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On Discrete Wrapped Exponential Distribution– Characteristics

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***Abstract*:** Here an attempt is made to propose new discrete circular model by employing the method of discretization of existing continuous circular model. Here considering the Wrapped Exponential model we discretized it to make it discrete Wrapped Exponential distribution. The probability mass function, distribution function and characteristic function of the new discrete circular model are obtained and population characteristics are studied .

***Keywords*:** Characteristic function, Distribution function, Trigonometric moments.

I. INTRODUCTION

Circular models play vital role in data analysis of angular or periodic data. A good number of continuous circular models are available in the literature. Wrapped circular models [13], stereographic circular and semicircular models [17] and Offset circular and semicircular models [1] and *l* – axial models [6] provide a rich and very useful class of models for circular as well as *l*-axial data. Scant attention was paid in investigation of discrete circular models. There is a need of developing discrete circular models which are invariant of zero direction and sense of rotation. Wrapped Discrete Binomial model derived in [14] and characteristics of Wrapped Poisson model were studied in [15]. In this paper, a new discrete circular model called Discrete Wrapped Exponential model is constructed by employing discretization on Wrapped Exponential model.

II. CIRCULAR DISTRIBUTIONS

A circular distribution is a probability distribution whose total probability is defined on the unit circle. Since each point on the unit circle represents a direction, it is a way of assigning probabilities to different directions. The range of a circular random variable θ measured in radians, can be taken as $[0, 2\pi)$ or $[-\pi, \pi)$.

Circular distributions are of two types, they may be discrete, assigning probabilities to a finite number of directions or continuous assigning probabilities to a infinite number of directions. One of the construction methods of circular models is wrapping a linear distribution. Here wrapped discrete circular random variables are discussed.

Wrapped Discrete Circular Random Variables

If *X* is a discrete random variable on the set of integers, then reduction modulo $2\pi m\left(m \in \mathbf{Z}^+\right)$ wraps the integers on to the group of m^{th} roots of unity which is a sub group of unit circle.

$$\theta = 2\pi x\left(\operatorname{mod} 2\pi m\right)$$

More precisely θ is a mapping from a set of integers *G* which is a group with respect to '+' to the set of m^{th} roots of unity G' which is a group with respect to '.' is defined as

$$\theta(x) = e^{\frac{2\pi i x}{m}} \quad \text{where } x \in G, \quad e^{\frac{2\pi i x}{m}} \in G'$$

Then θ is called wrapped discrete circular random variable.

Clearly θ is a homomorphism

1) $\theta(x+y) = e^{\frac{2\pi i(x+y)}{m}} = e^{\frac{2\pi i x}{m}} e^{\frac{2\pi i y}{m}} = \theta(x)\theta(y)$

2) $\theta(0) = e^{\frac{2\pi i(0)}{m}} = e^0 = 1$ where $0 \in G,\ 1 \in G'$

Since θ contains a finite number of elements they are denoted by $\theta = \left\{ \frac{2\pi r}{m} \Big/ r = 0,1,2,...m-1 \right\}$ which is lattice on the unit circle.

Probability Mass Function

Suppose if θ is a wrapped discrete circular random variable then probability mass function of θ is denoted by $Pr\left(\theta = \frac{2\pi r}{m}\right)$ which satisfies the following properties

1. $Pr\left(\theta = \frac{2\pi r}{m}\right) \geq 0$
2. $\sum_{r=0}^{m-1} Pr\left(\theta = \frac{2\pi r}{m}\right) = 1$
3. $Pr(\theta) = Pr(\theta + 2\pi l)$ For any integer l (i.e Pr is periodic)

Distribution Function

Suppose if θ is a wrapped discrete circular random variable then distribution function of θ is denoted by $F_w(\theta)$ and it is defined as

$$F_w(\theta) = \sum_{r=0}^{k} Pr\left(\theta = \frac{2\pi r}{m}\right) \quad \text{where } k = 0,1,...m-1$$

In mathematics, discretization is the process of transforming continuous functions, models, variables, and equations into its discrete counterparts. This process is usually carried out to model a given discrete phenomenon depending on the practical situation.

III. DISCRETIZATION OF A CONTINUOUS DISTRIBUTION

In this paper, we adopt discretization from [9] to construct discrete circular models by transforming existing continuous circular models. The process is as follows

Consider a continuous circular distribution with probability density function (pdf) $f(\theta)$ on the circle to construct a probability mass function $Pr\left(\theta = \frac{2\pi r}{m}\right)$ on a set of the equally spaced points for $r = 0,1,2...m-1$ with a fixed integer m equal or greater than 2.

$Pr\left(\theta = \frac{2\pi r}{m}\right) = f\left(\frac{2\pi r}{m}\right) \Big/ \sum_{r=0}^{m-1} f\left(\frac{2\pi r}{m}\right)$ is probability mass function(pmf) of respective discrete circular model. $m = 1$ is excluded as it is degenerate. Here we derive new discrete circular model by employing the discretization of a continuous circular model.

IV. DISCRETE WRAPPED EXPONENTIAL DISTRIBUTION

The probability density function of a Wrapped Exponential distribution [12] is $f(\theta) = \frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}$ where λ >0 is a parameter.

Now this model is converted into Discrete Wrapped Exponential distribution by normalizing the probability density function with $\frac{1-e^{\frac{-2\pi\lambda}{m}}}{\lambda}$

Then probability mass function of the Discrete Wrapped Exponential Distribution is defined as $Pr(\theta = \frac{2\pi r}{m}) = \frac{e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\left(1-e^{\frac{-2\pi\lambda}{m}}\right)$ where $m \in \mathbf{Z}^{+}$ and $r = 0,1,2,....m-1$

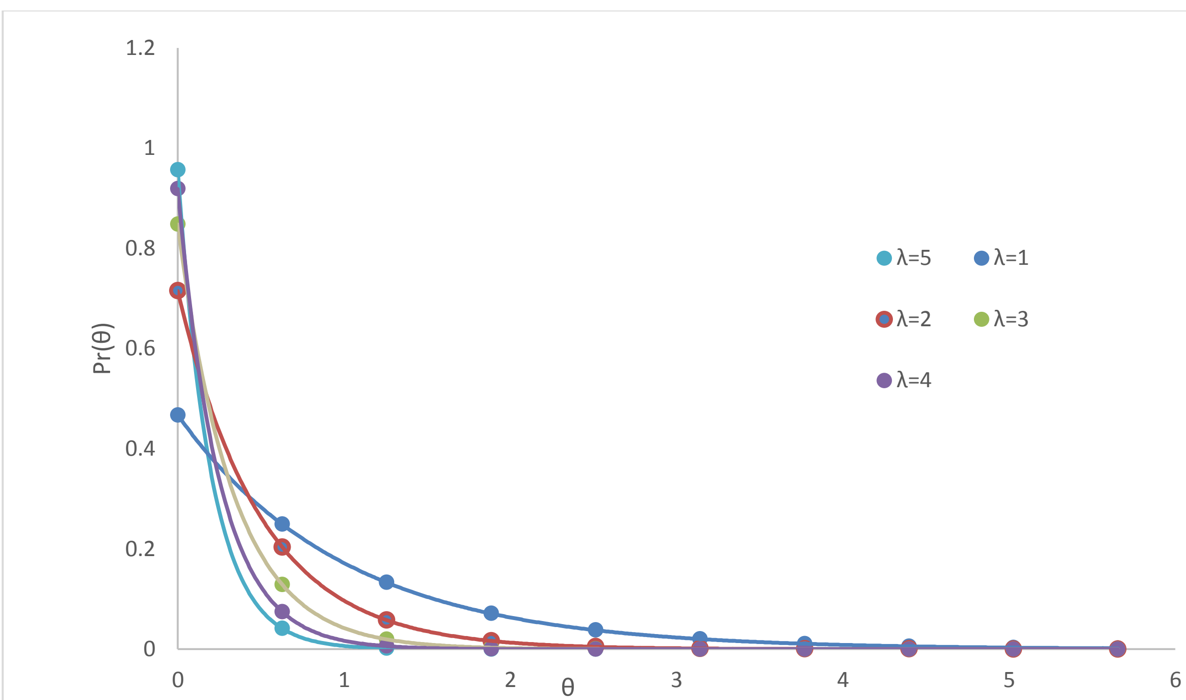

Fig.1: Graph for Probability mass function of Discrete Wrapped Exponential Distribution

V. DISTRIBUTION FUNCTION OF DISCRETE WRAPPED EXPONENTIAL DISTRIBUTION

The Distribution function of Discrete Wrapped Exponential Distribution is defined as $F_w(\theta)=\sum_{r=0}^{k} Pr\left(\theta=\frac{2\pi r}{m}\right)$

$$=\frac{1-e^{\frac{-2\pi\lambda}{m}}}{1-e^{-2\pi\lambda}}\sum_{r=0}^{k} e^{\frac{-2\pi r\lambda}{m}}$$

$$=\frac{1-e^{\frac{-2\pi\lambda}{m}}}{1-e^{-2\pi\lambda}}\left[1+e^{\frac{-2\pi\lambda}{m}}+e^{\frac{-4\pi\lambda}{m}}+---+e^{\frac{-2\pi\lambda k}{m}}\right]$$

$$=\frac{1-e^{\frac{-2\pi\lambda}{m}}}{1-e^{-2\pi\lambda}}\left[1+e^{\frac{-2\pi\lambda}{m}}+\left(e^{\frac{-2\pi\lambda}{m}}\right)^2+---+\left(e^{\frac{-2\pi\lambda}{m}}\right)^k\right]$$

Since $\left|e^{\frac{-2\pi\lambda}{m}}\right|<1$ then by the sum of n terms of Geometric progression

$$F_w(\theta)=\frac{1-e^{\frac{-2\pi\lambda(k+1)}{m}}}{1-e^{-2\pi\lambda}} \text{ Where } \quad k=0,1,2,...m-1$$

VI. CHARACTERISTIC FUNCTION OF THE DISCRETE WRAPPED EXPONENTIAL DISTRIBUTION

The Characteristic function of Discrete Wrapped Exponential distribution is defined as

$\varphi_\theta(p)=E(e^{ip\theta})$ where $p\in\mathbf{Z}$

$$=\sum_{r=0}^{m-1} e^{ip\theta}\frac{e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}(1-e^{\frac{-2\pi\lambda}{m}})=\frac{1-e^{\frac{-2\pi\lambda}{m}}}{1-e^{-2\pi\lambda}}\sum_{r=0}^{m-1} e^{ip\theta}e^{\frac{-2\pi r\lambda}{m}}$$

$$=\frac{1-e^{\frac{-2\pi\lambda}{m}}}{1-e^{-2\pi\lambda}}\sum_{r=0}^{m-1} e^{\frac{-2\pi r}{m}(\lambda-ip)}=\frac{1-e^{\frac{-2\pi\lambda}{m}}}{1-e^{-2\pi\lambda}}\left[1+e^{\frac{-2\pi}{m}(\lambda-ip)}+e^{\frac{-4\pi}{m}(\lambda-ip)}+------+e^{\frac{-2\pi(m-1)}{m}(\lambda-ip)}\right]$$

Since $\left|e^{\frac{-2\pi}{m}(\lambda-ip)}\right|<1$ then by sum of n terms in Geometric progression

$$\varphi_\theta(p)=\frac{1-e^{\frac{-2\pi\lambda}{m}}}{1-e^{-2\pi\lambda}}\left[\frac{1-e^{-2\pi(\lambda-ip)}}{1-e^{\frac{-2\pi}{m}(\lambda-ip)}}\right]=\frac{1-e^{\frac{-2\pi\lambda}{m}}}{1-e^{-2\pi\lambda}}\left[\frac{1-e^{-2\pi\lambda}e^{2\pi ip}}{1-e^{\frac{-2\pi\lambda}{m}}e^{\frac{2\pi ip}{m}}}\right]=\rho_p e^{i\mu_p}$$

$$=\frac{1-e^{\frac{-2\pi\lambda}{m}}}{1-e^{-2\pi\lambda}}\left[\frac{1-e^{-2\pi\lambda}\left(\cos 2\pi p+i\sin 2\pi p\right)}{1-e^{\frac{-2\pi\lambda}{m}}\left(\cos\frac{2\pi p}{m}+i\sin\frac{2\pi p}{m}\right)}\right]$$

$$=\frac{1-e^{\frac{-2\pi\lambda}{m}}}{1-e^{\frac{-2\pi\lambda}{m}}\left(\cos\frac{2\pi p}{m}+i\sin\frac{2\pi p}{m}\right)}$$

$$=\frac{1-e^{\frac{-2\pi\lambda}{m}}}{1-e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi p}{m}-ie^{\frac{-2\pi\lambda}{m}}\sin\frac{2\pi p}{m}}$$

$$=\frac{1-e^{\frac{-2\pi\lambda}{m}}}{\left(1-e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi p}{m}\right)-ie^{\frac{-2\pi\lambda}{m}}\sin\frac{2\pi p}{m}}\;\frac{\left(1-e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi p}{m}\right)+ie^{\frac{-2\pi\lambda}{m}}\sin\frac{2\pi p}{m}}{\left(1-e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi p}{m}\right)+ie^{\frac{-2\pi\lambda}{m}}\sin\frac{2\pi p}{m}}$$

$$=\left(1-e^{\frac{-2\pi\lambda}{m}}\right)\left(\frac{1-e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi p}{m}+ie^{\frac{-2\pi\lambda}{m}}\sin\frac{2\pi p}{m}}{1+e^{\frac{-4\pi\lambda}{m}}-2e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi p}{m}}\right)$$

$$=\frac{1-e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi p}{m}+ie^{\frac{-2\pi\lambda}{m}}\sin\frac{2\pi p}{m}-e^{\frac{-2\pi\lambda}{m}}+e^{\frac{-4\pi\lambda}{m}}\cos\frac{2\pi p}{m}-ie^{\frac{-4\pi\lambda}{m}}\sin\frac{2\pi p}{m}}{1+e^{\frac{-4\pi\lambda}{m}}-2e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi p}{m}}$$

$$=\alpha_p+i\beta_p$$

where $$\alpha_p=\frac{1-e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi p}{m}-e^{\frac{-2\pi\lambda}{m}}+e^{\frac{-4\pi\lambda}{m}}\cos\frac{2\pi p}{m}}{1+e^{\frac{-4\pi\lambda}{m}}-2e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi p}{m}}$$

and $\beta_p = \dfrac{e^{\frac{-2\pi\lambda}{m}} \sin\frac{2\pi p}{m} - e^{\frac{-4\pi\lambda}{m}} \sin\frac{2\pi p}{m}}{1+e^{\frac{-4\pi\lambda}{m}} - 2e^{\frac{-2\pi\lambda}{m}} \cos\frac{2\pi p}{m}}$ Here α_p, β_p are called p[th] trigonometric moments

Clearly $\rho_p = \sqrt{\alpha^2{}_p + \beta_p{}^2}$ and $\mu_p = \tan^{-1}\left(\dfrac{\beta_p}{\alpha_p}\right)$

The circular mean direction is denoted by μ_1 and it is defined as

$$\mu_1 = \tan^{-1}\left[\frac{e^{\frac{-2\pi\lambda}{m}} \sin\frac{2\pi}{m} - e^{\frac{-4\pi\lambda}{m}} \sin\frac{2\pi}{m}}{1-e^{\frac{-2\pi\lambda}{m}} \cos\frac{2\pi}{m} - e^{\frac{-2\pi\lambda}{m}} + e^{\frac{-4\pi\lambda}{m}} \cos\frac{2\pi}{m}}\right]$$

Now ρ_1 represents the concentration towards mean direction which is defined as

$$\rho_1 = \sqrt{\left[\frac{\left(1-e^{\frac{-2\pi\lambda}{m}} \cos\frac{2\pi}{m} - e^{\frac{-2\pi\lambda}{m}} + e^{\frac{-4\pi\lambda}{m}} \cos\frac{2\pi}{m}\right)^2 + \left(e^{\frac{-2\pi\lambda}{m}} \sin\frac{2\pi}{m} - e^{\frac{-4\pi\lambda}{m}} \sin\frac{2\pi}{m}\right)^2}{\left(1+e^{\frac{-4\pi\lambda}{m}} - 2e^{\frac{-2\pi\lambda}{m}} \cos\frac{2\pi}{m}\right)^2}\right]}$$

In general $\mu_1 = \mu$ and $\rho_1 = \rho$

The circular variance is denoted by V_o and it is defined as $V_o = 1-\rho$

$$V_o = 1 - \sqrt{\left[\frac{\left(1-e^{\frac{-2\pi\lambda}{m}} \cos\frac{2\pi}{m} - e^{\frac{-2\pi\lambda}{m}} + e^{\frac{-4\pi\lambda}{m}} \cos\frac{2\pi}{m}\right)^2 + \left(e^{\frac{-2\pi\lambda}{m}} \sin\frac{2\pi}{m} - e^{\frac{-4\pi\lambda}{m}} \sin\frac{2\pi}{m}\right)^2}{\left(1+e^{\frac{-4\pi\lambda}{m}} - 2e^{\frac{-2\pi\lambda}{m}} \cos\frac{2\pi}{m}\right)^2}\right]}$$

And the circular standard deviation is defined as

$$\sigma_o = \sqrt{-2\log(1-V_o)}$$

$$= \sqrt{-2\log\left[\sqrt{\left[\frac{\left(1-e^{\frac{-2\pi\lambda}{m}} \cos\frac{2\pi}{m} - e^{\frac{-2\pi\lambda}{m}} + e^{\frac{-4\pi\lambda}{m}} \cos\frac{2\pi}{m}\right)^2 + \left(e^{\frac{-2\pi\lambda}{m}} \sin\frac{2\pi}{m} - e^{\frac{-4\pi\lambda}{m}} \sin\frac{2\pi}{m}\right)^2}{\left(1+e^{\frac{-4\pi\lambda}{m}} - 2e^{\frac{-2\pi\lambda}{m}} \cos\frac{2\pi}{m}\right)^2}\right]}\right]}$$

VII. CENTRAL TRIGONOMETRIC MOMENTS

The pth central trigonometric moment of θ is defined as $\varphi_\theta^*(p) = E\left(e^{ip(\theta-\mu)}\right)$

$$= E\left(e^{ip\theta}e^{-ip\mu}\right) = e^{-ip\mu}E\left(e^{ip\theta}\right)$$

$$= (\cos p\mu - i\sin p\mu)\left(\alpha_p + i\beta_p\right)$$

$$= \left(\alpha_p \cos p\mu + i\beta_p \cos p\mu - i\alpha_p \sin p\mu + \beta_p \sin p\mu\right)$$

$$= \alpha_p \cos p\mu + \beta_p \sin p\mu + i(\beta_p \cos p\mu - \alpha_p \sin p\mu)$$

$$= \alpha_p^* + i\beta_p^*$$

where $\alpha_p^* = \alpha_p \cos p\mu + \beta_p \sin p\mu$

$$\beta_p^* = \beta_p \cos p\mu - \alpha_p \sin p\mu$$

Now the Skewness for the Discrete Wrapped Exponential Distribution is defined as

$$\gamma_1 = \frac{\beta_2^*}{(V_o)^{\frac{3}{2}}}$$

$$= \frac{\beta_2 \cos 2\mu - \alpha_2 \sin 2\mu}{\left[1 - \sqrt{\left[\frac{\left(1 - e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi}{m} - e^{\frac{-2\pi\lambda}{m}} + e^{\frac{-4\pi\lambda}{m}}\cos\frac{2\pi}{m}\right)^2 + \left(e^{\frac{-2\pi\lambda}{m}}\sin\frac{2\pi}{m} - e^{\frac{-4\pi\lambda}{m}}\sin\frac{2\pi}{m}\right)^2}{\left(1 + e^{\frac{-4\pi\lambda}{m}} - 2e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi}{m}\right)^2}\right]}\right]^{\frac{3}{2}}}$$

And the kurtosis for the Discrete Wrapped Exponential Distribution is defined as

$$\gamma_2 = \frac{\alpha_2^* - (1 - V_o)^4}{(V_o)^2}$$

$$= \frac{(\alpha_2 \cos 2\mu + \beta_2 \sin 2\mu) - \left[\sqrt{\left[\frac{\left(1 - e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi}{m} - e^{\frac{-2\pi\lambda}{m}} + e^{\frac{-4\pi\lambda}{m}}\cos\frac{2\pi}{m}\right)^2 + \left(e^{\frac{-2\pi\lambda}{m}}\sin\frac{2\pi}{m} - e^{\frac{-4\pi\lambda}{m}}\sin\frac{2\pi}{m}\right)^2}{\left(1 + e^{\frac{-4\pi\lambda}{m}} - 2e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi}{m}\right)^2}\right]}\right]^4}{\left[1 - \sqrt{\left[\frac{\left(1 - e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi}{m} - e^{\frac{-2\pi\lambda}{m}} + e^{\frac{-4\pi\lambda}{m}}\cos\frac{2\pi}{m}\right)^2 + \left(e^{\frac{-2\pi\lambda}{m}}\sin\frac{2\pi}{m} - e^{\frac{-4\pi\lambda}{m}}\sin\frac{2\pi}{m}\right)^2}{\left(1 + e^{\frac{-4\pi\lambda}{m}} - 2e^{\frac{-2\pi\lambda}{m}}\cos\frac{2\pi}{m}\right)^2}\right]}\right]^2}$$

VIII. CONCLUSION

On the lines of discretization of continuous circular model, we constructed Discrete Wrapped Exponential Distribution.Trigonometric moments are obtained from the characteristic function and we studied population characteristics using these trigonometric moments.

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