International Journal of Scientific Research in Mathematical and Statistical Sciences

Research Paper

Volume-4, Issue-1, pp.1-5, February (2017)

E-ISSN: 2348-4519

Shrinkage Estimator of the Parameters of Normal Distribution Based On K- Record Values

Vishal Mehta

Indian Statistical Institute, North- East Centre, Tezpur-784028, Assam, India
Email**:** visdewas@gmail.com.

Available online at: www.isroset.org

Received 22nd Nov 2016, Revised 10th Dec 2016, Accepted 15th Jan 2017, Online 28th Feb 2017

***Abstract*—In this paper, we discusses the problem of estimating the parameters of normal distribution based on k-record values in presence of guessed value (or apriori) of the parameters under investigation. We have suggested shrinkage estimators for estimating the parameters μ and σ based on best linear unbiased estimator (BLUE). The expressions of biases and mean squared errors (MSEs) of the suggested estimators are obtained. Under some realistic conditions it is shown that the proposed shrinkage estimators are better than [15] estimators.**

AMS Subject Classification: 62G30; 62H12.

Keywords- Best linear unbiased estimation; K- record values; Minimum mean square estimator; Normal distribution; Shrinkage estimator

I. Introduction

Let $X_1, X_2, ...$ be an infinite sequence of independent and identically distributed random variables having the same absolutely continuous cumulative distribution function (cdf) $F(x)$. An observation X_j will be called an upper record (or simply a record) if its value exceeds that of all previous observations. Thus X_j is a record if $X_j > X_i$ for every $i < j$. An analogous definition deals with lower record values. For a positive integer k, the upper K- record times $T_{n(k)}$ and the upper k- record value $R_{n(k)}$ are introduced by [24] as

$T_{0(k)} = k$, with probability one

and, for $n \geq 1$

$$T_{n(k)} = \min\left\{j : j > T_{n-1(k)}, X_j > X_{T_{n-1(k)}-k+1:T_{n-1(k)}}\right\}, \quad (1.1)$$

where $X_{i:m}$ denote the *i*-th order statistics of sample size m. The sequence of upper k- records are then defined by

$$R_{n(k)} = X_{T_{n(k)}-k+1:T_{n-1(k)}} \text{ for } n = 0,1,..., k \geq 1. \quad (1.2)$$

In a similar manner, we can define the *k*th lower record times and the *k*th lower record values. For $k = 1$, the usual records are recovered. The probability density function (pdf) of $R_{n(k)}$ is given by see [1],

$$f_{n(k)}(x) = \frac{k^{n+1}}{n!}\left[-\log\{1 - F(x)\}\right]^n \left[1 - F(x)\right]^{k-1} f(x),$$
$$-\infty < x < \infty, \quad (1.3)$$

and the joint pdf of *m*th and *n*th k- record values for $m < n$ is given by,

$$f_{m,n(k)}(x, y) = \frac{k^{n+1}}{m!(n-m-1)!}\left[-\log\{1 - F(x)\}\right]^m$$
$$\times\left[-\log\{1 - F(y)\} + \log\{1 - F(x)\}\right]^{n-m-1}$$
$$\times\frac{\left[1 - F(y)\right]^{k-1}}{1 - F(x)} f(x) f(y), \quad x < y. \quad (1.4)$$

Let $R^*_{0(k)}, R^*_{1(k)}, ..., R^*_{n(k)}$ be the first $(n+1)$ upper k-record values arising from a sequence of iid standard normal random variables. If we denote $\Phi(.)$ as the cdf and $\varphi(.)$ as the pdf of a standard normal random variable, then by using (1.3), the pdf of nth upper k-record value of $R^*_{n(k)}$ is given See, [15]

*Corresponding Author: Vishal Mehta, visdewas@gmail.com,
Indian Statistical Institute, North-East Centre, Tezpur-784028, Assam, India.

$$f^*_{n(k)}(x)=\frac{k^{n+1}}{n!}\left[-\log(1-\Phi(x))\right]^n\left[1-\Phi(x)\right]^{k-1}\varphi(x),$$
$$-\infty<x<\infty, n\ge 0. \tag{1.5}$$

From (1.4), the joint pdf of *m*th and *n*th upper k- record values, $R^*_{m(k)}$ and $R^*_{n(k)}$ for $m<n$ is given by

$$f^*_{m,n(k)}(x,y)=\frac{k^{n+1}}{m!(n-m-1)!}\left[-\log\{1-\Phi(x)\}\right]^m$$
$$\times\left[-\log\{1-\Phi(y)\}+\log\{1-\Phi(x)\}\right]^{n-m-1}$$
$$\times\frac{[1-\Phi(y)]^{k-1}}{1-\Phi(x)}\varphi(x)\varphi(y),\ \infty<x<y<\infty. \tag{1.6}$$

We used same notations of [15] as $E\left(R^*_{n(k)}\right)$ by $\alpha_{n(k)}$, $Var\left(R^*_{n(k)}\right)$ by $\beta_{n,n(k)}$, $E\left(R^*_{m(k)},R^*_{n(k)}\right)$ by $\alpha_{m,n(k)}$ and $Cov\left(R^*_{m(k)},R^*_{n(k)}\right)$ by $\beta_{m,n(k)}$. Then, the single moments of the *nth* upper k-record value for $n\ge 0$ is given by

$$\alpha_{n(k)}=\int_{-\infty}^{\infty}xf^*_{n(k)}(x)dx,$$

and the product moments of the *mth* and *nth* upper k-record values for $m<n$ is given by

$$\alpha_{m,n(k)}=\int_{-\infty}^{\infty}\int_{x}^{\infty}xyf^*_{m,n(k)}(x,y)dydx.$$

By using above notations [15] obtained BLUEs of mean μ and standard deviation σ respectively as

$$\mu^*=\frac{\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{\alpha}\boldsymbol{1}^T\boldsymbol{B}^{-1}-\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{1}\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}}{\left(\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{\alpha}\right)\left(\boldsymbol{1}^T\boldsymbol{B}^{-1}\boldsymbol{1}\right)-\left(\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{1}\right)^2}\boldsymbol{R}_{n(k)}=\sum_{i=0}^{n}a_iR_{i(k)} \tag{1.7}$$

and

$$\sigma^*=\frac{\boldsymbol{1}^T\boldsymbol{B}^{-1}\boldsymbol{1}\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}-\boldsymbol{1}^T\boldsymbol{B}^{-1}\boldsymbol{\alpha}\boldsymbol{1}^T\boldsymbol{B}^{-1}}{\left(\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{\alpha}\right)\left(\boldsymbol{1}^T\boldsymbol{B}^{-1}1\right)-\left(\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{1}\right)^2}\boldsymbol{R}_{n(k)}=\sum_{i=0}^{n}b_iR_{i(k)}. \tag{1.8}$$

Furthermore, the variance and covariance of the above estimators are respectively given by (see, [16], pp. 80-81)

$$Var\left(\mu^*\right)=\sigma^2\frac{\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{\alpha}}{\left(\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{\alpha}\right)\left(\boldsymbol{1}^T\boldsymbol{B}^{-1}\boldsymbol{1}\right)-\left(\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{1}\right)^2}=\sigma^2A_1(say) \tag{1.9}$$

$$Var\left(\sigma^*\right)=\sigma^2\frac{\boldsymbol{1}^T\boldsymbol{B}^{-1}\boldsymbol{1}}{\left(\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{\alpha}\right)\left(\boldsymbol{1}^T\boldsymbol{B}^{-1}\boldsymbol{1}\right)-\left(\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{1}\right)^2}=\sigma^2A_2(say) \tag{1.10}$$

and

$$Cov\left(\mu^*,\sigma^*\right)=-\sigma^2\frac{\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{1}}{\left(\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{\alpha}\right)\left(\boldsymbol{1}^T\boldsymbol{B}^{-1}\boldsymbol{1}\right)-\left(\boldsymbol{\alpha}^T\boldsymbol{B}^{-1}\boldsymbol{1}\right)^2}=-\sigma^2A_3(say) \tag{1.11}$$

There are some situations in which only records are observed, such as in destructive stress testing, meteorological analysis, hydrology, seismology, sporting and athletic events and oil and mining surveys. For a more specific example, consider the situation of testing the breaking strength of wooden beams as described in [17]. Interest in records has increased steadily over the years since [13] formulation. Useful reviews of literature are given in the books of [1], [18], [14] and the references cited therein. For current reference in this context the reader is referred to [2]-[9] and [19]-[23].

The prior point information (or guessed values) regarding parameters μ and σ can be obtained from the past data or experience gathered in due course of time for instance see, [2], [11] and [12]. The principle objective of this paper is to use the prior information μ_0 and σ_0 of the parameters μ and σ respectively in addition to sample observations in order to get better estimators than the usual estimators available in the literature.

The organization of the remaining part of the paper is as follows. In section 2, we have derived some improved shrinkage estimators of parameters μ of normal distribution based on BLUE while in section 3, some improved shrinkage estimators of σ are obtained based on BLUE using K-record values in presence guessed values μ_0 (for μ) and σ_0 (for σ). We have also obtained the biases and mean squared errors (MSEs) of the suggested estimators. Theoretically, it has been shown that the proposed shrinkage estimators are always superior to the one suggested by [15]. Section 4, concludes the paper with final remarks.

II. SHRINKAGE ESTIMATOR FOR MEAN μ

When the prior point estimate μ_0 of μ is available, we suggest the following class of estimators for the parameter μ as

$$\hat{\mu}_1 = P_1\mu_0 + (1-P_1)\mu^*, \tag{2.1}$$

where P_1 being a suitably chosen constant.

The bias and MSE of $\hat{\mu}_1$ are respectively, given by

$$B(\hat{\mu}_1) = \sigma\left(\frac{\phi P_1}{C}\right), \tag{2.2}$$

and

$$MSE(\hat{\mu}_1) = \sigma^2\left[P_1^2\left(A_1 + \frac{\phi^2}{C^2}\right) - 2P_1A_1 + A_1\right], \tag{2.3}$$

where $C = \frac{\sigma}{\mu}, \phi = \left(\frac{\mu_0}{\mu} - 1\right) = (\beta - 1)\ (say)$.

A. *Efficiency Comparison*

It is observed from (1.9) and (2.3) that

$$MSE(\hat{\mu}_1) < Var(\mu^*)$$

if

$$P_1 \in \left(0, \left(\frac{2A_1C^2}{A_1C^2 + \phi^2}\right)\right). \tag{2.4}$$

It is known that the optimum estimator (OE) of μ is given as

$$\hat{\mu}^*_{1(opt.)} = \mu^* + \frac{A_3}{(1+A_2)}\sigma^*, \tag{2.5}$$

in the class of estimators $\hat{\mu}_1^* = \mu^* + P\sigma^*$, P being a suitably chosen constant such that mean squared error (MSE) of $\hat{\mu}^*_{1(opt.)}$ is minimum.

The bias and MSE of $\hat{\mu}^*_{1(opt.)}$ are, respectively given by

$$B\left(\hat{\mu}^*_{1(opt.)}\right) = \sigma\frac{A_3}{(1+A_2)}, \tag{2.6}$$

$$MSE\left(\hat{\mu}^*_{1(opt.)}\right) = \sigma^2\left[A_1 - \frac{A_3^2}{(1+A_2)}\right]. \tag{2.7}$$

From (2.3) and (2.7) it follows that the suggested shrinkage estimator $\hat{\mu}_1$ is more efficient than the OE $\hat{\mu}^*_{1(opt.)}$ if

$$MSE(\hat{\mu}_1) < MSE\left(\mu^*_{1(opt.)}\right)$$

if

$$P_1 \in \left(0, \left(\frac{(2A_1 + 2A_1A_2 - A_3^2)C^2}{(A_1C^2 + \phi^2)(1+A_2)}\right)\right). \tag{2.8}$$

The proposed shrinkage estimator $\hat{\mu}_1$ is better than the BLUE μ^* and the OE $\hat{\mu}^*_{1(opt.)}$ if the conditions (2.4) and (2.8) are respectively satisfied. From (2.4) and (2.8) one can also calculate the ranges P_1 in which the suggested shrinkage estimator $\hat{\mu}_1$ is more efficient than the [15] BLUE μ^* and OE $\hat{\mu}^*_{1(opt.)}$.

III. SUGGESTED ESTIMATORS FOR σ

Based on prior estimate σ_0 of σ and the BLUE σ^*, we define a class of estimators of σ as

$$\hat{\sigma}_1 = \sigma_0 + P_2\sigma^*, \tag{3.1}$$

where P_2 is a constant to be determined such that the MSE of $\hat{\sigma}_1$ is minimum.

The bias and MSE of $\hat{\sigma}_1$ are respectively given by

$$B(\hat{\sigma}_1) = \sigma(\psi + P_2), \tag{3.2}$$

$$MSE(\hat{\sigma}_1) = \sigma^2\left[\psi^2 + P_2^2(1+A_2) + 2\psi P_2\right], \tag{3.3}$$

where

$$\psi = \left(\frac{\sigma_0}{\sigma} - 1\right) = (\beta^* - 1)\ (say).$$

The $MSE(\hat{\sigma}_1)$ at (3.3) is minimized for

$$P_2 = -\frac{\psi}{(1+A_2)}. \tag{3.4}$$

The value of P_2 at (3.4) depends on the unknown parameter σ, so an estimate of P_2 based on sample data is given by

$$P_2^* = -\frac{\psi^*}{(1+A_2)} = -\left(\frac{\sigma_0 - \sigma^*}{\sigma^*(1+A_2)}\right). \qquad (3.5)$$

Putting P_2^* in (3.1), we get a shrinkage estimator of σ as

$$\sigma_1^* = \sigma_0 - (1+A_2)^{-1}(\sigma_0 - \sigma^*). \qquad (3.6)$$

The bias and MSE of σ_1^* are respectively given by

$$B(\sigma_1^*) = \sigma \frac{\psi A_2}{1+A_2}, \qquad (3.7)$$

$$MSE(\sigma_1^*) = \sigma^2 \frac{A_2(\psi^2 A_2 + 1)}{(1+A_2)^2}. \qquad (3.8)$$

It is observed from (1.10) and (3.8) that

$$MSE(\sigma_1^*) < Var(\sigma^*)$$

if

$$\sigma \in \left(\frac{\sigma_0}{1+\sqrt{(2+A_2)}}, \infty\right). \qquad (3.9)$$

or

$$\sigma_0 \in \left(0, \sigma\left(1+\sqrt{(2+A_2)}\right)\right). \qquad (3.10)$$

It is interesting to note from (3.9) that the proposed shrinkage estimator σ_1^* is more efficient than the BLUE σ^* envisaged by [15] for a wider range of σ. It also suggests that the proposed shrinkage estimator σ_1^* is more efficient than the BLUE σ^* due to [15] even if the guessed value σ_0 of σ slides far away from the true value σ.

IV. CONCLUSION

In this paper we have suggested better estimators of parameters μ and σ of normal distribution based on k-record values in presence of guessed (or apriori) information μ_0 (for μ) and σ_0 (for σ). The expressions of biases and MSEs of the estimators $\hat{\mu}_1$ and σ_1^* of the parameters μ and σ respectively are obtained. Superiority of the proposed shrinkage estimators $(\hat{\mu}_1, \sigma_1^*)$ of the parameters (μ, σ) respectively are also comprehensively discussed.

Thus the use of prior information regarding the parameters (μ, σ) under investigation may be highly rewarding in terms of precision of the proposed shrinkage estimators $(\hat{\mu}_1, \sigma_1^*)$ over the usual estimators (μ^*, σ^*) envisaged by [15] of the parameters (μ, σ) respectively.

ACKNOWLEDGMENT

The authors are highly grateful to the referees for their constructive comments/ suggestions that helped in the improvement of the revised version of the paper.

REFERENCES

[1] B. C. Arnold, N. Balakrishnan and H. N. Nagaraja, “Records”, Wiley, New York-**1998**, ISBN: 978-0-471-08108-1.

[2] B. N. Pandey and B. P. Singh, “On Estimation of Population Variance in Normal Distribution”, Journal of Scientific Research, Volum-**27**, Page (**221-225**), **1977**.

[3] Housila P. Singh and Vishal Mehta, “An Improved Estimation of Parameters of Morgenstern Type Bivariate Logistic Distribution Using Ranked Set Sampling”, STATISTICA, Volum-**73**, Issue-**4**, Page No (**437-461**), **2013**.

[4] Housila P. Singh and Vishal Mehta, “An Alternative Estimation of the Scale Parameter for Morgenstern Type Bivariate Log-Logistic Distribution Using Ranked Set Sampling”, Journal of Reliability and Statistical Studies, Volum-**7**, Issue-**1**, Page No (**19-29**), **2014**.

[5] Housila P. Singh and Vishal Mehta, “Linear shrinkage estimator of scale parameter of Morgenstern type bivariate logistic distribution using ranked set sampling”, Model Assisted Statistics and Applications (MASA), Volum-**9**, Page No (**295-307**), **2014**.

[6] Housila P. Singh and Vishal Mehta, “Estimation of Scale Parameter of a Morgenstern Type Bivariate Uniform Distribution Using Censored Ranked Set Samples”, Model Assisted Statistics and Applications (MASA), Volum-**10**, Page No (**139-153**), **2015**.

[7] Housula P. Singh and Vishal Mehta, ”Some Classes of Shrinkage Estimators in the Morgenstern Type Bivariate Exponential Distribution Using Ranked Set Sampling”, Hacettepe Journal of Mathematics and Statistics, Volum-**45**, Issue-**2**, Page No (**575-591**), **2016**.

[8] Housila P. Singh and Vishal Mehta, “Improved Estimation of Scale Parameter of Morgenstern Type Bivariate Uniform Distribution Using Ranked Set Sampling”, Communications in Statistics- Theory and Methods, Volum-**45**, Issue-**5** Page No (**1466-1476**), **2016**.

[9] Housila P. Singh and Vishal Mehta, “A Class of Shrinkage Estimators of Scale Parameter of Uniform Distribution Based on K- Record Values”, National Academy Science Letters, Volume-**39**, Issue-**3**, Page No (**221-227**), **2016**.

[10] Housula P. Singh and Vishal Mehta,”Improved estimation of the scale parameter for log-logistic distribution using balanced ranked set sampling”, STATISTICS IN TRANSITION-New Series (Accepted), **2016**.

[11] J. R. Thompson, “Some shrinkage technique for estimating the mean”, Journal of the American Statistical Association, Volum-**63**, Issue-**321**, Page No (**113-122**), **1968**.

[12] J. S. Mehta and R. Srinivasan, “Estimation of the mean by shrinkage to a point”, Journal of the American Statistical Association, Volume- **66**, Page No (**86-90**), **1971**.

[13] K. N. Chandler, "The distribution and frequency of record values", Journal of Royal Statistical Society, Series B, Volum-**14**, Page No (**220-228**), **1952**.

[14] M. Ahsanullah,"Record Values- Theory and applications", University press of America, Lanham, MaryIand, USA, Page No (**224-232**), **2004**, ISBN: 978-0-7618-2794-8.

[15] M. Chacko and M. Shy Mary,"Estimation and prediction based on K-record values from normal distribution", STATISTICA, Volum-**73**, Issue-**4**, Page No (**505-516**), **2013**.

[16] N. Balakrishnan and A. C. Cohen, "Order Statistics and Inference: Estimation Method", Academic Press, San Diego, **1991**, ISBN: 978-0120769483.

[17] N. Glick, "Breaking records and breaking boards", American Mathematical Monthly, Volume- **85**, Page No (**2-26**), **1978**.

[18] V. Nevzorov, "Records: Mathematical Theory. Translation of Mathematical Monographs", American Mathematical Society, 194, Providence, Rhode IsIand, **2001**, ISBN: 978-0-8218-1945-6.

[19] Vishal Mehta and Housila P. Singh, "Shrinkage Estimators of Parameters of Morgenstern Type Bivariate Logistic Distribution Using Ranked Set Sampling", Journal of Basic and Applied Engineering Research (JBAER), Volium-**1**, Issue-**13**, Page No. (**1-6**), **2014**.

[20] Vishal Mehta and Housila P. Singh, "Minimum Mean Square Error Estimation of Parameters in Bivariate Normal Distribution Using Concomitants of Record Values", Statistics and Informatics in Agricultural Research, Excel India Publishers, 91A, Ground Floor, Pratik Market, Munirka, New Delhi, India. Page No (**162-174**), **2015**, ISBN: 978-93-84869-98-4.

[21] Vishal Mehta and Housila P. Singh, "Search for Better Estimators Based on k-Record Values from Normal Distribution When Apriori Information is Known", Proceeding of the 13th National Conference on Recent Trends in Mathematical and Computational Sciences (NCRTMCS-2015), University Department of Statistics and Computer Applications, T. M. Bhagalpur University, Bhagalpur, Bihar, India., **2015**.

[22] Vishal Mehta, "Estimation in Morgenstern Type Bivariate Exponential Distribution with Known Coefficient of Variation by Ranked Set Sampling", Proceeding of the "30th M. P. Young Scientist Congress" (MPYSC-2015), M. P. Council of Science and Technology, Vigyan Bhawan, Nehru Nagar, Bhopal, Madhya Pradesh, India, **2015**.

[23] Vishal Mehta, "Minimum Mean Squared Error Estimators of a Parameter of Morgenstern Type Bivariate Exponential Distribution Using Ranked Set Sampling", Proceeding of the "31th M. P. Young Scientist Congress" (MPYSC-2016), M. P. Council of Science and Technology, Vigyan Bhawan, Nehru Nagar, Bhopal, Madhya Pradesh, India, **2016**.

[24] W. Dziubdziela and B. Kopocinski, "Limiting properties of the K-th record values", Zastosowania Matematyki, Volum-**15**, Page No (**187-190**), **1976**.

AUTHORS PROFILE

DR. VIHAL MEHTA, Visiting Scientist, Indian Statistical Institute (ISI), North-East Center, Tezpur, Assam. My research interests are in the areas of Statistics, Sampling Theory; Statistical Inference-Use of Prior information in estimation procedure.