International Journal of Scientific Research in Mathematical and Statistical Sciences
Research Paper Volume-2, Issue-2 E-ISSN: 2348-4519

Effect of Markoff's model on Economic design of X- bar control chart for independent observations

Manzoor A. Khanday[1*] and J. R. Singh[2]

[1*,2] School of Studies in Statistics,
Vikram University, Ujjain-456010, M.P. India,
manzoorstat@gmail.com; *jrsinghstat@gmail.com*

Available online at www.isroset.org

Received: 26/Sep/2015 Revised: 18/Oct/2015 Accepted: 12/Nov/2015 Published: 30/Dec/2015

Abstract – In this paper, an attempt is made to determine the effect of Markoff's model on the economic design of $\overline{X}$ - control chart for independent observation. We have calculated the optimum value of sample size n and sampling interval h. Tables for n and h are available for different values of k. The presence of positive and negative autocorrelation parameters seriously affects the optimum sample size and sampling intervals. Thus, there is a need for a procedure which enables us to deal with the observations which are independent, to design control charts accordingly continue our search for the assignable causes of variation and for the optimum parameters. It may be inferred that autocorrelation seriously affects the optimum value of the sample size and optimum sampling interval. It is necessary to point out that the observations from the population should be taken in to account while designing a control chart as the optimum values of the control chart parameters are affected by the measurement of the population.

Key Words: Economic Design Control Chart; Autocorrelation; Markoff's model.

Introduction

Statistical quality control (SQC) techniques aim at improving the quality of manufactured products at a reasonable low cost. The two important tools of SQC are control charts and sampling plans. Measured quality of manufactured product is always subject to ascertain amount of variation as a result of chance. Some stable "system of chance causes" is inherent in any particular scheme of production and inspection. Variations within this stable pattern are inevitable. However, the reasons for variations outside this stable pattern may be discovered and corrected. In industry today, the form in which applied statistics is most widely used is that of control charts. A control chart is statistical device principally used to differentiate between the causes of variation in quality. Thus we say Statistical Quality Control refers to the use of statistically based methods to monitor, control evaluate, analyze and improve process in production system. The economic design of control charts is used to determine various design parameters that minimize total economic costs. The effect of production lot size on the quality of the product may also be significant. If the production process shifts to an out-of-control state at the beginning of the production run, the entire lot will contain more defective items. Hence, it is wiser to reduce the production cycle to decrease the fraction of defective items and, to improve output quality. On the other hand, reduction of the production cycle may result in an increase in costs due to frequent setups. A balance must be maintained so that the total cost is minimized. The production of quality goods depends upon the operating condition of the machine tools; however, the performance of machine tools depends upon the maintenance policy. It is assumed that the cost of maintaining the equipment increases with age, therefore, an age replacement strategy is needed to minimize the total cost of the system, which will simultaneously improve quality of the product and maintenance policy. The earlier pioneering work related to this paper was conducted by Duncan's (1956, 1971), then it was summarized by Montgomery (1980, 1991), Vance (1983). Svoboda (1991), Ho and Case (1994), Goel *et al*. (1968), Knappenberger and Grandage (1969), and Gibra (1971), provided extensive reviews of the economic design of process control charts. The objective of the economic design of an $\overline{X}$ control chart is to determine the optimal design parameter values of the sample size *n,* the sampling (inspection) interval *h,* and the control limit coefficient k to minimize the expected cost per unit time of operation. Furthermore, Banerjee and Rahim (1987) treated the sample size n and the control limit coefficient k as constants. The questioning of interaction between quality and manufacturing operation has been addressed recently by Gershwin and Kim (2005), and Colledani (2008). Their studies are the first investigations of how quality considerations can modify the production control. The design of control charts involves the selection of three parameters: sampling size (n), control frequency (h), and control limits (L) in order to detect earlier tools and processes shifts (Montgomery (2004)). Vispute and Singh (2014) discuss the problem of optimum process mean setting of product with rework process under second order autocorrelation. Thus, Economic design of control charts is a method which aims at determining these parameters of a control chart in optimizing a cost function of the process monitored. A breakthrough has been the generalization of all these models by Lorenzen and Vance (1986), it is nowadays a reference in economic design, as it can be easily implemented and adapted.

Mathematical Model for the cost function

Duncan (1956) obtained an approximate function for the average net income per hour of using the control chart for mean of normal variables as:

$$I = V_0 - \frac{\eta MB + (\alpha T/h) + \eta W}{1+\eta B} - \frac{b+cn}{h} \qquad (1)$$

Duncan's cost model indicates
(i) the cost of an out-of –control conditions
(ii) the cost of false alarms,
(iii) the cost of finding an assignable cause
(iv) the cost of sampling inspection, evolution, and plotting.

Notations

V_0 = the average income per hour when process is in control and process average is μ.

V_1 = the average income per hour when process is not in control and process average is $\mu' = \mu + \delta\sigma$.

$M = V_0 - V_1$

η = the average number of times the assignable cause occur within an interval of time.

$B = a\,h + Cn + D$

$a = \frac{1}{P} - \frac{1}{2} + \frac{\eta h}{12}$,

h = Sampling interval in hours

Cn = the time required to take and inspect a sample of size n.

D = average time taken to find the assignable cause after a point plotted on the chart falls outside the control limits,

P = Probability of detecting an assignable cause when it exists.

$$P = \int_{-\infty}^{\mu - k\sigma/\sqrt{n}} g(\bar{x}/\mu')d\,\bar{x} + \int_{\mu+k\sigma/\sqrt{n}}^{\infty} g(\bar{x}/\mu')d\,\bar{x}$$

$\cong 1 - \Phi(k - \delta\sqrt{n})$ for $\delta > 0$

Where $g(\bar{x}/\mu')$ is the density function of $\bar{x}$ when the true mean μ and $\Phi(x)$ is the normal probability

α = probability of wrongly indicating the presence of assignable cause.

$$= \int_{\mu-k\sigma/\sqrt{n}}^{\mu+k\sigma/\sqrt{n}} g(\bar{x}/\mu)d\,\bar{x} = 2\Phi(-k) \qquad (2)$$

T = The cost per occasions of looking for an assignable cause when no assignable cause exists,

W = the average cost per occasion of finding the assignable cause when it exist,

b = per sample cost of sampling and plotting, that is independent of sample size,

And c= the cost per unit of measuring an item in a sample.

The average cost per hour involved for maintaining the control chart is $\frac{(b+cn)}{h}$. The average net income per hour of the process under the surveillance of the control chart for mean can be rewritten as,

$I = V_0 - L$

Where, $L = \frac{\eta MB + (\alpha T / h) + \eta W}{1+\eta B} + \frac{b+cn}{h}$ (3)

L Can now be treated as the per hour cost due to the surveillance of the process under the control chart. The probability density function for Markoffs model is represented by the first two terms of Edgewoth series and P and α' are determined from the sampling distribution of mean and are written as.

$$P = 1 - \Phi(\xi) \quad (4)$$

Where $\xi = (k - \delta\sqrt{n})$

Derivation for optimum value of sample size n and sampling interval h:

One can determine the optimum value of sample size n and sampling interval h either by maximizing the gain function I or by minimizing the cost function L with respect to n and h. and we get,

$$\frac{\partial L}{\partial n} = \frac{\left\{ \begin{array}{l} (1+\eta B)\left(\eta M \frac{\partial B}{\partial n} + \frac{T}{h}\frac{\partial \alpha}{\partial n}\right) \\ -\left(\eta MB + \frac{\alpha T}{h} + \eta W\right)\eta \frac{\partial B}{\partial n} \end{array} \right\}}{(1+\eta B)^2} + \frac{c}{h} = 0 \quad (5)$$

$$\frac{\partial L}{\partial h} = \frac{\left\{ \begin{array}{l} (1+\eta B)\left(\eta M \frac{\partial B}{\partial h} - \frac{\alpha T}{h^2}\right) \\ -\left(\eta MB + \frac{\alpha T}{h} + \eta W\right)\eta \frac{\partial B}{\partial n} \end{array} \right\}}{(1+\eta B)^2} - \left(\frac{b+cn}{h^2}\right) = 0 \quad (6)$$

Where,

$$\frac{\partial B}{\partial n} = \frac{-h}{P^2}\frac{\partial P}{\partial n} + c \quad (7)$$

$$\frac{\partial L}{\partial h} = \frac{1}{P} - \frac{1}{2} + \frac{\eta h}{6} \quad (8)$$

$$\frac{\partial \alpha}{\partial n} = 0 \quad (9)$$

$$\frac{\partial P}{\partial n} = \frac{\delta}{2\sqrt{n}}\phi(\xi), \quad (10)$$

The solutions of the equations (5) and (6) for n and h are

$$\begin{array}{l} \eta h\left(M - \eta MB - \frac{\alpha T}{h} - \eta W\right)\frac{\partial B}{\partial n} + \frac{T}{h}\frac{\partial \alpha}{\partial n} \\ + \eta B\left(\eta M \frac{\partial B}{\partial n} + \frac{T}{h}\frac{\partial \alpha}{\partial n}\right) + c(1+\eta B)^2 = 0 \end{array} \quad (11)$$

$$\begin{array}{l} \eta h^2\left(M - \eta MB - \frac{\alpha T}{h} - \eta W\right)\frac{\partial B}{\partial h} - \alpha T(1+\eta B) \\ + \eta^2 h^2 MB \frac{\partial B}{\partial n} - (b+cn)(1+\eta B)^2 = 0 \end{array} \quad (12)$$

By assuming η to be small and noting that the optimum h is roughly of order of $\frac{1}{\sqrt{\eta}}$, we neglect terms with ηB containing ηWc , $\frac{\alpha' T}{h}$ and the terms equating higher powers of η . The equations (2.7) and (2.8) are simplified and put in the following form

$$\frac{-\eta h^2 M}{P^2}\frac{\partial P}{\partial n} - \eta \alpha T + c = 0 \quad (13)$$

$$\eta M h^2\left(\frac{1}{P} - \frac{1}{2}\right) - (\alpha T + b + cn) = 0 \quad (14)$$

From the equation (14) we get

$$h = \left\{ \frac{\alpha T + b + cn}{\eta M\left(\frac{1}{P} - \frac{1}{2}\right)} \right\}^{\frac{1}{2}} \quad (15)$$

By eliminating h from the equation (13), we get,

$$-\frac{\alpha T + b + cn}{P^2\left(\frac{1}{P} - \frac{1}{2}\right)}.\frac{\partial P}{\partial n} - \eta \alpha T + c = 0 \quad (16)$$

The values of n for which the equation (16) satisfy yield us the required optimum value of sample size n. Substituting this value of n in equation (15), we find the optimum value of the sampling interval h.

Markoff's Model Description

Consider a Markoff's process given by the following model

$$X_t = \mu + \zeta_t , \qquad t = 1,2,\ldots n \tag{17}$$

where X_t is the response at time t, μ is a population mean and ξ_t can be expressed as

$$\xi_t = \alpha_1 \xi_{t-1} + \varepsilon_t \tag{18}$$

When the correlation is present in the data, we have for the distribution of the sample mean $\bar{x}$,

$$\left. \begin{array}{l} E(\bar{x}) = \mu \\ Var(\bar{x}) = \dfrac{\sigma^2}{n}\lambda(\alpha_1, n), \end{array} \right\}$$

Where, $\left[\dfrac{1+\alpha_1}{1-\alpha_1} - \dfrac{2\alpha_1}{n}\dfrac{(1-\alpha_1^n)}{(1-\alpha_1)^2}\right] = \lambda(\alpha_1, n) = g^2$,

$$\sigma^2 = \frac{\sigma_\varepsilon^2}{1-\alpha_1^2}$$

Therefore the probability density function under Markoff's model for independent case is

$$P = 1 - \Phi(\xi) \tag{19}$$

Where $\xi = \dfrac{(k - \delta\sqrt{n})}{g}$ (20)

$\alpha = 2\Phi(-\dfrac{k}{g})$, Under Markoff's model equation (5) and (6) will reduce in following form

$$\frac{\partial L}{\partial n} = \frac{\left\{ \begin{array}{l} (1+\eta B)\left(\eta M \dfrac{\partial B}{\partial n} + \dfrac{T}{h}\dfrac{\partial \alpha}{\partial n}\right) \\ -\left(\eta MB + \dfrac{\alpha T}{h} + \eta W\right)\eta \dfrac{\partial B}{\partial n} \end{array} \right\}}{(1+\eta B)^2} + \frac{c}{h} = 0 \tag{21}$$

$$\frac{\partial L}{\partial h} = \frac{\left\{ \begin{array}{l} (1+\eta B)\left(\eta M \dfrac{\partial B}{\partial h} - \dfrac{\alpha T}{h^2}\right) \\ -\left(\eta MB + \dfrac{\alpha T}{h} + \eta W\right)\eta \dfrac{\partial B}{\partial n} \end{array} \right\}}{(1+\eta B)^2} - \left(\frac{b+cn}{h^2}\right) = 0 \tag{22}$$

where,

$$\frac{\partial B}{\partial n} = \frac{-h}{P^2}\frac{\partial P}{\partial n} + c$$

$$\frac{\partial L}{\partial h} = \frac{1}{P} - \frac{1}{2} + \frac{\eta h}{6}$$

$$\frac{\partial \alpha}{\partial n} = 0$$

$$\frac{\partial P}{\partial n} = \frac{\delta}{g(2\sqrt{n})}\phi(\xi) , \tag{23}$$

Similarly by using the above procedure we get

$$h = \left\{ \frac{\alpha T + b + cn}{\eta M\left(\dfrac{1}{P} - \dfrac{1}{2}\right)} \right\}^{\frac{1}{2}} \tag{24}$$

and,

$-\dfrac{\alpha T + b + cn}{P^2\left(\dfrac{1}{P} - \dfrac{1}{2}\right)} \cdot \dfrac{\partial P}{\partial n} - \eta\alpha' T + c = 0$ (25) The values of n for which the equation (25) satisfy yield us the required optimum value of sample size *n*. Substituting this value of n in equation (24), we find the optimum value of the sampling interval *h*.

Numerical Illustration and Result

For the purpose of numerical illustration, we take k=3, 2.5, 2, 1.5, and 1, δ = 1.0, 1.5, and 2.0, η=0.01, M=100, W=25, T=50, c=0.05, D=2,b=0.5, we determine the optimum values of sample size n and sampling interval h which are presented in the table.

The sample size required to detect given shift increases with the increase in the value of α although the sampling interval is not much affected but has a dramatic effect, this is more marked for detecting small shifts in the process average. In most of the industrial situations, data follow normal distribution. We may be confronted with an industrial situation where the assumption of normality and error free measurements are achievable or desirable. Thus, there is a need for a procedure which enables us to deal with the data which are independent, to design control charts accordingly and to continue our search for the assignable causes of variation. It may be inferred that independent observations affects considerably the optimum value of the sample size and optimum sampling interval. The presence of positive and negative autocorrelation seriously affects the optimum sample size and sampling intervals. It is necessary to point out that the errors of the population should be taken in to account while designing a control chart as the optimum values of the control chart parameters are affected by the independent observations under Markoff's model.

Table : Optimum sample size (n) and sampling interval (h) under AR (1) model for independent case.

α=0	k= 3		k= 2.5		k= 2		k= 1.5		k= 1	
δ↓	**n**	**h**	**n**	**h**	**n**	**h**	**n**	**h**	**n**	**h**
1	23	2.3371	20	2.4134	19	3.0258	35	4.6216	29	6.2076
1.5	11	1.8026	9	1.9963	9	2.7064	16	4.1978	14	5.9551
2	6	1.5648	6	1.8157	6	2.5740	9	4.0314	8	5.8593
α=-0.5	k= 3		k= 2.5		k= 2		k= 1.5		k= 1	
δ↓	**n**	**h**	**n**	**h**	**n**	**h**	**n**	**h**	**n**	**h**
1	17	2.0714	13	1.8724	10	1.7088	8	1.8672	9	3.4353
1.5	8	1.5929	6	1.4823	5	1.4224	4	1.7244	5	3.4367
2	5	1.3837	4	1.3128	3	1.3125	2	1.7290	4	3.5540
α=0.5	k= 3		k= 2.5		k= 2		k= 1.5		k= 1	
δ↓	**n**	**h**	**n**	**h**	**n**	**h**	**n**	**h**	**n**	**h**
1	52	4.3259	90	5.7040	80	6.3380	82	7.4299	78	8.4832
1.5	23	3.4476	38	4.5090	36	5.4164	34	6.6671	36	7.9074
2	12	2.8790	23	3.9909	20	4.9195	19	6.3005	19	7.5832
α=0.8	k= 3		k= 2.5		k= 2		k= 1.5		k= 1	
δ↓	**n**	**h**	**n**	**h**	**n**	**h**	**n**	**h**	**n**	**h**
1	193	8.1392	249	9.2428	187	9.1151	247	10.4570	169	10.3237
1.5	76	6.0757	65	6.4149	68	7.2377	75	8.4807	66	9.1235
2	34	4.6382	38	5.4999	30	6.2699	33	7.6530	31	8.4982

References

[1] Banerjee, P.K. and Rahim, M.A., "The economic design of control charts, A renewal theory approach" *Engineering Optimization,* 12, 63-73. (**1987**).

[2] Colledani, M., "Integrated analysis of quality and production logistics performance in manufacturing lines, In: *Proceedings of IFAC world congress, 6–11 July Seoul, Korea, 8368, 1–7.* **(2008).**

[3] Duncan, A. J. "The economic design of *X*-bar chart used to maintain current control of a process", *Journal of the American Statistical Association,* 51, 228-242. (**1956**)

[4] Duncan, A. J., "The economic design of *X* charts when there is a multiplicity of assignable causes. *Journal of the American Statistical Association*, 66, 107- 121. (**1971**).

[5] Gershwin, S. and Kim, J., "Integrated quality and quantity modelling of a production line. OR *Spectrum*, 27 (2-3), 287–314. (**2005**).

[6] Gibra, I. N., "Economically optimal determination of the parameters of an $\overline{X}$ control chart", *Management Science*, 17, 635- 646. (**1971**).

[7] Goel, A. L., Jain, S. D. and Wu, S. M., "An algorithm for the determination of the economic design of *X* charts based on Duncan's model", *Journal of the American Statistical Association*, 63, 304-320. (**1968**).

[8] Ho, C. and Case, K. E., "Economic design of control charts: A literature review for 1981-1991" *Journal of Quality Technology*,26, 1- 78. (**1994**).

[9] Knappenberger, H. A. and Grandage, A. H. E., "Minimum cost quality control tests", *AIIE Transactions*, 1, 24-32. (**1969**).

[10] Lorenzen, T. J. Vance, L. C., "The economic design of control charts: a unified approach", *Technometrics,* 28(1), 3–10. (**1986**).

[11] Montgomery, D. C., "Introduction to statistical quality control", New York: *John Wiley & Sons,* (**2004**).

[12] Montgomery, D.C. "The economic design of control charts", A review and literature survey. *Journal of Quality Technology.* 12*,* 75–87. (**1980**).

[13] Montgomery, D.C. "Introduction to Statistical Process Control*,* 2nd edn." *John Wiley & Sons, New York,* (**1991**).

[14] Svoboda, L. "Economic design of control charts: a review and literature survey (1979-1989), in Statistical Process Control in Manufacturing, Keats, J.B. and Montgomery, D.C. (eds)", *Marcel Dekker, New York.* (**l991**).

[15] Vance, L. C. "A bibliography of statistical quality control chart techniques, 1970-1980", *Journal of Quality Technology, 15, 59- 62*. (**1983**).

[16] Vispute S. and Singh J. R. "Optimum Process Mean Setting for Product with Rework Process Under Second order autocorrelation" *International Journal of Scientific Research in Mathematical and Statistical Sciences*, 1 (3), 1-12, (**2014**).