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Inverse and Saturation Results for Modified Baskakov Operators in Simultaneous Approximation

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***Abstract*:** In this paper, we discuss mixed type i.e. summation-integral type operators having the Baskakov basis function in summation and integration both. Especially, we consider here the linear combination of modified Baskakov operators to get better order of approximation. We find central moments and some other basic results for these operators, and obtain the Inverse and Saturation results with better approximation.

***Keywords*:** Baskakov type operators; Linear combination; Steklov mean; Simultaneous approximation; Inverse Theorem; Saturation Theorem.

I. INTRODUCTION

In the year 1996, Gupta-Srivastava [3] and Sinha et al. [5] estimated the rate of convergence for these operators for $x \in [0,\infty)$

$$P_n(f,x) = (n-1)\sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} p_{n,v}(t)\, f(t)\,dt, \qquad \ldots.(1.1)$$

where $p_{n,v}(x) = \binom{n+v-1}{v} . \frac{x^v}{(1+x)^{n+v}}$ is Baskakov basis function.

These operators $P_n(f,x)$ are termed as modified Baskakov operators. To get better approximation in simultaneous approximation, we extend this study by taking linear combination $P_n(f,v,x)$ of the operators $P_{d_jn}(f,x)$ as described in Agrawal-Mohammad [1] such as

$$P_n(f,v,x) = \sum_{j=0}^{v} C(j,v) P_{d_jn}(f,x) \qquad \ldots (1.2)$$

where, we have

$$C(j,v) = \prod_{\substack{i=0,\\ i\neq j}}^{v} \frac{d_i}{d_i - d_j}, \qquad C(0,0) = 1$$

and $d_0, d_1, d_2, \ldots d_v$ are arbitrary but fixed distinct positive integers. Now, let $C_\gamma[0,\infty) = \{f \in C[0,\infty) : |f(t)| \leq Mt^\gamma;\ M > 0, \gamma > 0\}$ and the norm of 'f' on this space is defined by $\|f\|_\gamma = \max_{0 \leq t < \infty} |f(t)| t^{-\gamma}$. It can easily be verified that for $f \in C_\gamma[0,\infty)$, the operators (1.2) are well defined.

We take kernel of P_n as follows

$$K_n(x,t) = (n-1)\sum_{v=0}^{\infty} p_{n,v}(x) p_{n,v}(t).$$

Then we have our operators (1.1) with kernel

$$P_n(f,x) = \int_0^{\infty} K_n(x,t)\, f(t)\,dt, \qquad 0 \leq x < \infty.$$

It is clear that $P_n(1,x) = 1$. In this paper, we study simultaneous approximation properties for the operators $P_n(f,v,x)$ defined by (1.2) i.e. the approximation of derivative of function 'f' by the corresponding order derivatives of operators $P_n(f,v,x)$. The inverse theorem infers the nature of smoothness of functions from its order of approximation and the saturation theorem provides upper bounds on the possible approximation order. In this paper these results are proved for $P_n(f,v,x)$ in simultaneous approximation using the technique of Steklov Mean.

II. AUXILIARY RESULTS

In this section we obtain some basic results necessary to prove our main results.

Lemma1. If for $m = 0,1,2,\ldots$ the m^{th} order moment is defined as

$$T_{n,m}(x) = (n-1)\sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} p_{n,v}(t)\,(t-x)^m\,dt,$$

then $T_{n,0}(x) = 1$, $T_{n,1}(x) = \frac{1+2x}{n-2}$, $T_{n,2}(x) = \frac{2(n+3)(x^2+x)+2}{(n-2)(n-3)}$

and for $n > m + 2$, there is the recurrence relation

$$\begin{aligned}(n-m-2)T_{n,m+1}(x) &= x(1+x)\left[T'_{n,m}(x) + 2mT_{n,m-1}(x)\right] + (1+2x) \\ &\quad \times (m+1)T_{n,m}(x)\end{aligned}$$

for all $x \in [0,\infty)$. Further, order of $T_{n,m}(x)$ is of $O\left(n^{-[m+1]/2}\right)$, where $[\xi]$ is the greatest positive integer.

Proof of this lemma is left on forthcoming researchers.

Corollary 2.1: Let $\gamma, \delta > 0$. Moreover, let $x \in (0,\infty)$ be fixed. Then for every positive integer m, there exists a constant M_m independent of n such that

$$\int_{|t-x|\geq\delta} K_n(x,t)t^{\gamma}dt \leq M_m n^{-m}$$

Lemma2. If $0 < a < b < \infty$ then for $f \in C_{\gamma}[0,\infty)$ and $g \in C_0^{\infty}$ with $\sup g \subset (a,b)$, we have

$$\left|n^{v+1}\langle\left[P_{2n}^{(r)}(f,v,.) - P_n^{(r)}(f,v,.)\right]g(.)\rangle\right| \leq K\|f\|_{\gamma}$$

where K is a constant independent of f and n and $\langle h,g\rangle = \int_0^1 h(t)\,g(t)\,dt.$

Lemma3. There exists polynomial $Q_{i,j,r}(x)$, independent of n and v, s.t.

$$x^r(1+x)^rD^r\left[p_{n,v}(x)\right] = \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} n^i(v-nx)^jQ_{i,j,r}(x)p_{n,v}(x), \qquad D \equiv \frac{d}{dx}$$

Theorem 2.1: For $f \in C_{\gamma}[0,\infty)$, if $f^{(2v+r+2)}$ exists at a fixed point $x \in (0,\infty)$ then for n being sufficiently large and a certain polynomial $Q(j,v,r,x)$ in x of degree at the most j

$$\lim_{n\to\infty} n^{v+1}\left\{P_n^{(r)}(f,v,.) - f^{(r)}(x)\right\} = \sum_{j=1}^{2v+2} Q(j,v,r,x)f^{(j+r)}(x)$$

$$\lim_{n\to\infty} n^{v+1}\left\{P_n^{(r)}(f,v+1,.) - f^{(r)}(x)\right\} = 0.$$

The proof of this lemma can be seen in earlier studies.

Theorem 2.2: If for $f \in C_{\gamma}[0,\infty)$, $f^{(p+r)}, 1 \leq p \leq 2v+2$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$ then for sufficiently large n, we have

$$\left\|P_n^{(r)}(f,v,.) - f^{(r)}(x)\right\|_{C[a,b]} = max\left\{C_1n^{-(v+1)}, C_2\omega\left(f^{(p+r)}, n^{-1/2}\right)\right\}.$$

Here $C_1 = C_1(v,p,r), C_2 = C_2(v,p,r,f)$ and $\omega\left(f^{(p+r)}, n^{-1/2}\right)$ is the modulus of continuity on the interval $(a-\eta, b+\eta)$.

Its proof also can be seen in earlier studies for these operators.

III. MAIN RESULTS

In this section we show the inverse and saturation estimates as in [3] [4] for the linear combination of modified Baskakov operators in the theory of simultaneous approximation.

Inverse Theorem

Theorem 3.1: If $0 < \alpha < 2$ and $f \in C_{\gamma}[0,\infty)$, the following statements hold the assertion: $(i) \Longrightarrow (ii)$,

(i) If $f^{(r)}$ exists in the interval $[a_1, b_1]$ and

$$\left\|P_n^{(r)}(f,v,.) - f^{(r)}(x)\right\|_{C[a,b]} = O\left(n^{-\alpha(v+1)/2}\right),$$

(ii) $f^{(r)} \in Liz(\alpha, v+1, a_2, b_2)$,

where $[a_2, b_2] \subset (a_1, b_1)$.

Proof: Let us take $a', a", b', b"$ in such a way that $a_1 < a' < a" < a_2 < b_2 < b" < b' < b_1$. Also we suppose $g \in C_0^{\infty}$ with $\operatorname{supp} g \subset (a", b")$ and $g(x) = 1$ on $[a_2, b_2]$.

To prove the theorem, it is sufficient to show that the condition

$$\left\|P_n^{(r)}(fg,v,.) - (fg)^{(r)}(x)\right\|_{C[a,b]} = O\left(n^{-\alpha(v+1)/2}\right) \qquad \text{... (3.1)}$$

implies (ii). For it, we substitute $fg = \bar{f}$ for small values of m and get

$$\begin{aligned}&\left\|\Delta_m^{2v+r+2}\bar{f}\right\|_{C[a",b"]} \\ &= \left\|\Delta_m^{2v+r+2}\left(\bar{f} - P_n(\bar{f},v,.)\right)\right\|_{C[a",b"]} \\ &\quad + \left\|\Delta_m^{2v+r+2}P_n(\bar{f},v,.)\right\|_{C[a",b"]} \qquad \text{... (3.2)}\end{aligned}$$

By the definition of Δ_m^{2v+r+2}, we have

$$\begin{aligned}&\left\|\Delta_m^{2v+r+2}P_n(\bar{f},v,.)\right\|_{C[a",b"]} \\ &= \left\|\int_0^m\int_0^m \dots \int_0^m \Delta_m^{2v+r+2}P_n\left(\bar{f}, v, x + \sum_{i=1}^{2v+r+2} x_i\right) dx_1dx_2dx_{2v+r+2}\right\|_{C[a",b"]} \\ &\leq m^{2v+r+2}\left\|P_n^{2v+r+2}(\bar{f},v,.)\right\|_{C[a",b"+(2v+r+2)m]}\end{aligned}$$

Using linearity property, we have

$$\left\|\Delta_m^{2v+r+2}P_n(\bar{f},v,.)\right\|_{C[a",b"]} \leq m^{2v+r+2} \times$$

$$\left\{\begin{aligned}&\left\|P_n^{(2v+r+2)}\left(\bar{f} - \bar{f}_{\eta,2v+r+2}, v,.\right)\right\|_{C[a",b"+(2v+r+2)m]} \\ &+ \left\|P_n^{(2v+r+2)}\left(\bar{f}_{\eta,2v+r+2}, v,.\right)\right\|_{C[a",b"+(2v+r+2)m]}\end{aligned}\right\} \qquad \text{... (3.3)}$$

where $\bar{f}_{\eta,2v+r+2}$ is the Steklov mean of $O(2v+r+2)$ corresponding to $\bar{f}$.

Now, by using Lemma 3, we have

$$\int_0^\infty \left|\frac{\partial^{2v+r+2}}{\partial x^{2v+r+2}} K_n(x,t)\right| dt$$
$$\leq (n-1) \sum_{\substack{2i+j\geq 2v+r+2 \\ i\neq j}} \sum_{v=0}^{\infty} n^i |v-nx|^j \frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r} p_{n,v}(x) \int_0^\infty p_{n,v}(t)dt$$

Since $\int_0^\infty p_{n,v}(t)dt = (n-1)^{-1}$; therefore using Lemma 3, Corollary 2.1 and Schwaz inequality, we obtain

$$\left\|P_n^{(2v+r+2)}\left(\bar{f}-\bar{f}_{\eta,2v+r+2},v,.\right)\right\|_{C[a",b"+(2v+r+2)m]}$$
$$\leq M_1 n^{v+1}\left\|\bar{f}^{(r)} - \bar{f}^{(r)}_{\eta,2v+r+2}\right\|_{C[a",b"]}, \qquad \ldots(3.4)$$

where M_1 is a constant. Next by Taylor's expansion of $\bar{f}$, we have

$$\bar{f}^{(r)}_{\eta,2v+r+2}(t) = \sum_{i=1}^{2v+r+1} \frac{\bar{f}^{(i)}_{\eta,2v+r+2}(x)}{i!}(t-x)^i + \frac{\bar{f}^{(2v+r+2)}_{\eta,2v+r+2}(\eta)}{(2v+r+2)!}(t-x)^{2v+r+2},$$

where $x<\eta<t$. Using this expansion, we obtain

$$\left\|P_n^{(2v+r+2)}\left(\bar{f}_{\eta,2v+r+2},v,.\right)\right\|_{C[a",b"+(2v+r+2)m]}$$
$$\leq \sum_{j=0}^{v} \frac{C(j,v)}{(2v+r+2)!}\left\|\bar{f}^{(2v+r+2)}_{\eta,2v+r+2}\right\|_{C[a",b"]} \times \left\|\int_0^\infty \left[\frac{\partial^{2v+r+2}}{\partial x^{2v+r+2}} K_{d_j n}(x,t)\right](t-x)^{2v+r+2}\,dt\right\|_{C[a",b"]}$$

Since from Lemma 3, we have

$$\int_0^\infty \left[\frac{\partial^{2v+r+2}}{\partial x^{2v+r+2}} K_{d_j n}(x,t)\right](t-x)^{2v+r+2}\,dt$$
$$\leq (n-1) \sum_{\substack{2i+j\geq 2v+r+2 \\ i\neq j}} \sum_{v=0}^{\infty} n^i |v-nx|^j \frac{|Q_{i,j,2v+r+2}(x)|}{x^r(1+x)^{2v+r+2}} \times p_{n,v}(x)\int_0^\infty p_{n,v}(t)(t-x)^{2v+r+2}dt$$

so that

$$\left\|\int_0^\infty \left[\frac{\partial^{2v+r+2}}{\partial x^{2v+r+2}} K_{d_j n}(x,t)\right](t-x)^{2v+r+2}\,dt\right\|_{C[a",b"]} = O(1)$$

Hence

$$\left\|P_n^{(2v+r+2)}\left(\bar{f}_{\eta,2v+r+2},v,.\right)\right\|_{C[a",b"+(2v+r+2)m]} \leq M_2\left\|\bar{f}^{(2v+r+2)}_{\eta,2v+r+2}\right\|_{C[a",b"]} \quad \ldots(3.5)$$

Collecting the estimates from (3.2)-(3.5), we obtain

$$\left\|\Delta_m^{2v+r+2}\bar{f}\right\|_{C[a",b"]}$$
$$= \left\|\Delta_m^{2v+r+2}\left(\bar{f}-P_n\left(\bar{f},v,.\right)\right)\right\|_{C[a",b"]} + M_3 m^{2v+r+2}\left\{n^{v+1}\left\|\bar{f}^{(r)} - \bar{f}^{(r)}_{\eta,2v+r+2}\right\|_{C[a",b"]} + \left\|\bar{f}^{(r)}_{\eta,2v+r+2}\right\|_{C[a",b"]}\right\}$$

It holds for sufficiently small values of m, Therefore from (3.1) and Steklov
Mean, we have

$$\omega^{(r)}_{\eta,2v+r+2}\left(\bar{f},h,a",b"\right) \leq M_4\left\{n^{-\alpha(v+1)/2} + h^{2v+r+2}\left(n^{v+1} + \eta^{-(2v+r+2)}\right)\omega_{2v+r+2}\left(\bar{f},\eta,a",b"\right)\right\}$$

Here we choose η in such a way that $n<\eta^{-2}<2n$ and by the definition of Zygmund class $Liz(\alpha,v+1,a_2,b_2)$ of a function, we get

$$\omega^{(r)}_{\eta,2v+r+2}\left(\bar{f},h,a",b"\right) = O\left(h^{\alpha(v+1)}\right) \qquad \ldots(3.6)$$

ince $\bar{f}=fg$ for $[a_2,b_2]$, we have

$$(fg)^{(r)} \in Liz(\alpha,v+1,a_2,b_2)$$

.

Thus to prove our inverse theorem, we have shown the validity of (3.1) under the hypothesis (i).

Saturation Theorem

Theorem3.2: If $f\in C_\gamma[0,\infty)$ and $0<a_1<a_2<a_3<b_3<b_2<b_1<\infty$, the following statements then have implications- $(i)\Rightarrow(ii)\Longrightarrow(iii)$ and $(iv)\Rightarrow(v)\Longrightarrow(vi)$

(i) $f^{(r)}$ exists on $[a_1,b_1]$ and $n^{v+1}\left\|P_n^{(r)}(f,v,.)-f^{(r)}(.)\right\|_{C[a_1,b_1]} = O(1)$.

(ii) $f^{(2v+r+1)}\in A.C.[a_2,b_2]$ and $f^{(2v+r+2)}\in P_\infty[a_2,b_2]$.

(iii) $n^{v+1}\left\|P_n^{(r)}(f,v,.)-f^{(r)}(.)\right\|_{C[a_3,b_3]} = O(1)$.

(iv) $n^{v+1}\left\|P_n^{(r)}(f,v,.)-f^{(r)}\right\|_{C[a_1,b_1]} = o(1)$.

(v) $f^{(2v+r+2)}\in C[a_2,b_2]$ and for certain polynomial $Q(j,v,r,x)$ in x

$$\sum_{j=r}^{2v+r+2} Q(j,v,,r,x)f^{(j)}(x) = 0, \qquad x\in[a_2,b_2].$$

(vi) $n^{v+1}\left\|P_n^{(r)}(f,v,.)-f^{(r)}(.)\right\|_{C[a_3,b_3]} = o(1)$.

where O-o show orders with respect to n as $n\to\infty$.

Proof: Assuming (i), by the previous theorem, it is clear that $f^{(2v+r+1)}$ exists and is continuous on $[a_1,b_1]$ and moreover

$$\left\|P_{2n}^{(r)}(f,v,.)-P_{n}^{(r)}(f,v,.)\right\|_{C[a_1,b_1]} = O\left(n^{-(v+1)}\right) \qquad \dots (3.7)$$

In order to show $(i) \Rightarrow (ii)$, we have to prove that $(3.7) \Rightarrow (ii)$.

From (3.7), it is clear that $\left[n^{v+1}\left\{P_{2n}^{(r)}(f,v,x)-P_{n}^{(r)}(f,v,x)\right\}\right]$ is bounded in $C[a_1,b_1] \cap P_\infty[a_1,b_1]$. But $P_\infty[a_1,b_1]$ is the dual space of $P_1[a_1,b_1]$ so it can be stated that $\left[n^{v+1}\left\{P_{2n}^{(r)}(f,v,x)-P_{n}^{(r)}(f,v,x)\right\}\right]$ is weak compact i.e. there exists an $h \in P_\infty[a_1,b_1]$ and a subset $\{n_i\}_{i=1}^{\infty}$ of $\{n\}$ such that $\left[{n_i}^{v+1}\left\{P_{2n_i}^{(r)}(f,v,x)-P_{n_i}^{(r)}(f,v,x)\right\}\right]$ converges to h in the weak topology. In particular, $g \in C_0^\infty$ with $\operatorname{supp} g \subset (a_1,b_1)$, we have

$$\lim_{n_i\to\infty}\langle\left\{P_{2n_i}^{(r)}(f,v,x)-P_{n_i}^{(r)}(f,v,x)\right\}.g(.)\rangle = \langle h,g\rangle \qquad \dots (3.8)$$

Since $C^{2v+r+2}[a_1,b_1] \cap C_\gamma[a_1,b_1]$ is dense in $C_\gamma[a_1,b_1]$, there exists a sequence $\{f_\sigma\}_{\sigma=1}^{\infty}$ in $C^{2v+r+2}[a_1,b_1] \cap C_\gamma[a_1,b_1]$ converging to f in $\|.\|_\gamma$. Therefore, for any $g \in C_0^\infty$ with $\operatorname{supp} g \subset (a_1,b_1)$ and each function f_σ, by using Theorem 2.3, we have

$$\lim_{n_i\to\infty} n_i^{v+1}\langle\left\{P_{2n_i}^{(r)}(f_\sigma,v,x)-P_{n_i}^{(r)}(f_\sigma,v,x)\right\}.g(.)\rangle.$$

$$\langle-\left(1-2^{-(v+1)}\sum_{i=r}^{2v+r+2} Q(i,v,,r,x)\, f_\sigma^{(i)}(x).g(.)\right)\rangle.$$

$$\langle P_{2v+r+2}(D) f_\sigma^{(i)}(x).g(.)\rangle$$

$$= \langle f_\sigma, P_{2v+r+2}^{*}(D).g(.)\rangle \qquad \dots (3.9)$$

where $P_{2v+r+2}^{*}(D)$ is the dual operator of $P_{2v+r+2}(D)$, and by Lemma 2

$$\lim_{n_i\to\infty} n_i^{v+1}\langle\begin{Bmatrix} P_{2n_i}^{(r)}(f-f_\sigma,v,x) \\ -P_{n_i}^{(r)}(f-f_\sigma,v,x)\end{Bmatrix}.g(.)\rangle \le k\|f-f_\sigma\|_\gamma \qquad \dots (3.10)$$

Collecting (3.8)-(3.10), we obtain

$$\langle f, P_{2v+r+2}^{*}(D).g(.)\rangle = \lim_{\sigma\to\infty}\langle f_\sigma, P_{2v+r+2}^{*}(D).g(.)\rangle$$

$$\lim_{\sigma\to\infty}\left[\begin{matrix}\lim\limits_{n_i\to\infty} n_i^{v+1}\langle\begin{Bmatrix} P_{2n_i}^{(r)}(f-f_\sigma,v,x) - \\ P_{n_i}^{(r)}(f-f_\sigma,v,x)\end{Bmatrix}.g(.)\rangle \\ +\langle f_\sigma(.), P_{2v+r+2}^{*}(D).g(.)\rangle\end{matrix}\right]$$

$$= \lim_{n_i\to\infty}\langle\left\{P_{2n_i}^{(r)}(f,v,x)-P_{n_i}^{(r)}(f,v,x)\right\}.g(.)\rangle$$

$$= \langle h,g\rangle, \qquad (3.11)$$

where $h(x) = P_{2v+r+2}(D) f(x)$.

Now by Lemma 3, we have $Q(2v+r+2,v,x) \neq 0$. Therefore we can write (3.11) as a first order linear differential equation for $f^{(2v+r+2)}$, from which we conclude that $f^{(2v+r+2)} \in A.C.[a_2,b_2]$ and also $f^{(2v+r+2)} \in P_\infty[a_2,b_2]$.

Thus $(i) \Rightarrow (ii)$. Further from (ii), it follows that

$$f^{(2v+r+2)} \in Lip_M(1,a_2,b_2)$$

with $M = \left\|f^{(2v+r+2)}\right\|_{P_\infty[a_2,b_2]}$. Therefore $(ii) \Rightarrow (iii)$ using Theorem 2.2 and hence $(i) \Rightarrow (ii) \Rightarrow (iii)$.

The proof of $(iv) \Rightarrow (v)$ is similar to $(i) \Rightarrow (ii)$ and $(v) \Rightarrow (vi)$ is followed by Theorem 2.1, and hence $(iv) \Rightarrow (v) \Rightarrow (vi)$.

$(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v) \Rightarrow (vi)$ together complete the required proof of our saturation theorem.

REFERENCES

[1]. P.N. Agrawal, A.J. Mohammad, “Linear combination of a new sequence of linear positive operators,” Revista de la Union Mat. Argent. 44(1) (2003), 33-42.

[2]. V. Gupta, G.S. Srivastava, “An estimate of the rate of convergence of modified Baskakov operators for functions of bounded variation,” Kyungpook Math. J. 36(2) (1996), 237-237.

[3]. P. Maheshwari., “An inverse result in simultaneous approximation by modified Beta operators,” Georg. Math. J. 13(2) (2006), 297-306.

[4]. C.P. May, “Saturation and Inverse theorems for combinations of a class of exponential type operators,” Canad. J. Math. 28(6) (1976), 1224-1250.

[5]. R.P. Sinha, P.N. Agrawal and V. Gupta, “On simultaneous approxima- tion by modified Baskakov operators,” Bull. Soc. Math. Belg. Ser. B, 43(2) (1991),