b p_{bc}P_{n_a,1,n_c-1}+\mu_b p_b P_{n_a,1,n_c}\\ &+\mu_c p_{ca}P_{n_a-1,0,n_c+1}+\mu_c p_c P_{n_a,0,n_c+1}\end{aligned} \tag{3}$$

Considering $n_c = 0$; the Eq (1) can be given as

$$\begin{aligned}(\lambda_a + \lambda_b + \lambda_c + \mu_a + \mu_b)P_{n_a,n_b,0} &= \lambda_a P_{n_a-1,n_b,0} + \lambda_b P_{n_a,n_b-1,0} \\ &+ \mu_a p_{ab} P_{n_a+1,n_b-1,0} + \mu_a p_a P_{n_a+1,n_b,0} \\ &+ \mu_b p_{ba} P_{n_a-1,n_b+1,0} + \mu_b p_b P_{n_a,n_b+1,0} \\ &+ \mu_c p_{cb} P_{n_a,n_b-1,1} + \mu_c p_{ca} P_{n_a-1,n_b,1} + \mu_c p_c P_{n_a,n_b,1}\end{aligned} \tag{4}$$

Considering $n_a = 0$ & $n_b = 0$; the Eq (1) can be given as

$$\begin{aligned}(\lambda_a + \lambda_b + \lambda_c + \mu_c)P_{0,0,n_c} &= \lambda_c P_{0,0,n_c-1} + \mu_a p_{ac} P_{1,0,n_c-1} + \mu_a p_a P_{1,0,n_c} \\ &+ \mu_b p_{bc} P_{0,1,n_c-1} + \mu_b p_b P_{0,1,n_c} \\ &+ \mu_c p_c P_{0,0,n_c+1}\end{aligned} \tag{5}$$

Considering $n_a = 0$ & $n_c = 0$; the Eq (1) can be given as

$$\begin{aligned}(\lambda_a + \lambda_b + \lambda_c + \mu_b)P_{0,n_b,0} &= \lambda_b P_{0,n_b-1,0} + \mu_a p_{ab} P_{1,n_b-1,0} + \mu_a p_a P_{1,n_b,0} \\ &+ \mu_b p_b P_{0,n_b+1,0} \\ &+ \mu_c p_{cb} P_{0,n_b-1,1} + \mu_c p_c P_{0,n_b,1}\end{aligned} \tag{6}$$

Considering $n_b = 0$ & $n_c = 0$; the Eq (1) can be given as

$$\begin{aligned}(\lambda_a + \lambda_b + \lambda_c + \mu_a)P_{n_a,0,0} &= \lambda_a P_{n_a-1,0,0} + \mu_a p_a P_{n_a+1,0,0} \\ &+ \mu_b p_{ba} P_{n_a-1,1,0,n_d} + \mu_b p_b P_{n_a,1,0} \\ &+ \mu_c p_{ca} P_{n_a-1,0,1} + \mu_c p_c P_{n_a,0,1}\end{aligned} \tag{7}$$

Considering $n_a = 0, n_b = 0$ & $n_c = 0$; the Eq (1) can be given as

$$(\lambda_a + \lambda_b + \lambda_c)P_{0,0,0} = \mu_a p_a P_{1,0,0} + \mu_b p_b P_{0,1,0} + \mu_c p_c P_{0,0,1} \tag{8}$$

using preliminary condition

$$P_{n_a,n_b,n_c}(0) = \begin{cases} 1, & for \;\; n_a, n_b, n_c \neq 0 \\ 0, & else \end{cases} \tag{9}$$

IV: GOVERNING EQUATIONS AND SOLUTION METHODOLOGY

To solve the governing Equations, Generating function is assumed as

$$F(X_1, X_2, X_3) = \sum_{n_a=0}^{\infty} \sum_{n_b=0}^{\infty} \sum_{n_c=0}^{\infty} X_1^{n_a} X_2^{n_b} X_3^{n_c} \tag{10}$$

such that $|X_1| = |X_2| = |X_3| = 1$

Also, taking partial generating function as

$$F_{n_b,n_c}(X_1) = \sum_{n_a=0}^{\infty} P_{n_a,n_b,n_c} X_1^{n_a} \tag{11}$$

$$F_{n_c}\left(X_1, X_2\right) = \sum_{n_b=0}^{\infty} F_{n_b,n_c}\left(X_1\right).X_2^{\,n_b} \tag{12}$$

$$F\left(X_1, X_2, X_3\right) = \sum_{n_c=0}^{\infty} F_{n_c}\left(X_1, X_2\right).X_3^{\,n_c} \tag{13}$$

On solving equations from (1) to (8) with the assistance of (10), (11), (12),(13), we get [2]

Multiplying (1) by $X_1^{\,n_a}$ and summing over n_a from 0 to ∞ using (2) with the assistance of (11), we get

$$\begin{aligned}(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_b+\mu_c)F_{n_b,n_c}(X_1)-\mu_a P_{0,n_b,n_c} &= \lambda_a X_1 F_{n_b,n_c}(X_1)\\ &+\lambda_b F_{n_b-1,n_c}(X_1)+\lambda_c F_{n_b,n_c-1}(X_1) + \frac{\mu_a p_{ab}}{X_1}\left[F_{n_b-1,n_c}(X_1)-P_{0,n_b-1,n_c}\right]\\ &+\frac{\mu_a p_{ac}}{X_1}\left[F_{n_b,n_c-1}(X_1)-P_{0,n_b,n_c-1}\right]+\frac{\mu_a p_a}{X_1}\left[F_{n_b,n_c}(X_1)-P_{0,n_b,n_c}\right]\\ &+\mu_b p_{ba} X_1 F_{n_b+1,n_c}(X_1)+\mu_b p_{bc} F_{n_b+1,n_c-1}(X_1)+\mu_b p_b F_{n_b+1,n_c}(X_1)\\ &+\mu_c p_{ca} X_1 F_{n_b,n_c+1}(X_1)+\mu_c p_{cb} F_{n_b-1,n_c+1}(X_1)+\mu_c p_c F_{n_b,n_c+1}(X_1)\end{aligned} \tag{14}$$

Multiplying (3) by $X_1^{\,n_a}$ and summing over n_a from 0 to ∞ using (5) with the assistance of (11), we get

$$\begin{aligned}(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_b+\mu_c)F_{0,n_c}(X_1)-\mu_a P_{0,0,n_c} &= \lambda_a X_1 F_{0,n_c}(X_1)+\lambda_c F_{0,n_c-1}(X_1)\\ &+\frac{\mu_a p_{ac}}{X_1}\left[F_{0,n_c-1}(X_1)-P_{0,0,n_c-1}\right]+\frac{\mu_a p_a}{X_1}\left[F_{0,n_c}(X_1)-P_{0,0,n_c}\right]\\ &+\mu_b p_{ba} X_1 F_{1,n_c}(X_1)+\mu_b p_{bc} F_{1,n_c-1}(X_1)+\mu_b p_b F_{1,n_c}(X_1)\\ &+\mu_c p_{ca} X_1 F_{0,n_c+1}(X_1)+\mu_c p_c F_{0,n_c+1}(X_1)\end{aligned} \tag{15}$$

Multiplying (4) by $X_1^{\,n_a}$ and summing over n_a from 0 to ∞ using (6) with the assistance of (11), we get

$$\begin{aligned}(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_b+\mu_c)F_{n_b,0}(X_1)-\mu_a P_{0,n_b,0} &= \lambda_a X_1 F_{n_b,0}(X_1)+\lambda_b F_{n_b-1,0}(X_1)\\ &+\frac{\mu_a p_{ab}}{X_1}\left[F_{n_b-1,0}(X_1)-P_{0,n_b-1,0}\right]+\frac{\mu_a p_a}{X_1}\left[F_{n_b,0}(X_1)-P_{0,n_b,0}\right]\\ &+\mu_b p_{ba} X_1 F_{n_b+1,0}(X_1)+\mu_b p_b F_{n_b+1,0}(X_1)\\ &+\mu_c p_{ca} X_1 F_{n_b,1}(X_1)+\mu_c p_{cb} F_{n_b-1,1}(X_1)+\mu_c p_c F_{n_b,1}(X_1)\end{aligned} \tag{16}$$

Multiplying (7) by $X_1^{\,n_a}$ and summing over n_a from 0 to ∞ using (8) with the assistance of (11), we get

$$\begin{aligned}(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_b+\mu_c)F_{0,0}(X_1)-\mu_a P_{0,0,0} &= \lambda_a X_1 F_{0,0}(X_1)\\ &+\frac{\mu_a p_a}{X_1}\left[F_{0,0}(X_1)-P_{0,0,0}\right]\\ &+\mu_b p_{ba} X_1 F_{1,0}(X_1)+\mu_b p_b F_{1,0}(X_1)\\ &+\mu_c p_{ca} X_1 F_{0,1}(X_1)+\mu_c p_c F_{0,1}(X_1)\end{aligned} \tag{17}$$

Multiplying (14) by $X_2^{\,n_b}$ and summing over n_b from 0 to ∞ using (15) with the assistance of (12), we get

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_b+\mu_c)F_{n_c}(X_1,X_2)-\mu_b F_{0,n_c}(X_1)-\mu_a F_{n_c}(X_2)=\lambda_a X_1 F_{n_c}(X_1,X_2)\\
+\lambda_b X_2 F_{n_c}(X_1,X_2)+\lambda_c F_{n_c-1}(X_1,X_2)\\
+\frac{\mu_a p_{ab}}{X_1}X_2\left[F_{n_c}(X_1,X_2)-F_{n_c}(X_2)\right]+\frac{\mu_a p_{ac}}{X_1}\left[F_{n_c-1}(X_1,X_2)-F_{n_c-1}(X_2)\right]\\
+\frac{\mu_a p_a}{X_1}\left[F_{n_c}(X_1,X_2)-F_{n_c}(X_2)\right]+\frac{\mu_b p_{ba}}{X_2}X_1\left[F_{n_c}(X_1,X_2)-F_{0,n_c}(X_1)\right]\\
+\frac{\mu_b p_{bc}}{X_2}\left[F_{n_c-1}(X_1,X_2)-F_{0,n_c-1}(X_1)\right]+\frac{\mu_b p_b}{X_2}\left[F_{n_c}(X_1,X_2)-F_{0,n_c}(X_1)\right]\\
+\mu_c p_{ca}X_1F_{n_c+1}(X_1,X_2)+\mu_c p_{cb}X_2F_{n_c+1}(X_1,X_2)+\mu_c p_c F_{n_c+1}(X_1,X_2)
\end{aligned}
\tag{18}
$$

Multiplying (16) by $X_2^{n_b}$ and summing over n_b from 0 to ∞ using (17) with the assistance of (12), we get

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_b+\mu_c)F_0(X_1,X_2)-\mu_b F_{0,0}(X_1)-\mu_a F_0(X_2)=\lambda_a X_1 F_0(X_1,X_2)\\
+\lambda_b X_2 F_0(X_1,X_2)+\frac{\mu_a p_{ab}}{X_1}X_2\left[F_0(X_1,X_2)-F_0(X_2)\right]\\
+\frac{\mu_a p_a}{X_1}\left[F_0(X_1,X_2)-F_0(X_2)\right]+\frac{\mu_b p_{ba}}{X_2}X_1\left[F_0(X_1,X_2)-F_{0,0}(X_1)\right]\\
+\frac{\mu_b p_b}{X_2}\left[F_0(X_1,X_2)-F_{0,0}(X_1)\right]+\mu_c p_{ca}X_1F_1(X_1,X_2)\\
+\mu_c p_{cb}X_2F_1(X_1,X_2)+\mu_c p_c F_1(X_1,X_2)
\end{aligned}
\tag{19}
$$

Multiplying (18) by $X_3^{n_c}$ and summing over n_c from 0 to ∞ using (19) with the assistance of (13), we get

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_b+\mu_c)F(X_1,X_2,X_3)-\mu_c F(X_1,X_2)-\mu_b F(X_1,X_3)-\mu_a F(X_2,X_3)=\\
\lambda_a X_1 F(X_1,X_2,X_3)+\lambda_b X_2 F(X_1,X_2,X_3)+\lambda_c X_3 F(X_1,X_2,X_3)\\
+\frac{\mu_a p_{ab}}{X_1}X_2\left[F(X_1,X_2,X_3)-F(X_2,X_3)\right]+\frac{\mu_a p_{ac}}{X_1}X_3\left[F(X_1,X_2,X_3)-F(X_2,X_3)\right]\\
+\frac{\mu_a p_a}{X_1}\left[F(X_1,X_2,X_3)-F(X_2,X_3)\right]+\frac{\mu_b p_{ba}}{X_2}X_1\left[F(X_1,X_2,X_3)-F(X_1,X_3)\right]\\
+\frac{\mu_b p_{bc}}{X_2}X_3\left[F(X_1,X_2,X_3)-F(X_1,X_3)\right]+\frac{\mu_b p_b}{X_2}\left[F(X_1,X_2,X_3)-F(X_1,X_3)\right]\\
+\frac{\mu_c p_{ca}}{X_3}X_1\left[F(X_1,X_2,X_3)-F(X_1,X_2)\right]+\frac{\mu_c p_{cb}}{X_3}X_2\left[F(X_1,X_2,X_3)-F(X_1,X_2)\right]\\
+\frac{\mu_c p_c}{X_3}\left[F(X_1,X_2,X_3)-F(X_1,X_2)\right]
\end{aligned}
\tag{20}
$$

on simplify (20), we get

$$F\left(X_1,X_2,X_3\right)=\frac{\begin{array}{c}\mu_a\left\{1-\frac{p_{ab}X_2}{X_1}-\frac{p_{ac}X_3}{X_1}-\frac{p_a}{X_1}\right\}F\left(X_2,X_3\right)+\mu_b\left\{1-\frac{p_{ba}X_1}{X_2}-\frac{p_{bc}X_3}{X_2}-\frac{p_b}{X_2}\right\}F\left(X_1,X_3\right)\\+\mu_c\left\{1-\frac{p_{ca}X_1}{X_3}-\frac{p_{cb}X_2}{X_3}-\frac{p_c}{X_3}\right\}F\left(X_1,X_2\right)\end{array}}{\begin{array}{c}\lambda_a\left(1-X_1\right)+\lambda_b\left(1-X_2\right)+\lambda_c\left(1-X_3\right)+\mu_a\left\{1-\frac{p_{ab}X_2}{X_1}-\frac{p_{ac}X_3}{X_1}-\frac{p_a}{X_1}\right\}\\+\mu_b\left\{1-\frac{p_{ba}X_1}{X_2}-\frac{p_{bc}X_3}{X_2}-\frac{p_b}{X_2}\right\}+\mu_c\left\{1-\frac{p_{ca}X_1}{X_3}-\frac{p_{cb}X_2}{X_3}-\frac{p_c}{X_3}\right\}\end{array}}$$

Assuming $F\left(X_2,X_3\right)=F_a,\ \ F\left(X_1,X_3\right)=F_b,\ \ F\left(X_1,X_2\right)=F_c$

$$F\left(X_1,X_2,X_3\right)=\frac{\begin{array}{c}\mu_a\left\{1-\frac{p_{ab}X_2}{X_1}-\frac{p_{ac}X_3}{X_1}-\frac{p_a}{X_1}\right\}F_a+\mu_b\left\{1-\frac{p_{ba}X_1}{X_2}-\frac{p_{bc}X_3}{X_2}-\frac{p_b}{X_2}\right\}F_b\\+\mu_c\left\{1-\frac{p_{ca}X_1}{X_3}-\frac{p_{cb}X_2}{X_3}-\frac{p_c}{X_3}\right\}F_c\end{array}}{\begin{array}{c}\lambda_a\left(1-X_1\right)+\lambda_b\left(1-X_2\right)+\lambda_c\left(1-X_3\right)+\mu_a\left\{1-\frac{p_{ab}X_2}{X_1}-\frac{p_{ac}X_3}{X_1}-\frac{p_a}{X_1}\right\}\\+\mu_b\left\{1-\frac{p_{ba}X_1}{X_2}-\frac{p_{bc}X_3}{X_2}-\frac{p_b}{X_2}\right\}+\mu_c\left\{1-\frac{p_{ca}X_1}{X_3}-\frac{p_{cb}X_2}{X_3}-\frac{p_c}{X_3}\right\}\end{array}} \tag{21}$$

Since $F(1,1,1)\ =\ 1$, the total probability. Considering $X_1=1$ as $X_2\to 1,\ \ X_3\to 1$,

$F(X_1,X_2,X_3)$ is of (0/0) indeterminate form. Therefore, using L- Hospital rule,

differentiating numerator and denominator separately w.r.t. X_1, we get

$$1=\frac{\mu_a\left(p_{ab}+p_{ac}+p_a\right)F_a+\mu_b\left(-p_{ba}\right)F_b+\mu_c\left(-p_{ca}\right)F_c}{-\lambda_a+\mu_a\left(p_{ab}+p_{ac}+p_a\right)+\mu_b\left(-p_{ba}\right)+\mu_c\left(-p_{ca}\right)}$$

where $p_{ab}+p_{ac}+p_a=1$

$$\mu_aF_a-\mu_bp_{ba}F_b-\mu_cp_{ca}F_c=-\lambda_a+\mu_a-\mu_bp_{ba}-\mu_cp_{ca} \tag{22}$$

Again differentiating numerator and denominator separately w.r.t. X_2 by taking $X_2=1$

as $X_1\to 1,\ \ X_3\to 1$, we get

$$1=\frac{\mu_a\left(-p_{ab}\right)F_a+\mu_b\left(p_{ba}+p_{bc}+p_b\right)F_b+\mu_c\left(-p_{cb}\right)F_c}{-\lambda_b+\mu_a\left(-p_{ab}\right)+\mu_b\left(p_{ba}+p_{bc}+p_b\right)+\mu_c\left(-p_{cb}\right)}$$

where $p_{ba}+p_{bc}+p_b=1$

$$-\mu_ap_{ab}F_a+\mu_bF_b-\mu_cp_{cb}F_c=-\lambda_b-\mu_ap_{ab}+\mu_b-\mu_cp_{cb} \tag{23}$$

Again differentiating numerator and denominator separately w.r.t. X_3 by taking $X_3=1$

as $X_1\to 1,\ \ X_2\to 1$, we get

$$1=\frac{\mu_a\left(-p_{ac}\right)F_a+\mu_b\left(-p_{bc}\right)F_b+\mu_c\left(p_{ca}+p_{cb}+p_c\right)F_c}{-\lambda_c+\mu_a\left(-p_{ac}\right)+\mu_b\left(-p_{bc}\right)+\mu_c\left(p_{ca}+p_{cb}+p_c\right)}$$

where $p_{ca}+p_{cb}+p_c=1$

$$-\mu_a p_{ac}F_a-\mu_b p_{bc}F_b+\mu_c F_c=-\lambda_c-\mu_a p_{ac}-\mu_b p_{bc}+\mu_c \tag{24}$$

on solving (22), (23), (24), we get

$$F_a=1-\frac{\lambda_a\left(1-p_{cb}p_{bc}\right)+\lambda_b\left\{p_{ba}(1-p_{cb}p_{bc})+p_{bc}\left(p_{ca}+p_{cb}p_{ba}\right)\right\}+\lambda_c\left(p_{ca}+p_{cb}p_{ba}\right)}{\mu_a\left\{\left(1-p_{ab}p_{ba}\right)\left(1-p_{cb}p_{bc}\right)-\left(p_{ac}+p_{ab}p_{bc}\right)\left(p_{ca}+p_{cb}p_{ba}\right)\right\}}$$

$$F_b=1-\frac{\lambda_a\left(p_{ab}+p_{ac}p_{cb}\right)+\lambda_b\left(1-p_{ac}p_{ca}\right)+\lambda_c\left\{p_{ca}\left(p_{ab}+p_{ac}p_{cb}\right)+p_{cb}\left(1-p_{ac}p_{ca}\right)\right\}}{\mu_b\left\{\left(1-p_{bc}p_{cb}\right)\left(1-p_{ac}p_{ca}\right)-\left(p_{ba}+p_{bc}p_{ca}\right)\left(p_{ab}+p_{ac}p_{cb}\right)\right\}}$$

$$F_c=1-\frac{\lambda_a\left\{p_{ab}\left(p_{bc}+p_{ba}p_{ac}\right)+p_{ac}\left(1-p_{ab}p_{ba}\right)\right\}+\lambda_b\left(p_{bc}+p_{ba}p_{ac}\right)+\lambda_c\left(1-p_{ab}p_{ba}\right)}{\mu_c\left\{\left(1-p_{ac}p_{ca}\right)\left(1-p_{ab}p_{ba}\right)-\left(p_{cb}+p_{ab}p_{ca}\right)\left(p_{bc}+p_{ba}p_{ac}\right)\right\}}$$

The solution (Joint Probability) of the model is written as

$$P_{n_a,n_b,n_c}=\left(1-F_a\right)^{n_a}\left(1-F_b\right)^{n_b}\left(1-F_c\right)^{n_c}F_aF_bF_c \tag{25}$$

$$P_{n_a,n_b,n_c}=\rho_a^{\,n_a}\ \rho_b^{\,n_b}\ \rho_c^{\,n_c}\ \left(1-\rho_a\right)\left(1-\rho_b\right)\left(1-\rho_c\right)$$

where $\rho_a=1-F_a,\ \rho_b=1-F_b,\ \rho_c=1-F_c$

$$\rho_a=\frac{\lambda_a\left(1-p_{cb}p_{bc}\right)+\lambda_b\left\{p_{ba}(1-p_{cb}p_{bc})+p_{bc}\left(p_{ca}+p_{cb}p_{ba}\right)\right\}+\lambda_c\left(p_{ca}+p_{cb}p_{ba}\right)}{\mu_a\left\{\left(1-p_{ab}p_{ba}\right)\left(1-p_{cb}p_{bc}\right)-\left(p_{ac}+p_{ab}p_{bc}\right)\left(p_{ca}+p_{cb}p_{ba}\right)\right\}}$$

$$\rho_b=\frac{\lambda_a\left(p_{ab}+p_{ac}p_{cb}\right)+\lambda_b\left(1-p_{ac}p_{ca}\right)+\lambda_c\left\{p_{ca}\left(p_{ab}+p_{ac}p_{cb}\right)+p_{cb}\left(1-p_{ac}p_{ca}\right)\right\}}{\mu_b\left\{\left(1-p_{bc}p_{cb}\right)\left(1-p_{ac}p_{ca}\right)-\left(p_{ba}+p_{bc}p_{ca}\right)\left(p_{ab}+p_{ac}p_{cb}\right)\right\}}$$

$$\rho_c=\frac{\lambda_a\left\{p_{ab}\left(p_{bc}+p_{ba}p_{ac}\right)+p_{ac}\left(1-p_{ab}p_{ba}\right)\right\}+\lambda_b\left(p_{bc}+p_{ba}p_{ac}\right)+\lambda_c\left(1-p_{ab}p_{ba}\right)}{\mu_c\left\{\left(1-p_{ac}p_{ca}\right)\left(1-p_{ab}p_{ba}\right)-\left(p_{cb}+p_{ab}p_{ca}\right)\left(p_{bc}+p_{ba}p_{ac}\right)\right\}}$$

The solution of this model exists if $\rho_a,\ \rho_b,\ \rho_c<1$ (26)

V. COMPUTATION OF VARIOUS QUEUING MODEL CHARACTERISTICS

(1) Mean queue length (average number of customers)

$$L_Q=L_a+L_b+L_c$$

$$L_Q=\frac{\rho_a}{1-\rho_a}+\frac{\rho_b}{1-\rho_b}+\frac{\rho_c}{1-\rho_c}$$

Where $L_a=\frac{\rho_a}{1-\rho_a},\quad L_b=\frac{\rho_b}{1-\rho_b},\quad L_c=\frac{\rho_c}{1-\rho_c}$

(2) Fluctuation (Variance) in queue length

$$V_{ar}=V_a+V_b+V_c$$

$$V_{ar}=\frac{\rho_a}{\left(1-\rho_a\right)^2}+\frac{\rho_b}{\left(1-\rho_b\right)^2}+\frac{\rho_c}{\left(1-\rho_c\right)^2}$$

Where $V_a = \frac{\rho_a}{(1-\rho_a)^2}$, $V_b = \frac{\rho_b}{(1-\rho_b)^2}$, $V_c = \frac{\rho_c}{(1-\rho_c)^2}$

(3) Average waiting time for customer

$$E_{wt} = \frac{L_Q}{\lambda} \quad , \quad where \quad \lambda = \lambda_a + \lambda_b + \lambda_c$$

VI. RESULTS AND DISCUSSION

In section 2 and 3, description of the mathematical model and associated governing equations have been presented. In the present example, ten customers have been considered to be processed through the network of three queues Q_a, Q_b and Q_c associated with servers Sr_a, Sr_b and Sr_c respectively. The details of various input parameters used in the computation of various characteristics have been presented in Table 1.

Table 2 demonstrates the joint probability and utilization of servers for various mean arrival rates (λ_a, λ_b and λ_c) and various probabilities ($p_{ab}, p_{ac}, p_a, p_{ba}, p_{bc}, p_b, p_{ca}, p_{cb}, p_c$). It is manifest from the calculated results that the values utilization of servers is less than 1 which also confirms the basic conditions given in Eq. (26). In Table 3 and Table 4, the average waiting time, queue lengths and variances have been deliberated considering different mean arrival rate (λ_a, λ_b and λ_c) and various probabilities ($p_{ab}, p_{ac}, p_a, p_{ba}, p_{bc}, p_b, p_{ca}, p_{cb}, p_c$) same as considered in the case of Table 2.

Table 1: Various input parameters considered in computation of results

μ_a	μ_b	μ_c	n_a	n_b	n_c
14	15	16	3	4	3

Table 2: Joint Probability and utilization of servers for various mean arrival rates and probabilities

λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	ρ_a	ρ_b	ρ_c	P
2	5	6	0.2	0.3	0.5	0.2	0.3	0.5	0.2	0.3	0.5	0.43	0.63	0.66	2.5E-04
3												0.51	0.65	0.69	4.2E-04
4												0.60	0.68	0.72	6.0E-04
5												0.68	0.70	0.75	7.6E-04
6												0.76	0.72	0.78	8.4E-04
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	ρ_a	ρ_b	ρ_c	P
4	2	6	0.2	0.3	0.5	0.2	0.3	0.5	0.2	0.3	0.5	0.52	0.43	0.63	1.3E-04
	3											0.55	0.51	0.66	2.5E-04
	4											0.57	0.59	0.69	4.1E-04
	5											0.60	0.68	0.72	6.0E-04
	6											0.62	0.76	0.75	7.6E-04
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	ρ_a	ρ_b	ρ_c	P
4	5	2	0.2	0.3	0.5	0.2	0.3	0.5	0.2	0.3	0.5	0.50	0.56	0.41	1.1E-04
		3										0.52	0.59	0.49	2.0E-04
		4										0.55	0.62	0.57	3.3E-04
		5										0.57	0.65	0.64	4.7E-04
		6										0.60	0.68	0.72	6.0E-04
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	ρ_a	ρ_b	ρ_c	P
4	5	6	0.1	0.45	0.45	0.2	0.3	0.5	0.2	0.3	0.5	0.61	0.64	0.79	5.5E-04
			0.2	0.4	0.4							0.62	0.70	0.79	6.7E-04
			0.3	0.35	0.35							0.63	0.76	0.78	7.7E-04
			0.4	0.3	0.3							0.64	0.82	0.77	8.0E-04
			0.5	0.25	0.25							0.65	0.88	0.76	7.2E-04

λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	ρ_a	ρ_b	ρ_c	P
4	5	6	0.2	0.3	0.5	0.1	0.45	0.45	0.2	0.3	0.5	0.55	0.69	0.81	5.3E-04
						0.2	0.4	0.4				0.62	0.71	0.80	6.8E-04
						0.3	0.35	0.35				0.70	0.72	0.79	7.9E-04
						0.4	0.3	0.3				0.78	0.73	0.78	8.3E-04
						0.5	0.25	0.25				0.86	0.74	0.77	7.3E-04
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	ρ_a	ρ_b	ρ_c	P
4	5	6	0.2	0.3	0.5	0.2	0.3	0.5	0.1	0.45	0.45	0.54	0.79	0.74	6.2E-04
									0.2	0.4	0.4	0.62	0.77	0.76	7.8E-04
									0.3	0.35	0.35	0.71	0.76	0.77	8.7E-04
									0.4	0.3	0.3	0.81	0.74	0.79	8.1E-04
									0.5	0.25	0.25	0.91	0.72	0.82	5.3E-04

Table 3: Average waiting time and queue lengths for various mean arrival rates and probabilities

λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	L_a	L_b	L_c	L_Q	E_{wt}
2	5	6	0.2	0.3	0.5	0.2	0.3	0.5	0.2	0.3	0.5	0.75	1.67	1.97	4.39	0.34
3												1.05	1.86	2.25	5.16	0.37
4												1.47	2.08	2.59	6.14	0.41
5												2.11	2.33	3.00	7.44	0.47
6												3.20	2.63	3.52	9.36	0.55
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	L_a	L_b	L_c	L_Q	E_{wt}
4	2	6	0.2	0.3	0.5	0.2	0.3	0.5	0.2	0.3	0.5	1.10	0.77	1.74	3.60	0.30
	3											1.21	1.06	1.97	4.24	0.33
	4											1.33	1.47	2.25	5.05	0.36
	5											1.47	2.08	2.59	6.14	0.41
	6											1.63	3.09	3.00	7.72	0.48
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	L_a	L_b	L_c	L_Q	E_{wt}
4	5	2	0.2	0.3	0.5	0.2	0.3	0.5	0.2	0.3	0.5	1.00	1.27	0.70	2.97	0.27
		3										1.10	1.43	0.96	3.49	0.29
		4										1.21	1.61	1.31	4.13	0.32
		5										1.33	1.83	1.81	4.97	0.36
		6										1.47	2.08	2.59	6.14	0.41
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	L_a	L_b	L_c	L_Q	E_{wt}
4	5	6	0.1	0.45	0.45	0.2	0.3	0.5	0.2	0.3	0.5	1.53	1.81	3.87	7.21	0.48
			0.2	0.4	0.4							1.60	2.34	3.71	7.64	0.51
			0.3	0.35	0.35							1.68	3.14	3.55	8.37	0.56
			0.4	0.3	0.3							1.76	4.52	3.40	9.67	0.64
			0.5	0.25	0.25							1.85	7.41	3.25	12.52	0.83
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	L_a	L_b	L_c	L_Q	E_{wt}
4	5	6	0.2	0.3	0.5	0.1	0.45	0.45	0.2	0.3	0.5	1.20	2.28	4.30	7.78	0.52
						0.2	0.4	0.4				1.63	2.40	4.07	8.10	0.54
						0.3	0.35	0.35				2.31	2.54	3.84	8.69	0.58
						0.4	0.3	0.3				3.50	2.70	3.64	9.83	0.66
						0.5	0.25	0.25				6.15	2.87	3.44	12.46	0.83
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	L_a	L_b	L_c	L_Q	E_{wt}
4	5	6	0.2	0.3	0.5	0.2	0.3	0.5	0.1	0.45	0.45	1.17	3.72	2.82	7.71	0.51
									0.2	0.4	0.4	1.66	3.39	3.10	8.15	0.54
									0.3	0.35	0.35	2.49	3.09	3.44	9.02	0.60
									0.4	0.3	0.3	4.18	2.82	3.87	10.87	0.72
									0.5	0.25	0.25	9.62	2.57	4.41	16.60	1.11

Table 4: Variances for various mean arrival rates and probabilities

λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	V_a	V_b	V_c	V_{ar}
2	5	6	0.2	0.3	0.5	0.2	0.3	0.5	0.2	0.3	0.5	1.31	4.46	5.86	11.64
3												2.15	5.32	7.31	14.78
4												3.63	6.40	9.27	19.31
5												6.57	7.78	12.00	26.35
6												13.44	9.57	15.92	38.93
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	V_a	V_b	V_c	V_{ar}
4	2	6	0.2	0.3	0.5	0.2	0.3	0.5	0.2	0.3	0.5	2.31	1.36	4.75	8.42
	3											2.68	2.18	5.86	10.72
	4											3.11	3.62	7.31	14.05
	5											3.63	6.40	9.27	19.31
	6											4.27	12.65	12.00	28.91
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	V_a	V_b	V_c	V_{ar}
4	5	2	0.2	0.3	0.5	0.2	0.3	0.5	0.2	0.3	0.5	2.00	2.87	1.20	6.08
		3										2.31	3.46	1.89	7.66
		4										2.68	4.21	3.03	9.91
		5										3.11	5.16	5.09	13.36
		6										3.63	6.40	9.27	19.31
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	V_a	V_b	V_c	V_{ar}
4	5	6	0.1	0.45	0.45	0.2	0.3	0.5	0.2	0.3	0.5	3.89	5.08	18.81	27.78
			0.2	0.4	0.4							4.17	7.79	17.43	29.40
			0.3	0.35	0.35							4.49	13.00	16.15	33.64
			0.4	0.3	0.3							4.86	24.90	14.95	44.71
			0.5	0.25	0.25							5.28	62.35	13.83	81.46
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	V_a	V_b	V_c	V_{ar}
4	5	6	0.2	0.3	0.5	0.1	0.45	0.45	0.2	0.3	0.5	2.64	7.46	22.80	32.90
						0.2	0.4	0.4				4.31	8.17	20.60	33.07
						0.3	0.35	0.35				7.64	8.99	18.63	35.26
						0.4	0.3	0.3				15.72	9.97	16.85	42.54
						0.5	0.25	0.25				43.99	11.13	15.25	70.37
λ_a	λ_b	λ_c	p_{ab}	p_{ac}	p_a	p_{ba}	p_{bc}	p_b	p_{ca}	p_{cb}	p_c	V_a	V_b	V_c	V_{ar}
4	5	6	0.2	0.3	0.5	0.2	0.3	0.5	0.1	0.45	0.45	2.54	17.59	10.77	30.89
									0.2	0.4	0.4	4.42	14.91	12.70	32.03
									0.3	0.35	0.35	8.68	12.67	15.28	36.62
									0.4	0.3	0.3	21.69	10.77	18.81	51.28
									0.5	0.25	0.25	102.07	9.17	23.88	135.11

VII. CONCLUSION

In the present study, an attempt has been made to investigate a queueing network modeled using three servers connected in parallel in tri-cum bi-series way. Detailed mathematical description of the presented model has been given which can be implemented in multi-tasking problems. Various queuing characteristics such as queue length, variance, joint probabilities and average waiting time for customers have been computed using proposed mathematical model.

REFERENCES

[1] I. Adan, J. Resing, "*Queuing theory department of mathematics and computing science"*, Eindhoren University of Technology Eindhoren, 1–123, 2002.

[2] P.L. Maggu, "*Phase type service queues with two servers in biseries"*, Journal of Operational Research Society of Japan, 13(1), 1-6, 1970.

[3] P.L. Maggu and G. Das, "*Equivalent jobs for job block in job sequencing*", *Opsearch*, 14(4), 277-281, 1977.

[4] Singh, Man, *"Steady state behaviour of serial queuing processes with impatient customers"*, Math, Operations forsch. U. statist. Ser. 15(2),289-298, 1984.
[5] D. Gupta, S. Sharma, and S. Sharma, *"On linkage of a flowshop scheduling model including job block criteria with a parallel biserial queue network"*, Computer Engineering and Intelligent System, 3(2), 17-28, 2012.
[6] R.R.P. Jackson, *"Queuing systems with phase-type service*", Operational Research Quarterly, 5, 109-120, 1954.
[7] S.K. Agrawal, and B.K. Singh, "*Computation of various queue characteristics using tri-cum biserial queuing model connected with a common server"*, International Journal of Mathematics Trends and Technology (IJMTT) – Volume 56 Number 1, 81-90, 2018. doi: 10.14445/22315373/IJMTT-V56P510.

Appendix

symbol	notations
Servers	Sr_a, Sr_b, Sr_c
Joint Probability	P_{n_a,n_b,n_c}
Mean arrival rates	$\lambda_a, \lambda_b, \lambda_c$
Mean Service Rates	μ_a, μ_b, μ_c
probabilities	$p_{ab}, p_{ac}, p_a, p_{ba}, p_{bc}, p_b, p_{ca}, p_{cb}, p_c$
No. of Customers	n_a, n_b, n_c
Traffic intensity or utilization of servers	ρ_a, ρ_b, ρ_c
Length of queues	L_a, L_b, L_c, L_Q
Variances	V_a, V_b, V_c, V_{ar}
Average waiting time for customer	E_{wt}

