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Numerical Study of a Steady State Two Dimensions Heat equation using TDMA Technique

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Abstract -- **In this paper, a steady 2-D heat equation was solved numerically using TDMA technique. A steady state two dimensional heat flow is governed by Laplace Equation. Using TDMA technique numerical solution for Laplace equation (heat equation) with constant thermal conductivity has been obtained. Finite volume method is used to obtain system of linear algebraic equations for TDMA technique. The numerical solution obtained is compared with the solution obtained by using Gauss Seidal method. The significant finding of this study is to establish TDMA technique is very efficient for two dimensional problems. A very good agreement between TDMA and Gauss Seidal solutions has also been observed.**

Keywords: Finite volume method, Heat equation, Laplace equation, Steady state, TDMA technique.

I. INTRODUCTION

In many industrial applications, a number of efficient, accurate and stable numerical methods are used for solving fluid flow problems. In many problems, the computer simulation provides a very cost effective, quick and sufficiently reliable results.

The mathematical models of fluid flow and heat transfer problems are often solved numerically [1], [2], [3]. There are number of discretization techniques used to convert the governing equations of such problems into a system of linear algebraic equations which are solved numerically. These techniques include finite difference method (FDM), finite volume method (FVM) and finite element method (FEM). Each of these techniques has its own advantages and disadvantages [4], [5]. The factors which influence the number of linear equations in the system and its complexity are the number of grid nodes taken and the choice of discretization technique. There are two types of techniques to solve such a system of linear algebraic equations, namely, direct methods and iterative methods. Direct methods such as Cramer's rule, Gauss Elimination method, Matrix Inversion method etc. yields the solution after a certain amount of fixed computations. On the other hand, iterative methods like Gauss Jacobi Method, Gauss Seidal Method etc. provides an approximate solution which improves with increasing number of computation cycles. Iterative methods are self-correcting methods and they require only non-zero coefficients of the equations to be stored in the core memory of the computer. These are the major advantages of iterative process. But, these methods do not ensure the convergence of the solution. The system of equations obtained using Finite volume method has a vast majority of zero entries. Also, the system obtained from realistic fluid flow and heat transfer problems can be very large. Hence, iterative methods are more economical than direct methods.

Thomas algorithm or the Tri Diagonal Matrix Algorithm (TDMA) is a direct method for one dimensional problem. But, for two or three dimensional problems it can be applied iteratively, in a line by line fashion. The technique is quite adaptable and proficient for solving problems in fluid dynamics [6], [7], [8], [9], [10]. It is very cost effective technique and requires very small amount of storage. In this paper, TDMA is used to study a steady state two dimensional heat transfer problem with constant thermal conductivity.

In Section 2, a short review of Thomas Algorithm is given. In Section 3, formulation of the two dimensional heat flow problem with Dirichlet boundary conditions is done. In Section 4, Numerical calculations and comparison with Gauss Seidal solutions are made. Finally, Section 5 concludes the paper.

II. TDMA Algorithm

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TDMA is a simplified form of Gauss elimination method that can be used to solve tri diagonal system of equations. A tri-diagonal system for n unknowns may be written as

$$\begin{aligned}
&x_1 &&= C_1\\
&-b_2x_1 + d_2x_2 - a_2\,x_3 &&= C_2\\
&-b_3x_2 + d_3x_3 - a_3\,x_4 &&= C_3\\
&-b_4x_3 + d_4x_4 - a_4\,x_5 &&= C_4\\
&\qquad\ldots\ldots\ldots\ldots\ldots &&\\
&-b_nx_{n-1} + d_nx_n - a_n\,x_{n+1} &&= C_n\\
&x_{n+1} &&= C_{n+1}
\end{aligned}$$

In the above set of equations x_1 and x_{n+1} are known boundary values. The general form of any single equation is

$$-b_ix_{i-1} + d_ix_i - a_i\,x_{i+1} = C_i \qquad (1)$$

The equations of the above system can be rewritten as

$$x_2 = \frac{a_2}{d_2}x_3 + \frac{b_2}{d_2}x_1 + \frac{C_2}{d_2} \qquad (2)$$

$$x_3 = \frac{a_3}{d_3}x_4 + \frac{b_3}{d_3}x_2 + \frac{C_3}{d_3} \qquad (3)$$

$$x_4 = \frac{a_4}{d_4}x_5 + \frac{b_4}{d_4}x_3 + \frac{C_4}{d_4} \qquad (4)$$

$$x_n = \frac{a_n}{d_n}x_{n+1} + \frac{b_n}{d_n}x_{n-1} + \frac{C_n}{d_n} \qquad (5)$$

These equations can be solved using forward elimination and backward substitution. The forward elimination process start by removing x_2 from Eq. (3) by substitution from Eq. (2) to get

$$x_3 = \left(\frac{a_3}{d_3 - b_3\frac{a_2}{d_2}}\right)x_4 + \left(\frac{b_3\left(\frac{b_2}{d_2}x_1 + \frac{C_2}{d_2}\right) + C_3}{d_3 - b_3\frac{a_2}{d_2}}\right) \qquad (6)$$

Adopting the notation

$$A_2 = \frac{a_2}{d_2} \quad \text{and} \quad C_2' = \frac{b_2}{d_2}x_1 + \frac{C_2}{d_2} \qquad (7)$$

Equation (6) can be written as

$$x_3 = \left(\frac{a_3}{d_3 - b_3A_2}\right)x_4 + \left(\frac{b_3C_2' + C_3}{d_3 - b_3A_2}\right) \qquad (8)$$

Letting

$$A_3 = \frac{a_3}{d_3 - b_3A_2} \quad \text{and} \quad C_3' = \frac{b_3C_2' + C_3}{d_3 - b_3A_2}$$

Equation (8) can be rewritten as

$$x_3 = A_3x_4 + C_3' \qquad (9)$$

Equation (9) can be used for eliminating x_3 from (4) and the procedure can be repeated up to the last equation of the set. This completes the forward elimination process. For the back substitution we use general form of recurrence relationship

$$x_i = A_jx_{j+1} + C_i' \qquad (10)$$

Where $$A_i = \frac{a_i}{d_i - b_iA_{i-1}} \qquad (11)$$

$$C_i' = \frac{b_iC_{i-1}' + C_i}{d_i - b_iA_{i-1}} \qquad (12)$$

The formulae can be made to apply at the boundary points i=1 and i=n+1 by setting the following values for A and C':

$$A_1 = 0\ ;\ C_1' = x_1$$

$$A_{n+1} = 0\ ;\ C_{n+1}' = x_{n+1}$$

In order to solve a system of equations it is first arranged in the form of equation (1) and a_i, b_i, d_i and C_i is identified. The values of A_i and C_i' are subsequently calculated starting at i=2 and going up to i=n using (11) and (12).Since the value of x is known at boundary location (n+1)the values for x_i can be obtained in reverse order $x_n, x_{n-1}, x_{n-2}, \ldots\ldots x_2$ by means of recurrence formula (10). This method can be applied iteratively to solve a system of equations obtained from a two dimensional problem. Consider a two dimensional problem with the domain as shown in Figure 1.

Let the discretized equation be given by $a_Px_P = a_Wx_W + a_Ex_E + a_Sx_S + a_Nx_N + S_u$

We choose north-south lines to solve the system of equations and rearrange the equations into the form

$$-a_S x_S + a_P x_P - a_N x_N = a_W x_W + a_E x_E + S_u \quad (13)$$

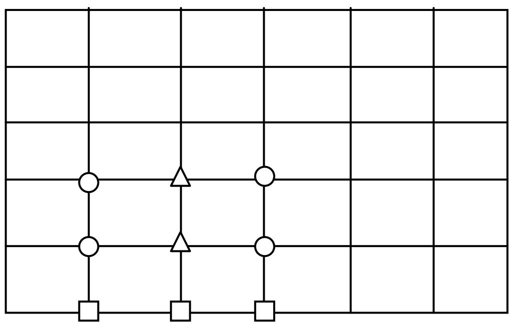


Figure- 1: Line by line application of TDMA

We assume the right hand side of Eqn. (13) to be known temporarily. Now Eqn. (13) is same as Eqn. (1).We can solve along north-south lines for all values of i = 2, 3, 4, …n as shown in Figure- 1. Next, we move the calculation to next north-south line. This sequence in which the lines are moved is called the sweep direction. If we sweep from west to east direction, the values of x_W to the west of a point P are known from the calculation on the previous line. The values of x_E to its east are unknown so the solution process must be iterative. At each iteration cycle, x_E is taken to have its values at the end of the previous iteration or a given initial value at the first iteration. The line by line calculation procedure is repeated several times until we obtain a converged solution.

The above technique is applied on two dimensional heat flow in a body of constant thermal conductivity.

III. MATHEMATICAL FORMULATION

We consider a two dimensional diamond strip of length 10 cm and breadth 1m as shown in Figure-2. The strip has thickness of 1 cm. The strip receives a steady heat flux of 1000 kW/m^2 along one of its breadth and the consecutive length is maintained at a constant temperature of 300 K. The two remaining sides of the strip are insulated.

The two dimensional steady state heat transfer in the plate is governed by

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=0 \quad (14)$$

where k is thermal conductivity of the plate. Integrating the Eqn. (14),we get

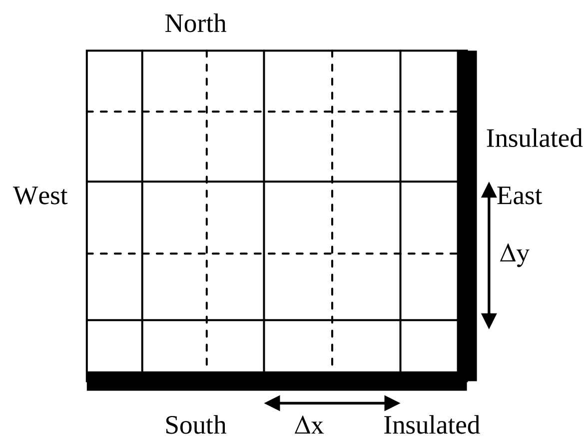


Figure- 2: Two dimensional domain

$$\int_{\Delta V}\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx.dy+\int_{\Delta V}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)dx.dy=0$$

Since $A_e = A_w = \Delta y$ and $A_n = A_s = \Delta x$,we obtain

$$\left[k_e A_e\left(\frac{\partial T}{\partial x}\right)_e - k_w A_w\left(\frac{\partial T}{\partial x}\right)_w\right]+\left[k_n A_n\left(\frac{\partial T}{\partial x}\right)_n - k_s A_s\left(\frac{\partial T}{\partial x}\right)_s\right]=0 \quad (15)$$

Using approximations for the flux through control volume faces, we write

$$k_w A_w\left(\frac{\partial T}{\partial x}\right)_w = k_w A_w\frac{(T_P - T_W)}{\delta x_{WP}},$$

$$k_e A_e\left(\frac{\partial T}{\partial x}\right)_e = k_e A_e\frac{(T_E - T_P)}{\delta x_{PE}}$$

$$k_s A_s\left(\frac{\partial T}{\partial y}\right)_s = k_s A_s\frac{(T_P - T_S)}{\delta y_{SP}},$$

$$k_n A_n\left(\frac{\partial T}{\partial y}\right)_n = k_n A_n\frac{(T_N - T_P)}{\delta y_{PN}}$$

Substituting he above expressions into Eqn. (15), we obtain

$$k_e A_e \frac{(T_E - T_P)}{\delta x_{PE}} - k_w A_w \frac{(T_P - T_W)}{\delta x_{WP}} + k_n A_n \frac{(T_N - T_P)}{\delta y_{PN}} - k_s A_s \frac{(T_P - T_S)}{\delta y_{SP}} = 0$$

This equation can be rearranged as

$$\left(\frac{k_w A_w}{\delta x_{WP}} + \frac{k_e A_e}{\delta x_{PE}} + \frac{k_s A_s}{\delta y_{SP}} + \frac{k_n A_n}{\delta y_{PN}} \right) T_P = \left(\frac{k_w A_w}{\delta x_{WP}} \right) T_W + \left(\frac{k_e A_e}{\delta x_{PE}} \right) T_E + \left(\frac{k_s A_s}{\delta y_{SP}} \right) T_S + \left(\frac{k_n A_n}{\delta y_{PN}} \right) T_N$$

which can be written in discretized form as

$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N \qquad (16)$$

Where

$$a_W = \frac{k_w A_w}{\delta x_{WP}},\ a_E = \frac{k_e A_e}{\delta x_{PE}},\ a_S = \frac{k_s A_s}{\delta y_{SP}},\ a_N = \frac{k_n A_n}{\delta y_{PN}}$$

and $a_P = a_W + a_E + a_S + a_N$ (17)

The temperature distribution in the strip can be obtained by writing discretized equation of the form (16) at each grid node of the sub-divided domain.

IV. NUMERICAL CALCULATIONS

To study the temperature distribution in the strip, the domain is divided uniformly into 1000 grid nodes by taking $\delta x = \delta y = 0.01$ m. The thermal conductivity of diamond is taken as k = 1000W/m. K. Since, the cross sectional area is same everywhere and $\delta x = \delta y$, therefore, the values of all neighbor coefficients are equal.

$$a_W = a_E = a_S = a_N = \frac{1000}{0.01} \times (0.01 \times 0.01) = 10$$

At the boundary nodes the discretized equations takes the form

$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N + S_u \qquad (18)$$

Where

$$a_P = a_W + a_E + a_S + a_N - S_p \qquad (19)$$

The boundary conditions are taken together with the discretized equations by setting relevant coefficients to zero and including source term through S_p and S_u.. The algebraic equations are obtained at each of 1000 node points using Eqns. (16) and (18).These equations constitutes a system of linear equations. To solve this system we use TDMA technique for two dimensional domain already described under Section II. The same algorithm has been implemented in MATLAB programming language. The converged solution is obtained only after 14,000 iterations. The temperature distribution is obtained at all grid nodes and is plotted in Figure- 3. These equations are also solved by using Point Iterative Gauss Seidal (GS) method and the solution obtained is plotted in Figure-4. It takes about 24,000 iterations to obtain a convergent solution using Gauss Seidal method.

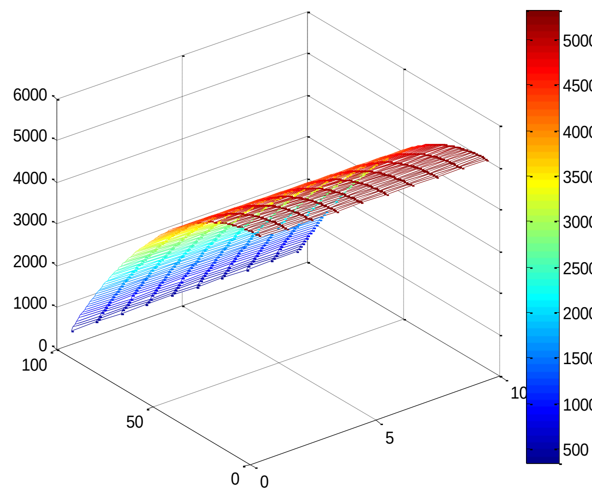


Figure-3: Temperature distribution obtained using TDMA technique

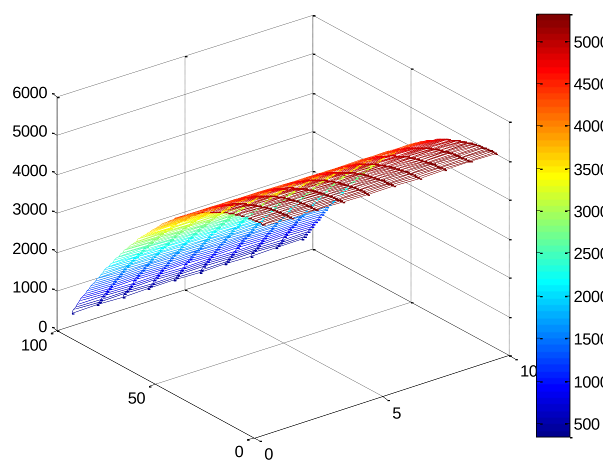
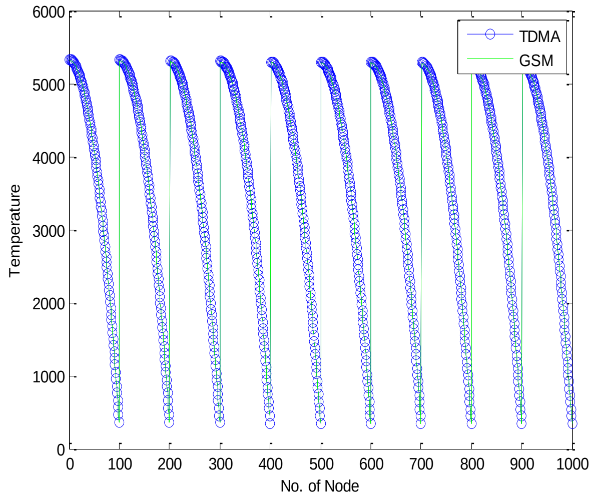


Figure- 4. Temperature distribution obtained using GS method

Figure-5(a) and Figure- 5(b) shows the comparison of the solution obtained by both the methods for 50 nodes and 1000 nodes, respectively for the same two dimensional domain. It can be observed that the difference in values of the temperature obtained by using TDMA technique and Gauss Seidal method in both cases is negligible.



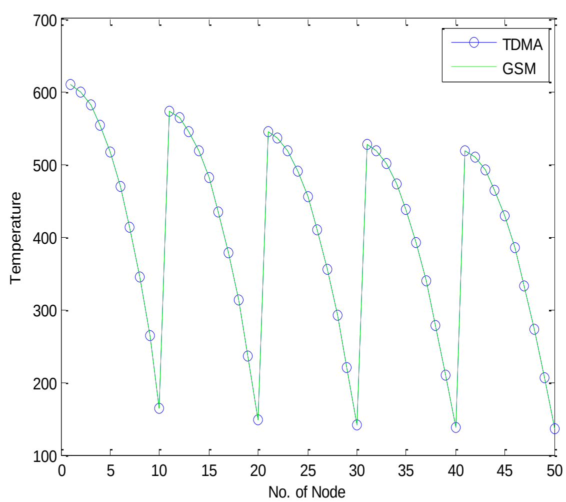


Figure. 5(a) & (b) Comparison of the solution obtained by both the methods for 50 nodes and 1000 nodes

Figure-6 and Figure-7 show the temperature variation near the insulated side parallel to Y axis and X axis respectively. The temperature decreases sharply along the insulated boundary parallel to Y axis, the direction of flow of heat whereas temperature remains almost invariable along the insulated boundary parallel to X axis, indicating no heat transfer taking place towards X-axis. The results are in accordance with the concept of thermodynamics.

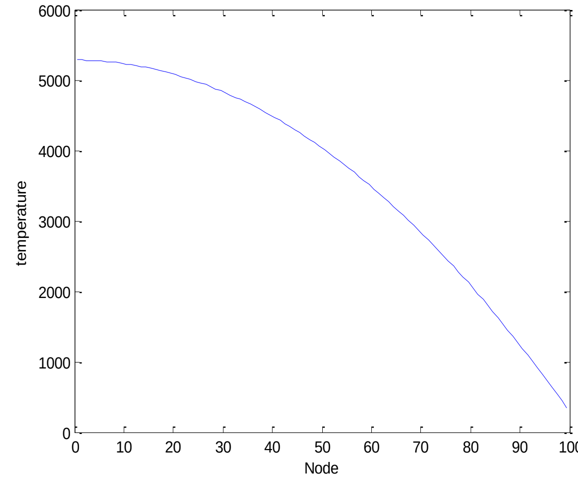


Figure-6: Temperature variation near the insulated side parallel to Y axis

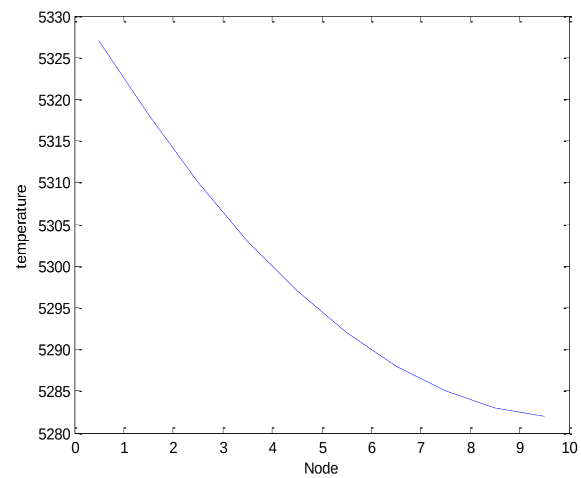


Figure-7: Temperature variation near the insulated side parallel to X axis

V. CONCLUSIONS

TDMA technique is known to be an efficient direct method for the solution of discretization equation for one dimensional situation. In the present study, TDMA technique has been applied for two dimensional heat equation on a body of constant thermal conductivity. The discretized system of linear equations for the problem was obtained using finite volume method. The temperature distribution was obtained at all grid points in conformity with the thermodynamical considerations. The results were compared with solutions obtained using Gauss Seidal method. It is concluded that the convergence in case of TDMA is much faster than Gauss Seidal method. Though Gauss Seidal is simple and powerful method but convergence is obtained only after satisfaction of

Scarborough criteria. The rate of convergence in Gauss Seidal method is slow when large number of grid points is involved. The TDMA technique has a fast convergence once boundary-conditions information is transmitted, whereas Gauss Seidal method transmits boundary –condition information at an interval of one grid per iteration. The comparison of results between TDMA and Gauss Seidal techniques shows a very good agreement .The difference is just negligible. Thus, TDMA technique is efficient, reliable, accurate and easier to implement in Microsoft excel as compared to other costly techniques.

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