International Journal of Scientific Research in Mathematical and Statistical Sciences **Research Paper**
Vol.6, Issue.5, pp.01-06, October (2019) **E-ISSN:** 2348-4519
DOI: https://doi.org/10.26438/ijsrmss/v6i5.16

Relation between Domination Number, Energy, Laplacian Energy of Graph and Rank

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Available online at: www.isroset.org

Received: 27/Sept/2019, Accepted: 13/Oct/2019, Online: 31/Oct/2019

Abstract—Graph theory has seen an hot-headed evolution due to its application in different arena of sciences in the last 40 years. Imaginably the fasting growing areas within graph theory are Domination theory, energy of graphs etc. Matrix representation of graph &. Rank of matrix is most familiar in graph theory In this paper we find few bounds which relate domination number of G, energy of G, laplacian energy of G and rank of the incident matrix of the graph G, and we pretence some open problems for further research. The motivation for this paper is to establish a link between these parameters.

Keywords— Domination Number, Energy of Graph, Laplacian energy of graph, Rank of the incidence matrix.

I. INTRODUCTION

In 1958, Laude Berge presented the theory of Domination. The encouragement for this concept was strained from the conventional problem of covering chessboard with minimum number of chess pieces. The most common definition given of a dominating set is that it is a set of vertices D/ V in a graph G = (V, E) having the property that V-D's every vertex is adjacent to at least one vertex in D. The domination number g (G) is the cardinality of a smallest dominating set of G.

Eigenvalues and Eigenvectors provide insight into the geometry of the associated linear transformation. Energy of graph and laplacian energy are originated from theoretical chemistry. The energy of a graph is the sum of the absolute values of the Eigen values of its adjacency matrix. The laplacian energy is also the sum of the absolute values of rank of the matrix we come to recognize several properties about this linear transformation. Rank of the Matrix equals the dimension of the linear manifold spanned by vectors x_1 , x_2 , x_3 ,..., x_k .

In this paper we find few bounds which relate domination number of G, energy of G, laplacian energy of G and rank of the incident matrix of the graph G, and pretence some open problems for further research. The Enthusiasm for this paper is to create a link between these parameters.

This is the paper motivated from the paper [11] by CLEMENS BRAND,NORBERT SEIFTER, Eigenvalues and $(\lambda - \frac{2m}{n})$ where λ is the eigenvalue of the Laplacian matrix ,

m is the number of edges and n is the number of vertices of the graph G. From the pioneering work of Coulson [1] there exists a continuous interest towards the general Mathematical properties of the total _ electron energy e as calculated within the framework of the Huckel Molecular Orbital (HMO) model. These efforts enabled one to get an insight into the dependence of e on molecular structure. The properties of E(G) and LE(G) are discussed in detail in [4], [6], [7].

The number of non-zero rows in the row reduced form (echelon form) of a matrix A is called the rank of A denoted by rank (A) or r(A) or $\rho(A)$. Rank of the matrix is the number of linearly independent rows or the number of linearly independent columns. A matrix always signifies a linear transformation between two vector spaces. From the domination in graphs, Mathematica Slovaca, Vol. 46 (1996), No. 1, 33–39, where the authors had found the relation between largest Eigen value of the Laplacian Matrix and Domination number.

II. RELATED WORK

Definition:- Rank and Nullity- A graph G with n vertices, m edges and k components has the rank $\rho(G) = n - k$. The nullity of the graph is $\mu(G) = m - n + k$. We see that $\rho(G) \geq 0$ and $\rho(G) + \mu(G) = m$.

Definition:-The energy, E(G), of a graph G is defined to be the sum of the absolute values of its eigen values. Hence if A(G) is the adjacency matrix of G, and $\lambda,...,\lambda_n$ are the eigen values of A(G), then E(G) = $\sum_{i=1}^{n}|\lambda_i|$. The set $\{\lambda_1,...,\lambda_n\}$ is the spectrum of G and denoted by Spec G

Definition:- If G is an (n, m)-graph, and let d_i be the degree of the i^{th} vertex of G, i = 1, 2,...,n. The spectrum of the graph G, consisting of the numbers $\lambda_1, \lambda_2,...,\lambda_n$, is the spectrum of its adjacency matrix [4]. The Laplacian spectrum of the graph G, consisting of the numbers $\mu_1, \mu_2,...,\mu_n$, is the spectrum of its Laplacian matrix[8], then the **Laplacian energy of G**, denoted by LE(G), is equal to

$$\sum_{i=1}^{n}|\gamma_i|\text{, i.e., LE(G)} = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| .$$

III. METHODOLOGY

Finding the rank , energy, laplacian energy of complete graphs, paths and cycles with vertices n=,3,4,5,….and also verifying the relation these parameters with the domination number of a complete graphs, paths and cycles with suitable formulae.

IV. RESULTS AND DISCUSSION

Theorem-1

Let G be a complete graph without loops and multiple edges, E(G) is the energy of graph G, LE(G) is the Laplacian energy of graph G, I (G) is the incident matrix of graph G, ρ(G) = Rank I (G), γ(G) is the domination number of G then (i) $\gamma(G) = \left\lfloor \frac{E(G)}{Rank\ I(G)} \right\rfloor$ (ii) $\gamma(G) = \left\lfloor \frac{LE(G)}{Rank\ I(G)} \right\rfloor$

Proof

The proof can be done in two ways
a) Direct method
b) Mathematical Induction
a) Direct Method

TABLE I COMPLETE GRAPH

G	$\gamma(G)$	E(G)	LE(G)	$\rho(G)$	$\Delta(G)$	Eigen Values	Laplacian eigenvalues
K_2	1	2	2	2	-1	-1, 1	0,1
K_3	1	4	4	3	2	-1,-1,2	1,1,4
K_4	1	6	6	4	-3	-1,-1,-1,3	2, 2,2, 6
K_5	1	8	8	5	4	-1,-1,-1,-1,4	3,3,3,3,8
K_6	1	10	10	6	-5	-1,-1,-1,-1,-1,5	4,4,4,4,4,10
K_7	1	12	12	7	6	-1,-1,-1,-1,-1,-1,6	5,5,5,5,5,5,12
K_8	1	14	14	8	-7	-1,-1,-1,-1,-1,-1,-1,7	6,6,6,6,6,6,6,14
K_9	1	16	16	9	8	-1,-1, -1,-1,-1,-1,-1,-1,-1,8	7,7,7,7,7,7,7,7,16
K_{10}	1	18	18	10	-9	-1,-1,-1, -1,-1,-1,-1,-1,-1,9	8,8,8,8,8,8,8,8,8,18
K_{11}	1	20	20	11	10	-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,10	9,9,9,9,9,9,9,9,9,9,20
K_{12}	1	22	22	12	-11	-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,,11	10,10,10,10,10,10,10, 10,10,10,10,22
K_{13}	1	24	24	13	12	-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,12	11,11,11,11,11,11,11,11,11,11,11,11,24
K_{14}	1	26	26	14	-13	-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,13	12,12,12,12,12,12,12,12,12,12,12,12,12,26
K_{15}	1	28	28	15	14	-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,14	13,13,13,13,13,13,13,13,13,13,13,13,13,13,28
…	….	…..	….	…	…	…	
…	…	…	…	…	…	…	
K_n	1	2(n-1)	2(n-1)	n	$(-1)^n n$	(n-1)(-1)’s &(n-1)	(n-1)’s of (n-2) & 2(n-1)

Here we observe that $\gamma(G) = 1$, where G is complete graph , and also we have

$$E(K_n) = 2(n-1) \quad \longrightarrow (1)$$

$$LE(K_n) = 2(n-1) \quad \longrightarrow (2)$$

From the above table , we observe that

$$\frac{E(G)}{Rank\, I(G)} = \frac{2(n-1)}{n} < \frac{2n}{n} < 2 \quad \longrightarrow (3)$$

$\left\lfloor \frac{E(G)}{Rank\, I(G)} \right\rfloor = \left\lfloor \frac{2(n-1)}{n} \right\rfloor = 1 = \gamma(G)$, Where G is a Complete graph.

Similar prof for (ii)

b) Mathematical Induction to prove that $\gamma(G) = \left\lfloor \frac{E(G)}{Rank\, I(G)} \right\rfloor$.

For the Complete graph K_2,

$$\gamma(G) = \left\lfloor \frac{E(G)}{Rank\, I(G)} \right\rfloor = \left\lfloor \frac{2}{2} \right\rfloor = 1$$

For the Complete graph K_3,

$$\gamma(G) = \left\lfloor \frac{E(G)}{Rank\, I(G)} \right\rfloor = \left\lfloor \frac{3}{2} \right\rfloor = 1$$

For LHS=RHS, and n=k, ie $\gamma(G) = \left\lfloor \frac{E(G)}{Rank\, I(G)} \right\rfloor$

To prove by induction for n=k+1, ie $\gamma(G_{k+1}) = \left\lfloor \frac{E(G_{k+1})}{Rank\, I(G_{k+1})} \right\rfloor$.

So we know that $E(G_k) \le E(G_{k+1}),\ and\ Rank\, I(G_k) \le Rank\, I(G_{k+1})$

By verifying the above that $Rank\, I(G_{k+1}) > E(G_{k+1})$.

Hence the proof by Mathematical Induction. Also similar idea for the proof of (ii).

Theorem-2

Let path P be a connected graph with no loops and multiple edges, then

(i) $\gamma(P_n) = \left\lceil \frac{E(P_n)}{Rank\, I(P_n)} \right\rceil + \left\lfloor E(P_n) - Rank\, I(P_n) \right\rfloor$

(ii)

$$\gamma(P_n) = \left\lceil \frac{LE(P_n)}{Rank\, I(P_n)} \right\rceil + \left\lfloor LE(P_n) - Rank\, I(P_n) \right\rfloor.$$

Proof:-

For the proof of (i) , we refer

For proof of (ii), We check the results for few paths from the Table-2: Path

For n= 2, $1 \le \left\lceil \frac{2}{2} \right\rceil + \left\lfloor 2-2 \right\rfloor = 1$

For n= 3, $1 \le \left\lceil \frac{3.330}{2} \right\rceil + \left\lfloor 3.330-2 \right\rfloor = 3$

For n= 4, $2 \le \left\lceil \frac{4.8284}{2} \right\rceil + \left\lfloor 4.8284-2 \right\rfloor = 4$

To prove the above result in general we consider a complete graph

If we delete all the extra edges from a complete graph with n vertices in order to get a path n P_n, we write the following equations

$LE(P_n) < LE(K_n)$, $Rank\, I(P_n) \le Rank\, I(K_n)$

Therefore from the above equation we can write that

$$\frac{LE(P_n)}{Rank\, I(P_n)} \le \left\lceil \frac{LE(K_n)}{Rank\, I(K_n)} \right\rceil$$

$$\Rightarrow \frac{LE(P_n)}{Rank\, I(P_n)} + k = \left\lceil \frac{LE(K_n)}{Rank\, I(K_n)} \right\rceil$$

Where k is a constant which is chosen as $k = \left\lfloor LE(P_n) - Rank\, I(P_n) \right\rfloor$.

Therefore

$$\left\lfloor \frac{LE(K_n)}{Rank\, I(K_n)} \right\rfloor \le \left\lceil \frac{LE(P_n)}{Rank\, I(P_n)} \right\rceil + \left\lfloor LE(P_n) - Rank\, I(P_n) \right\rfloor$$

From the above Theorem 1, we also write that

$$\gamma(P_n) \le \left\lceil \frac{LE(P_n)}{Rank\, I(P_n)} \right\rceil + \left\lfloor LE(P_n) - Rank\, I(P_n) \right\rfloor$$

Hence the proof of the Theorem.

TABLE II PATH

G	γ (G)	E(G)	LE(G)	$\rho(G)$	$\Delta(G)$	Eigen values	Laplacian eigenvalues
P_2	1	2	2	2	-1	± 1	0,2
P_3	1	2.828	3.330	2	0	± 1.414, 0	0,1,3
P_4	2	4.472	4.8284	4	1	± 1.618, ± 1.618	0,0.5858,2,3.4142
P_5	2	5.564	6.0721	4	0	± 1.732, ± 1 , 0	0,0.382,1.382,

							2.618,3.618
P_6	2	6.988	7.4641	6	-1	$\pm$1.802, $\pm$1.247, $\pm$0.445	0,0.2679,1,2,3, 3.7321
P_7	3	8.054	8.7022	6	0	$\pm$1.848, $\pm$1.414, $\pm$0, 0.765	0,0.1981,0.753,1.555, 2.445,3.247,3.8019
P_8	3	9.516	10.0547	8	1	$\pm$1.879, $\pm$1.532, $\pm$1, $\pm$0.347	0, 0.1522 0.5858, 1.2346, 2, 2.7654, 3.4142, 3.8478
P_9	3	10.628	11.2953	8	0	$\pm$1.902, $\pm$1.618 $\pm$1.176, $\pm$0.618, 0	0, 0.1206, 0.4679, 1, 1.6527, 2.3473, 3, 3.5321, 3.8794
P_{10}	4	12.056	12.6275	10	-1	$\pm$1.919, $\pm$1.683, $\pm$1.31, $\pm$0.831, $\pm$0.285	0, 0.0979, 0.3820, 0.8244, 1.3820, 2.0000, 2.6180, 3.1756, 3.6180, 3.9021
P_{11}	4	13.192	13.8715	10	0	$\pm$1.932, $\pm$1.732, $\pm$1.414, $\pm$1, $\pm$0.518, 0	0 , 0.0810, 0.3175, 0.6903, 1.1692, 1.7154, 2.2846, 2.8308, 3.3097, 3.6825, 3.9190
P_{12}	4	14.529	15.1915	12	1	$\pm$1.942, $\pm$1.771, $\pm$1.497, $\pm$1.136, $\pm$0.709, $\pm$0.241	0, 0.0681, 0.2679, 0.5858, 1, 1.4824, 2, 2.5176, 3, 3.4142, 3.7321, 3.9319
P_{13}	5	15.752	16.4386	12	0	$\pm$1.350, $\pm$1.802, $\pm$1.564, $\pm$1.247, $\pm$0.868, $\pm$0.445, 0	-0.0000, 0.0581, 0.2291, 0.5030, 0.8639, 1.2908, 1.7589, 2.2411, 2.7092, 3.1361, 3.4970, 3.7709, 3.9419
P_{14}	5	17.132	17.7505	14	-1	$\pm$1.956, $\pm$1.827, $\pm$1.618, $\pm$1.338, $\pm$1, $\pm$0.618, $\pm$0.209	-0.0000, 0.0501, 0.1981, 0.4363, 0.7530, 1.1322, 1.5550, 2.0000, 2.4450, 2.8678, 3.2470, 3.5637, 3.8019, 3.9499
P_{15}	5	18.306	19.0002	14	0	+1.962, $\pm$1.848, $\pm$1.663, $\pm$1.414, $\pm$1.111, $\pm$0.765	-0.0000, 0.0437, 0.1729, 0.3820, 0.6617, 1.0000, 1.3820, 1.7909, 2.2091, 2.6180, 3.0000, 3.3383, 3.6180, 3.8271, 3.9563
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…

Theorem-3

Let a cycle C be a connected graph with no loops and multiple edges. Then

(i) $\gamma(C_n) \leq \left\lceil \frac{E(C_n)}{Rank\, I(C_n)} \right\rceil + \left\lfloor E(C_n) - Rank\, I(C_n) \right\rfloor$.

(ii)

$$\gamma(C_n) \leq \left\lceil \frac{LE(C_n)}{Rank\, I(C_n)} \right\rceil + \left\lfloor LE(C_n) - Rank\, I(C_n) \right\rfloor$$

Proof:-

For the proof of (i) , we refer

For proof of (ii),We can prove the above theorem similar to theorem-2

TABLE III CYCLE

G	γ (G)	E(G)	LE(G)	$\rho(G)$	$\Delta(G)$	Eigen values	Laplacian eigenvalues
C_3	1	4	4	3	2	-1,-1,2	1,1,4
C_4	2	4	4	2	0	-2,2,0,0	1,1,4,2
C_5	2	6.472	6.472	5	2	1.618,1.618, $\pm$ 0.618,2	0.3820, 0.3820 2.6180, 2.6180, 4
C_6	2	8	8	6	-4	$\pm$ 2, $\pm$ 1, $\pm$ 1	0,1,1,3,3,4
C_7	3	8.988	8.9879	7	2	-1.802,-1.802,-.445, -0.445, $\pm$ 1.247	0.1981, 0.1981 1.5550,1.5550 3.2470; 3.2470 4.0000
C_8	3	9.6569	9.6569	6	0	$\pm$ 2, $\pm$ 1.414, $\pm$ 1.414, 0,0	-0.0000, 0.5858 0.5858, 2, 2 3.4142, 3.4142, 4
C_9	3	11.5175	11.5175	9	2	-1.879,-1.879, -1, 0.347, 0.347,2,1.532, 1.532,-1	0.1206, 0.1206 1 , 1, 2.3473 2.3473, 3.5321 3.5321, 4
C_{10}	4	12.9443	12.9443	10	-4	$\pm$ 2, $\pm$ 1.616, $\pm$ 1.618, $\pm$ 0.618, $\pm$ 0.618	0.0000, 0.3820 0.3820, 1.3820 1.3820, 2.6180 2.6180, 3.6180 3.6180, 4.0000
C_{11}	4	14.0533	140533	11	2	-1.919,-1.919, -1.310,-1.310,-0.285, - 0.285, 0.831, 0.831, 2, 1.683,1.683	0.0810, 0.0810 0.6903, 0.6903 1.7154, 1.7154 2.8308, 2.8308 3.6825, 3.6825 4.0000
C_{12}	4	14.9282	14.9282	10	1	$\pm$ 2, $\pm$ 1, $\pm$ 1, $\pm$ 1.732, $\pm$ 1.732,0,0	-0.0000, 0.2679 0.2679, 1.0000 1.0000, 2.0000 2.0000, 3.0000 3.0000, 3.7321 3.7321, 4.0000
C_{13}	5	16.5925	16.5925	13	2	-1.942,-1.914, -1.497,-1.497, -0.709,-0.709, 0.241,0.241, 1.136, 1.136, 2,1.771,1.771	0.0581, 0.0581 0.5030, 0.5030 1.2908, 1.2908 2.2411, 2.2411 3.1361, 3.1361 3.7709, 3.7709 4.0000
C_{14}	5	17.9758	17.9758	14	-4	$\pm$ 2, $\pm$ 1.802,	-0.0000, 0.1981

						$\pm$ 1.802, $\pm$ 1.247, $\pm$ 1.247, $\pm$ 0.445, $\pm$ 0.445	0.1981, 0.7530 0.7530, 1.5550 1.5550, 2.4450 2.4450, 3.2470 3.2470, 3.8019 3.8019, 4.0000
C_{15}	5	19.1335	19.1335	15	2	-1.956,-1.618, -1.618,-1,-1, -0.209, -0.209, 0.618, 0.618, 1.338, 1.338, 2, 1.827,1.827, -1.956	0.0437, 0.0437 0.3820, 0.3820 1.0000, 1.0000 1.7909, 1.7909 2.6180, 2.6180 3.3383, 3.3383 3.8271, 3.8271 4.0000
…	…	…	…	…	…	…	…
	…	…	…	…	…	…	…

V. CONCLUSION AND FUTURE SCOPE

1. It can be exposed that for theorem-3 Cycles we can get better results for detailed cases of n- number of vertices being odd, even, prime etc.
2. We can extended the relation between these parameters to other classes of graphs and other types of Dominations.
3. Similar bounds i.e. relation between Domination number, Energy , Laplacian energy of graph and Rank of the incident matrix could be extended to any simply connected graphs, sub graphs, partitioning of graphs and other related topics.

REFERENCES

[1] C. A. Coulson, " On the calculation of the energy in unsaturated hydrocarbon molecules" Proc. Cambridge Phil. Soc., 36 (1940), 201 -203

[2] Georgi. E. Shilov, Linear Algebra, Dover publications,Inc. chapter

[3] Haynes, Teresa W, Fundamentals of Domination in graphs

[4] I. Gutman, "The energy of a graph," Ber. Math. Statist. Sekt.orschungszenturm Graz., 103 (1978), 1-22.

[5] I. Gutman, O. E. Polansky, Mathematical Concepts in Organic Chemistry, Springer, 1986.

[6] I. Gutman, "The energy of a graph: old and new results", in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer,(2000), 196-211.

[7] I. Gutman, "Topology and stability of conjugated hydrocarbons. The dependence of total ¼ - electron energy on molecular topology", J. Serb. Chem. Soc., 70 (2005), 441-456.

[8] Ivan Gutman a, Bo Zhou, "Laplacian Energy of a Graph", Linear Algebra and its Applications 414 (2006) 29–37.

[9] Teresa W., Haynes, Topics on Domination, Volume 86, Numbers1-3.

[10] Clemens brand , Norbert Seifter, "Eigenvalues and domination in graphs", Mathematica Slovaca, Vol. 46 (1996), No. 1, 33–39.

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