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Research Article

On the Presence of Positive Solutions for Generalized Fractional Boundary Value Problems with Green's Function

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Abstract—The branch of mathematics that deals with the study of non-integer order derivatives and integrals is called fractional calculus. The interesting thing about this subject is that in contrast to the classical derivatives, the fractional derivatives are not a point quantity. Indeed, the fractional derivative of a function of order α at some point is a local property only for α being an integer. In recent years, the study of positive solutions for fractional differential equation boundary value problems has attracted considerable attention, and fruits from research into it emerge continuously. In this paper, the existence of positive solutions is established for boundary value problems defined within generalized Riemann–Louville and Caputo fractional operators. Our approach is based on utilizing the technique of fixed point theorems. For the sake of converting the proposed problems into integral equations, we construct Green's function and study their properties for three different types of boundary value problems. An example is presented to demonstrate the validity of theoretical findings.

Keywords— Existence of positive solutions; Fractional Calculus; Riemann-Liouville Differintegral; Generalized Fractional Boundary Value Problems.

1. Introduction

Fractional differential equations arise in many fields, such as physics, mechanics, chemistry, economics, engineering and biological sciences, etc; see [1-3] for example. In recent years, the study of positive solutions for fractional differential equation boundary value problems (BVP) has attracted considerable attention, and fruits from research into it emerge continuously. For a small sample of such work, we refer the reader to [] and the references therein. In these papers, many authors have investigated the existence of positive solutions for nonlinear fractional differential equation boundary value problems. Their results are based on Schauder fixed point theorem, Leggett-Williams theorem, fixed point index theorems in cones, Krasnosel'skii fixed point theorem, the method of upper-lower solutions, fixed point theorems in cones and so on. On the other hand, the uniqueness of positive solutions for nonlinear fractional differential equation BVP has been studied by some authors, see [4,5] for example. The methods used in these papers are fixed point theorems for mixed monotone operators, u_0-concave operators and monotone operators in partially ordered sets.
Moreover, Fractional calculus is the theory of integrals and derivatives of arbitrary real (and even complex) order and was first suggested in works by mathematicians such as Leibniz, L'Hôpital, Abel, Liouville, Riemann, etc. The importance of fractional derivatives for modeling phenomena in different branches of science and engineering is due to their non-locality nature, an intrinsic property of many complex systems. Unlike the derivative of integer order, fractional derivatives do not take into account only local characteristics of the dynamics but considers the global evolution of the system; for that reason, when dealing with certain phenomena, they provide more accurate models of real-world behavior than standard derivatives [6].

The remainder of this paper is organized as follows; Section 2 contains some basic fractional calculus. Section 3 offers the preliminary results. Section 4, introduces a some main results. Section 5, demonstrate illustrative example that show consistency to the main theorems. In Section 6, the main conclusions obtained from this paper are discussed.

2. Basic Fractional Calculus

The main objects of classical calculus are derivatives and integrals of functions- these two operations are inverse to each other in some sense. If we start with a function *f(t)*, and put its derivatives on the left- hand side and on the right- hand side we continue with integrals, we obtain a both- side infinite sequence [4].

$$\frac{d^2f(t)}{dt^2}, \frac{df(t)}{dt}, f(t), \int_a^t f(\tau)d\tau, \int_a^t \int_a^{\tau_1 f(\tau)} d\tau_1, ... \quad (1)$$

Fractional calculus tries to interpolate this sequence so this operation unifies the classical derivatives and integrals and generalizes them for arbitrary order. We will usually speak of differintegral, but sometimes the name α-derivative (α is an arbitrary real number) which can mean also an integral if α < 0, is also used, or we talk directly about fractional derivative and fractional integral.

There are many ways to define the differintegral and these approaches are called according to their authors. For example the Grounwald-Letnikov definition of differintegral starts from classical definitions of derivatives and integrals based on infinitesimal division and limit. The disadvantages of this approach are its technical difficulty of the computations and the proofs and large restrictions on functions. Fortunately there are other, more elegant approaches like the Riemann-Liouville definition which includes the results of the previous one as a special case. In this paper we will focus on the Riemann-Liouville, the Caputo and the Miller- Ross definitions since they are the most used ones in applications. We will formulate the conditions of their equivalence and derive the most important properties.

This science is part of mathematical analysis and deals with the applications of integration and derivation in the case of the ordered derivative, and this field is concerned with generalizing the derivative of the association (function) of any derivative of integer order, for example: we usually deal with the first and second derivatives. As for this field (fractional differentiation), it helps us to find the derivative number one-half or 0.3 or 0.7 ...etc [7].

2.1 The Gamma Function

In the integer-order calculus the factorial plays an important role because it is one of the most fundamental combinatorial tools. The Gamma function has the same importance in the fractional-order calculus and it is basically given by integral; see e.g. [7-9].

<u>Definition (2.1):</u> Gamma Function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad (2)$$

for $Re(z) > 0$.

It is natural to expect a connection between the Gamma function and the factorial, by the fact that $\Gamma(1) = 1; \Gamma(n+1) = n!.$, for $n \in \mathbb{N}$.

The gamma function then is defined in the complex plane as the analytic continuation of this integral function: it is a meromorphic function that is holomorphic except at zero and the negative integers, where it has simple poles.

2.2 The Beta Function

The Beta function is very important for the computation of the fractional derivatives of the power function. It is defined by the two-parameter integral; see e.g. [7-10].

<u>Definition (2.2):</u> Beta Function

$$\beta(z, w) = \int_0^1 \tau^{(z-1)} (1-\tau)^{w-1} d\tau, \quad (3)$$

for z, w satisfying $Re(z) > 0$ and $Re(w) > 0$.

We get a relation between the Beta function and the Gamma function which implies;

$$\beta(z, w) = \beta(w, z), \beta(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad (4)$$

where $\Gamma(z)$ is a gamma function define by Eq. (2).

2.3 The Riemann-Liouville Differintegral

The Riemann-Liouville approach is based on the Cauchy formula (5) for the n^{th} integral which uses only a simple integration so it provides a good basis for generalization;

$$I_a^n f(t) = \int_a^t \int_a^{\tau_{n-1}} ... \int_a^{\tau_1} f(\tau) d\tau_1 ... d\tau_{n-1} = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau, \quad (5)$$

<u>Definition (2.3):</u> Riemann-Liouville Differintegral

Let a, T, α be real conatants a < T, n = max (0, [α] + 1) *and f* (*t*) *an integrable function on* (*a*, *T*) *for n* > 0 *additional assume that f* (*t*) *is n-times differentiable on* (*a*, *T*) *except on a set of measure zero. Then the Riemann-Liouville differintegral is defined for t* ∈ (*a*, *T*) *by the formula:*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad (6)$$

2.4 The Caputo Differintegral

We will denote the Caputo differintegral by the capital letter with upper-left index ${}^cD$. The fractional integral is given by the same expression like before, so for $\alpha > 0$ we have;

$${}^cD_a^{-\alpha} f(t) = D_a^{-\alpha} f(t). \quad (7)$$

The difference occurs for fractional derivative. A non-integer-order derivative is again defined by the help of the fractional integral, but now we first differentiate $f(t)$ in the common sense and then go back by fractional integrating up to the required order. This idea leads to the following definition of the Caputo differintegral.

Definition (2.4): Caputo Differintegral

Let a, T, α be real conatants a < T, n_c = max (0, [α] + 1) and f (t) an integrable function on (a, T) in case n_c = 0 and n_c-times differentiable on (a, T) except on a set of measure zero in case n_c > 0. Then the Caputo differintegral is defined for t ∈ (a, T) by the formula:

$$ {}^{c}D_a^{\alpha} f(t) = I_a^{n_c-\alpha}\left[\frac{d^{n_c} f(t)}{dt^{n_c}}\right] \tag{8}$$

Remark: For $\alpha > 0$, $\alpha \in N_0$, Eq. (8) is often written in the form:

$$ {}^{c}D_a^{\alpha} f(t) = \frac{1}{\Gamma(n_c-\alpha)}\int_a^t (t-\tau)^{n_c-\alpha-1} f^{(n_c)}(\tau)d\tau, \tag{9}$$

2.5 Riemann–Liouville integral and derivative

Definition (2.5): Riemann–Liouville integral and derivative

The Riemann-Liouville fractional integral of order α > 0 for a continuous function f (t) : (0, ∞) → $\mathbb{R}$ is defined as;

$$ I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \tag{10}$$

Provided the integral exists.

Clearly, the Caputo derivative can also be written by the help of fractional integrals of the Riemann-Liouville type:

$$ {}^{c}D_a^{\alpha} f(t) = D_a^{-(n_c-\alpha)}\left[\frac{d^{n_c} f(t)}{dt^{n_c}}\right], \tag{11}$$

2.6 Left and right Riemann-Liouville fractional derivatives

All definitions given in the previous sections were so called left differ-integrals. The origin of this name is clear because we calculate the value of the differ-integral at point t by the help of points on the left of it. If *t* means time, it seems to be logical since we use in fact the history of the function *f* (*t*) and the future does not need to be known yet. On the other side, if *t* plays the role of a spatial variable, there is no reason why events on the left should be more important than those on the right, see [11].

In this paper, we may usually consider *t* to be time, so mostly we will not need right differ-integrals. The only exception occurs only in the chapter about applications where we will use the right Riemann-Liouville derivative according to the spatial variable. This is the reason why we do not introduce right differ-integrals for all approaches, but only the following formula for the right Riemann-Liouville derivative (we will denote the right fractional derivative by left bottom index −), *n* = [*α*] + 1:

Definition (2.6): Left and right Riemann-Liouville fractional derivatives

The left and right Riemann-Liouville fractional derivatives ${}_aD_t^{\alpha}$ and ${}_tD_b^{\alpha}$ of order $\alpha \in \mathbb{R}_+$, are defined by;

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau,\ t>a, \tag{12}$$

and

$$ {}_tD_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)d\tau,\ ,b>t. \tag{13}$$

2.7 Left and right Riemann-Liouville fractional integrals

Definition (2.7): Left and right Riemann-Liouville fractional integrals

Let J = [a, b], (−∞ < a < b < ∞) be a finite interval of $\mathbb{R}$. The left and right Riemann- Liouville fractional integrals ${}_aD_t^{-\alpha} f(t)$ and ${}_tD_b^{-\alpha} f(t)$ of order $\alpha \in \mathbb{R}_+$, are defined by;

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{1-\alpha} f(\tau)d\tau, t>a, a>0, \tag{14}$$

and

$$ {}_tD_b^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{1-\alpha} f(\tau)d\tau, b>t, a>0, \tag{15}$$

Lemma (2.1):

Assuming arbitrary function f (x) and m, n > 0 the following equations hold;

1. *Semi-group property:*

$$ I_a^m I_a^n f = I_a^{m+n} f,$$

2. *Commutative property:*

$$ I_a^m I_a^n f(t) = I_a^n I_a^m f(t),$$

3. Preliminary results

Theorem (3.1):

(Fixed point theorem): Let a,b∈ R *and a < b if F:[a,b]→[a,b] is continuous then there is a fixed point in F.*

Definition (3.1):

Let E be a vector space on the field $\mathbb{K}(\mathbb{R} or \mathbb{C})$ Let $N{:}E\rightarrow \mathbb{R}^{+}$ we say that N is a norm on E if it satisfies the following properties:

(1) *N(X)=0 ⇔ X=0.*

(2) *$N(\lambda X)=|\lambda|\, N(X)$, $\forall \lambda \in \mathbb{K}$, and $\forall x \in E$.*

(3) *$N(X+Y) \leqslant N(X)+N(Y)$, $\forall X \in E$ and $\forall y \in E$.*

In general N is donted by $\|\cdot\|$, and (E,N) is called a normed vector space (n.v.s).

Definition (3.2):

We say that $(E,\|\cdot\|)$ is complete if any cauchy sequence in E is convergent in E. A complete normed vector space is called a Banach space.

Let $C \in [0,1]$ be the Banach space endowed with the norm $\|\zeta\|=\sup\{|\zeta(t)|;\ t\in[0,1]\}$ for $\zeta \in C[0,1]$, and define the cone

$K=\{\zeta \in C[0,1]: \zeta \geq 0,\ t\in[0,1]\}$.

The positive solution that we take account in this work is such that $\zeta(t) \geq 0$, $0 \leqslant t \leqslant 1$, $\zeta \in C[0,1]$.

Definition (3.3):

Let $0< \upsilon$, $\rho : [a, b] \to \mathbb{R}$ be an integrable function and $\vartheta : [a, b] \to \mathbb{R}$ an increasing function with $\vartheta'(t) \neq 0$, for all $t \in [a, b]$ .The ϑ -RL fractional integral of ρ of order υ is given by;

$$I_{a+}^{\upsilon,\vartheta}\rho(t) = \frac{1}{\Gamma(\upsilon)}\int_a^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}\rho(s)ds. \tag{16}$$

Definition (3.4):

Let $n-1< \upsilon <n$, and $\rho, \vartheta \in C^n[a, b]$ such that ϑ is an increasing function with $\vartheta'(t) \neq 0$ for all $t \in [a,b]$.

Then the ϑ - RL fractional derivative of order υ is given by;

$$D_{a+}^{\upsilon,\vartheta}\rho(t)=D^{n,\vartheta} I_{a+}^{n-\upsilon,\vartheta}\rho(t) .$$

where:

$$D^{n,\vartheta} = \left[\frac{1}{\vartheta'(t)}\frac{d}{dt}\right]^n \text{ and } n=[\upsilon]+1.$$

Lemma (3.1):

Let $r \in \mathbb{R}$ with $r > n$. Then ϑ -fractional integral and derivative of the function $\rho(t) = (\vartheta(t) - \vartheta(a))^{r-1}$ are;

$$I_{a+}^{\upsilon,\vartheta}\rho(t)=\frac{\Gamma(r)}{\Gamma(r+\upsilon)}(\vartheta(t)-\vartheta(a))^{\upsilon+r-1}$$

and

$$D_{a+}^{\upsilon,\vartheta}\rho(t) = \frac{\Gamma(r)}{\Gamma(r-\upsilon)}(\vartheta(t)-\vartheta(a))^{\upsilon-r-1}$$

Lemma (3.2): *Let $\upsilon, r > 0$ and $\rho \in L[0,1]$. Then we have;*

- $D_{a+}^{\upsilon,\vartheta} I_{a+}^{\upsilon,\vartheta}\rho(t)=\rho(t).$
- $I_{a+}^{\upsilon,\vartheta} I_{a+}^{r,\vartheta}\rho(t)= I_{a+}^{\upsilon+r,\vartheta}\rho(t).$
- $D_{a+}^{k,\vartheta} I_{a+}^{\upsilon,\vartheta}\rho(t)=I_{a+}^{\upsilon-k,\vartheta}\rho(t),\ k\in \mathbb{N}, \upsilon > k .$

Lemma (3.3): *Let $\upsilon > 0$. If we suppose $\zeta \in C(0, 1) \cap L(0,1)$, then the equation $D_{0+}^{\upsilon,\vartheta}\zeta(t) = 0$ has a unique solution;*

$$\zeta(t) = c_1[\vartheta(t)-\vartheta(0)]^{\upsilon-1} + c_2[\vartheta(t)-\vartheta(0)]^{\upsilon-2} + \ldots + c_n[\vartheta(t)-\vartheta(0)]^{\upsilon-n},$$

Moreover,if ζ, $D_{0+}^{\upsilon,\vartheta}\zeta \in C(0, 1)\cap L(0,1)$,then;

$$I_{0+}^{\upsilon,\vartheta} D_{0+}^{\upsilon,\vartheta}\zeta(t) = \zeta(t) + c_1[\vartheta(t)-\vartheta(0)]^{\upsilon-1} + c_2[\vartheta(t)-\vartheta(0)]^{\upsilon-2} + \ldots + c_n[\vartheta(t)-\vartheta(0)]^{\upsilon-n}$$

for some $c_i \in \mathbb{R}$, $i=1,2,\ldots,n$.

Consider the non-linear generalized fractional BVP of the form;

$$\begin{cases} D_{0+}^{\upsilon,\vartheta}\zeta(t) + \beta(t,\zeta(t)) = 0, 0 < t < 1 \\ \zeta(0)=0 \text{ and } \zeta(1) = 0 \end{cases} \tag{17}$$

we will investigate the existence and uniqueness of positive solution of Eq.(17).

Theorem (3.2):

(Banach). Let X be a Banach space with a contraction mapping $T:X \to X$. Then T has a unique fixed - point x in X, see [12].

Theorem (3.3):

(schaefer fixed point theorem). Let X be a Banach space and let $F:X\to X$ be completely continuous mapping, see [13]. *Then either :*

(i) *The equation $X=\lambda FX$ has a solution for $\lambda = 1$.*

(ii) The set { x ∈ X :x=λ Fx for some λ ∈ (0,1)} is unbounded.

Theorem (3.4):

(Darbo-sadovskii fixed point theorem) If Ω is bounded closed and convex subset of Banach space X , the continuous mapping L: Ω → Ω is an α-contraction , then the mapping L has at least one fixed point in Ω, see [13].

Definition (3.5):

Let A⊂E we say that A is convex if : ∀x,y∈A ,∀t ∈[0,1] , it holds: tx +(1-t) y∈A, see [14].

Definition (3.6):

A set A is compact if every open cover of A contains a finite subcover of A, see [14].

Definition (3.7):

A set B in a metric space is relatively compact if it is closure $\bar{B}$ is compact, see [14].

Theorem (3.5):

(Schauder) Let X be a Banach space and let S be a closed ,convex,bounded subset of X. If T:S→ S is a continuous map such that the set {Ts:s∈S} is relatively compact in X , then T has at least one fixed point, see [12].

4. Main Results

This section is dedicated to demonstrating the developed Green's function corresponding to the problem (17) and proving the existence and uniqueness of positive solutions to a problem (17).

Lemma (4.1):

Let 1< υ ≤2 and ϕ : [0, 1] → $\mathbb{R}^+$ is continuous, see [15]. *Then we have;*

$$D_{0+}^{\upsilon,\vartheta}\zeta(t)+\phi(t)=0, 0<t<1, \quad (18)$$

$$\zeta(0)=\zeta(1)=0$$

has a unique solution $\zeta\in C[0,1]$ given by;

$$\zeta(t)=\int_0^1 \vartheta'(s)G(t,s)\phi(s)ds. \quad (19)$$

where;

$$G(t,s)=\begin{cases}\frac{Z_\vartheta^\upsilon(t,0)Z_\vartheta^\upsilon(1,s)-Z_\vartheta^\upsilon(1,0)Z_\vartheta^\upsilon(t,s)}{Z_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}, 0\le s\le t\le 1\\ \frac{Z_\vartheta^\upsilon(t,0)Z_\vartheta^\upsilon(1,s)}{Z_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}, 0\le t\le s\le 1\end{cases} \quad (20)$$

Here G(t,s) means the Green's function of BVP, Eq. (18) and the given notation is adopted for easiness;

$$Z_\vartheta^\upsilon(t,s)=[\vartheta(t)-\vartheta(s)]^{\upsilon-1}$$

proof . By applying Lemma (3.3) on first equation of Eq. (18) we obtain;

$$\zeta(t)=-\frac{1}{\Gamma(\upsilon)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}\phi(s)ds+C_1[\vartheta(t)-\vartheta(0)]^{\upsilon-1}+C_2[\vartheta(t)-\vartheta(0)]^{\upsilon-2} \quad (21)$$

for some $C_1, C_2\in\mathbb{R}$.

from the second equation of (18),we get $C_2 = 0$ and we have;

$$C_1=\frac{[\vartheta(1)-\vartheta(0)]^{1-\upsilon}}{\Gamma(\upsilon)}\int_0^1\vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}\phi(s)ds.$$

Substitute the values of C_1 and C_2 in Eq. (21), and we get;

$$\zeta(t)=-\frac{1}{\Gamma(\upsilon)}\int_0^t\vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}\phi(s)ds$$

$$+\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{1}{\Gamma(\upsilon)}\int_0^1\vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}\phi(s)ds.$$

Hence;

$$\begin{aligned}\zeta(t)&=-\frac{1}{\Gamma(\upsilon)}\int_0^t\vartheta'(s)Z_\vartheta^\upsilon(t,s)\phi(s)ds\\&+\frac{Z_\vartheta^\upsilon(t,0)}{Z_\vartheta^\upsilon(1,0)}\frac{1}{\Gamma(\upsilon)}\int_0^1\vartheta'(s)Z_\vartheta^\upsilon(1,s)\phi(s)ds\\&=-\int_0^t\vartheta'(s)\frac{Z_\vartheta^\upsilon(t,s)}{\Gamma(\upsilon)}\phi(s)ds\\&+\frac{Z_\vartheta^\upsilon(t,0)}{Z_\vartheta^\upsilon(1,0)}\int_0^t\vartheta'(s)\frac{Z_\vartheta^\upsilon(1,s)}{\Gamma(\upsilon)}\phi(s)ds\\&+\frac{Z_\vartheta^\upsilon(t,0)}{Z_\vartheta^\upsilon(1,0)}\frac{1}{\Gamma(\upsilon)}\int_t^1\vartheta'(s)Z_\vartheta^\upsilon(1,s)\phi(s)ds\\&=\int_0^t\vartheta'(s)\frac{Z_\vartheta^\upsilon(t,0)Z_\vartheta^\upsilon(1,s)-Z_\vartheta^\upsilon(1,0)Z_\vartheta^\upsilon(t,s)}{Z_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}\phi(s)ds\\&+\int_t^1\vartheta'(s)\frac{Z_\vartheta^\upsilon(t,0)Z_\vartheta^\upsilon(1,s)}{Z_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}\phi(s)ds\\&=\int_0^1\vartheta'(s)G(t,s)\phi(s)ds.\end{aligned}$$

Lemma (4.2):

For all υ ∈ (1, 2]. The Green's function given by Eq. (20) satisfies the properties:

(i) *G(t,s) is continuous on [0,1]×[0, 1].*

(ii) *G(t, s) > 0, 0 < t, s < 1.*

(iii) *For s ∈ (0, 1).*

$$\Gamma(\upsilon)\max_{t\in[0,1]}G(t,s)\le\frac{Z_\vartheta^\upsilon(1,s)}{Z_\vartheta^\upsilon(1,0)}. \quad (22)$$

Definition (4.1):

Let a,b ∈ $\mathbb{R}^+$ (b>a).For any ζ ∈[a,b],we say that β(t,.) is the upper control function if;

$$\bar{\beta}(t,\zeta) = \sup_{a\le\eta\le\zeta} \beta(t,\eta)$$

and is the lower-control function if;

$$\underline{\beta}(t,\zeta) = \inf_{\zeta\le\eta\le b} \beta(t,\eta),$$

Certainly, $\bar{\beta}(t,\zeta)$ and $\underline{\beta}(t,\zeta)$ are non-decreasing on ζ and;

$$\underline{\beta}(t,\zeta) \le \beta(t,\zeta) \le \bar{\beta}(t,\zeta).$$

Theorem (4.1):

Suppose β : [0, 1] × R+ → R+ *is continuous, see* [15]. *Then there exists at least one positive solution ζ*(*t*) *of Eq.(17). Moreover*

$$\underline{\zeta}(t) \le \zeta(t) \le \bar{\zeta}(t), \qquad t\in[0,1].$$

Where $\bar{\zeta}(t), \underline{\zeta}(t)$ *are upper and lower solutions of Eq.(17).*

Proof . Define *Q:K → K by*

$$(Q\zeta)(t) = \int_0^1 \vartheta'(s)G(t,s)\beta(s,\zeta(s))ds.$$

Lemma (4.1) shows that fixed points of *Q* are solutions of Eq.(17). Because *β*(*s, ζ*) and *G(t,s)* are non-negative and continuous , *Q:K→ K* is continuous. Define the ball;

$$B_r = \{\zeta\in K : ||\zeta|| \le r\} \subset K,$$

and set;

$$L := \max_{(t,\zeta)\in[0,1]\times[0,r]} |\beta(t,\zeta)| + 1.$$

Then for any *ζ* ∈ *B* we get;

$$|(Q\zeta)(t)| \le \int_0^1 \vartheta'(s)G(t,s)|\beta(s,\zeta(s))|ds$$

$$\le \int_0^1 \vartheta'(s)\max_{t\in[0,1]} G(t,s)|\beta(s,\zeta(s))|ds$$

$$\le \frac{L}{Z_\vartheta^v(1,0)}\frac{1}{\Gamma(v)}\int_0^1 \vartheta'(s)Z_\vartheta^v(1,s)ds$$

$$= \frac{L}{[\vartheta(1)-\vartheta(0)]^{v-1}}\frac{1}{\Gamma(v)}\int_0^1 \vartheta'(s)[\vartheta(1)-\vartheta(s)]^{v-1}ds$$

$$\le L\frac{[\vartheta(1)-\vartheta(0)]}{\Gamma(v+1)}.$$

This shows that *(Q Br)* is uniformly bounded.

Now , we prove that *Q* is equicontinuous. Let *ζ* ∈ B_r. Then for $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have;

$$|(Q\zeta)(t_2) - (Q\zeta)(t_1)| \le \int_0^1 \vartheta'(s)|G(t_2,s) - G(t_1,s)||\beta(s,\zeta(s))|ds. \tag{23}$$

Consider Δ=|$G(t_2, s) - G(t_1, s)$|.*Thus;*

$$\Delta = \Bigg|\frac{Z_\vartheta^v(t_2,0)Z_\vartheta^v(1,s) - Z_\vartheta^v(1,0)Z_\vartheta^v(t_2,s)}{Z_\vartheta^v(1,0)\Gamma(v)} + \frac{Z_\vartheta^v(t_2,0)Z_\vartheta^v(1,s)}{Z_\vartheta^v(1,0)\Gamma(v)}$$

$$-\frac{Z_\vartheta^v(t_1,0)Z_\vartheta^v(1,s) - Z_\vartheta^v(1,0)Z_\vartheta^v(t_1,s)}{Z_\vartheta^v(1,0)\Gamma(v)} - \frac{Z_\vartheta^v(t_1,0)Z_\vartheta^v(1,s)}{Z_\vartheta^v(1,0)\Gamma(v)}\Bigg|.$$

$$= \left|\frac{2Z_\vartheta^v(1,s)}{Z_\vartheta^v(1,0)\Gamma(v)}[Z_\vartheta^v(t_2,0) - Z_\vartheta^v(t_1,0)] + \frac{1}{\Gamma(v)}[Z_\vartheta^v(t_1,s) - Z_\vartheta^v(t_2,s)]\right|$$

$$\le \Bigg|\frac{2Z_\vartheta^v(1,s)}{Z_\vartheta^v(1,0)\Gamma(v)}[[\vartheta(t_2)-\vartheta(0)]^{v-1} - [\vartheta(t_1)-\vartheta(0)]^{v-1}]$$
$$+\frac{1}{\Gamma(v)}|[\vartheta(t_1)-\vartheta(s)]^{v-1} - [\vartheta(t_2)-\vartheta(s)]^{v-1}\Bigg|.$$

By applying the mean value theorem,then for *a,b* ∈ (t_1, t_2)

$$\Delta \le \frac{2Z_\vartheta^v(1,s)}{Z_\vartheta^v(1,0)\Gamma(v)}|t_2-t_1|\varnothing_1'(a) + \frac{1}{\Gamma(v)}|t_2-t_1|\varnothing_2'(b)$$
$$= |t_2-t_1|\left[\frac{2Z_\vartheta^v(1,s)}{Z_\vartheta^v(1,0)\Gamma(v)}\varnothing_1'(a) + \frac{1}{\Gamma(v)}\varnothing_2'(b)\right].$$

The estimation of Eq. (23) because;

$$|(Q\zeta)(t_2)-(Q\zeta)(t_1|\leq |t_2-t_1|\left[\frac{2\varnothing_1'(a)}{Z_\vartheta^\upsilon(1,0)}\frac{L}{\Gamma(\upsilon)}\int_0^1\vartheta'(s)[\vartheta(1)-\vartheta(s)]^{\upsilon-1}ds\right.$$
$$+\frac{\varnothing_2'(b)L}{\Gamma(\upsilon)}\int_0^1\vartheta'(s)ds]$$
$$=|t_2-t_1|\left[\frac{2\varnothing_1'(a)}{\Gamma(\upsilon+1)}[\vartheta(1)-\vartheta(0)]+\frac{\varnothing_2'(b)}{\Gamma(\upsilon)}[\vartheta(1)-\vartheta(0)]\right]L$$
$$=|t_2-t_1|\left[\frac{2\varnothing_1'(a)}{\Gamma(\upsilon+1)}+\frac{\varnothing_2'(b)}{\Gamma(\upsilon)}\right][\vartheta(1)-\vartheta(0)]L.$$

As $t_2-t_1\to 0$, $|(Q\zeta)(t_2)-(Q\zeta)(t_1)|\to 0$, which means that *(Q B_r)* is equicontinuous. So by Arzela-Ascoli theorem, we conclude that Q is completely continuous.

To apply Theorem (3.5),we need only to prove $Q:\Lambda\to\Lambda$ where;

$$\Lambda=\{w(t):w(t)\in K,\underline{\zeta}(t)\leq w(t)\leq \zeta(t),t\in[0,1]\},$$

and $\|W\| = max\{|W(t)|\leq b;\ t\in[0,1]\}$. Certainly,$\Lambda$ is bounded, closed ,and convex subset of *C[0,1]*. For any $W(t)\in\Lambda$, then $\underline{\zeta}(t)\leq w(t)\leq\bar{\zeta}(t)$, it follows from Definition (4.1) that;

$$(Qw)(t)=\int_0^1\vartheta'(s)G(t,s)\beta(s,w(s))ds\leq\int_0^1\vartheta'(s)G(t,s)\bar{\beta}(s,w(s))ds$$
$$\leq\int_0^1\vartheta'(s)G(t,s)\bar{\beta}(s,\bar{\zeta}(s))ds\leq\bar{\zeta}(t),$$

and;

$$Qw(t)=\int_0^1\vartheta'(s)G(t,s)\beta(s,w(s))ds\geq\int_0^1\vartheta'(s)G(t,s)\underline{\beta}(s,w(s))ds$$

$$\geq\int_0^1\vartheta'(s)G(t,s)\underline{\beta}(s,\underline{\zeta}(s))ds\leq\underline{\zeta}(t).$$

Thus , $Qw(t)\in\Lambda$,due to $\underline{\zeta}(t)\leq Qw(t)\leq\bar{\zeta}(t),$ $t\in[0,1]$.Hence , $Q:\Lambda\to\Lambda$. According to Theorem (3.5),Q has at least one fixed point $\zeta(t)\in\Lambda$ *for* $t\in[0,1]$. Therefor, the problem (2.16) has at least one positive solution $\zeta(t)\in C[0,1]$ and $\underline{\zeta}(t)\leq\zeta(t)\leq\bar{\zeta}(t),\in[0,1]$.

Corollary (4.1):

Let β:[0, 1] × $\mathbb{R}^+\to\mathbb{R}^+$ is continuous , and there exist two constants $L_1(\geq 0)$ and $L_2(\geq 0)$ such that;

$$L_1\leq\beta(t,k)\leq L_2,(t,k)\in[0,1]\times\mathbb{R}^+. \quad (24)$$

Then the problem (17) has at least one positive solution $\zeta(t)\in C[0,1]$. Moreover, for each t∈(0, 1),

$$\zeta(t)\geq L_1\left(\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{-1}-1\right)\frac{(\vartheta(t)-\vartheta(0))^\upsilon}{\Gamma(\upsilon+1)} \quad (25)$$

and

$$\zeta(t)\leq L_2\left(\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{-1}-1\right)\frac{(\vartheta(t)-\vartheta(0))^\upsilon}{\Gamma(\upsilon+1)} \quad (26)$$

proof. from Definition 4.1 and Eq. (24),we have;

$$L_1\leq\underline{\beta}(t,k)\leq\bar{\beta}(t,k)\leq L_2,(t,k)\in[0,1]\times\mathbb{R}^+. \quad (27)$$

Consider the following Fractional Differential Equation (FDE);

$$-D_{0+}^{\upsilon,\vartheta}\bar{\zeta}(t)=L_2,\qquad 0<t<1, \quad (28)$$

$$\bar{\zeta}(0) = \bar{\zeta}(1) = 0$$

Certainly, Eq. (28) has a positive solution;

$$\bar{\zeta}(t) = -\frac{L_2}{\Gamma(v)} \int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{v-1} ds$$

$$+ \left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right))^{v-1} \frac{L_2}{\Gamma(v)} \int_0^1 \vartheta'(s)(\vartheta(1) - \vartheta(s))^{v-1} ds$$

$$= -\frac{L_2(\vartheta(t) - \vartheta(0))^v}{\Gamma(v+1)} + \left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right)^{v-1} \frac{L_2(\vartheta(1) - \vartheta(0))^v}{\Gamma(v+1)}$$

$$= \left(\left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right)^{-1} - 1\right) \frac{L_2(\vartheta(t) - \vartheta(0))^v}{\Gamma(v+1)}. \qquad (29)$$

Taking into account Eq. (28), we can find that;

$$\bar{\zeta}(t) \geq -\frac{1}{\Gamma(v)} \int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{v-1} \bar{\beta}(s, \bar{\zeta}(s)) ds$$

$$+ \left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right)^{v-1} \frac{1}{\Gamma(v)} \int_0^1 \vartheta'(s)(\vartheta(1) - \vartheta(s))^{v-1} \bar{\beta}(s, \bar{\zeta}(s)) ds.$$

Consequently, $\bar{\zeta}$ is the upper solution of Eq. (17). Also, we consider the following FDE;

$$-D_{0+}^{v,\vartheta} \underline{\zeta}(t) = L_2 \qquad 0 < t < 1 \qquad (30)$$

$$\underline{\zeta}(0) = \underline{\zeta}(1) = 0.$$

Certainly, Eq. (30) has a positive solution;

$$\underline{\zeta}(t) = -\frac{L_1}{\Gamma(v)} \int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{v-1} ds$$

$$+ \left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right)^{v-1} \frac{L_1}{\Gamma(v)} \int_0^1 \vartheta'(s)(\vartheta(1) - \vartheta(s))^{v-1} ds$$

$$= -\frac{L_1(\vartheta(t) - \vartheta(0))^v}{\Gamma(v+1)} + \left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right)^{v-1} \frac{L_1(\vartheta(1) - \vartheta(0))^v}{\Gamma(v+1)}$$

$$= \left(\left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right)^{-1} - 1\right) \frac{L_1(\vartheta(t) - \vartheta(0))^v}{\Gamma(v+1)}.$$

Taking into account Eq. (27), we have;

$$\bar{\zeta}(t) \geq -\frac{1}{\Gamma(v)} \int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{v-1} \bar{\beta}(s, \bar{\zeta}(s)) ds$$

$$+ \left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right)^{v-1} \frac{1}{\Gamma(v)} \int_0^1 \vartheta'(s)(\vartheta(1) - \vartheta(s))^{v-1} \bar{\beta}(s, \bar{\zeta}(s)) ds.$$

Therefore, $\underline{\zeta}$ is the lower solution of Eq. (17). So, Theorem (4.1) yields that Eq. (17) has at least one positive solution *ζ(t) ∈ C[0, 1]* which satisfies the inequality Eq. (25) and Eq. (26).

Corollary (4.2):

Suppose β : [0, 1] × $\mathbb{R}^+$ → [*a* , +∞) *is continuous,* see [15]. *where a>0. If;*

$$a < \lim_{c \to +\infty} \beta(t, \zeta) < +\infty. \qquad (29)$$

Then the problem (17) has at least one positive solution;

Proof. By hypotheseis in Eq. (29), there exist $\Upsilon_1, \Upsilon_2 > 0$ such that if $\zeta > \Upsilon_2$,*we have* $\beta(t, \zeta) < \Upsilon_1$.

Let;

$$M = \max_{\substack{0 \leq t \leq 1 \\ 0 \leq \zeta \leq \Upsilon_2}} \beta(t, \zeta). \qquad (30)$$

Then;

$$a \leq \beta(t, \zeta) \leq \Upsilon_1 + M, for \quad 0 < \zeta < +\infty.$$

According to Corollary (4.1), the problem (17) has at least one positive solution *ζ ∈ C[0, 1]*.

Moreover, for each *t∈ (0, 1);*

$$\zeta(t) \geq a \left(\left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right)^{-1} - 1\right) \frac{(\vartheta(t) - \vartheta(0))^v}{\Gamma(v+1)}$$

and;

$$\zeta(t) \leq (\Upsilon_1 + M) \left(\left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right)^{-1} - 1\right) \frac{(\vartheta(t) - \vartheta(0))^v}{\Gamma(v+1)}.$$

The following result is based on the Theorem (3.2).

Theorem (4.2):

Suppose $\beta : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, and there exists a constant $\delta > 0$ such that;

$$|\beta(t,\zeta_1) - \beta(t,\zeta_2)| \le \delta|\zeta_1 - \zeta_2|, t \in (0,1), \quad \zeta_1, \zeta_2 \in \mathbb{R}^+$$

IF;

$$\Omega = \frac{\delta[\vartheta(1) - \vartheta(0)]}{\Gamma(\upsilon + 1)} < 1, \tag{31}$$

then the problem (17) has a unique positive solution $\zeta(t) \in C[0, 1]$.

Proof. Theorem (4.2) assures that Eq. (17) has at least one positive solution in K given by;

$$\zeta(t) = \int_0^1 \vartheta'(s) G(t,s) \beta(s, \zeta(s)) ds.$$

Hence, we need only to show that $Q: C[0, 1] \to C[0, 1]$ defined by;

$$(Q\zeta)(t) = \int_0^1 \vartheta'(s) G(t,s) \beta(s, \zeta(s)) ds.$$

is a contraction in $C[0,1]$. To this end, let $\zeta_1, \zeta_2 \in C[0, 1]$. Then by our assumption and Eq. (21), we have;

$$\begin{aligned}
|(Q\zeta_1)(t) - (Q\zeta_2)(t)| &\le \max_{t\in[0,1]} |(Q\zeta_1)(t) - (Q\zeta_2)(t)| \\
&\le \max_{t\in[0,1]} \int_0^1 \vartheta'(s) |G(t,s)| |\beta(s,\zeta_1(s)) - \beta(s,\zeta_2(s))| ds \\
&\le \frac{\delta}{Z_\vartheta^\upsilon} \frac{1}{\Gamma(\upsilon)} \int_0^1 \vartheta'(s) Z_\vartheta^\upsilon(1,s) |\zeta_1(s) - \zeta_2(s)| ds \\
&\le \frac{\delta}{(\vartheta(1) - \vartheta(0))^{\upsilon-1}} \frac{1}{\Gamma(\upsilon)} ||\zeta_1 - \zeta_2|| \frac{1}{\Gamma(\upsilon)} \int_0^1 \vartheta'(s) (\vartheta(1) - \vartheta(s))^{\upsilon-1} ds \\
&\le \frac{\delta[\vartheta(1) - \vartheta(0)]}{\Gamma(\upsilon+1)} ||\zeta_1 - \zeta_2|| = \Omega ||\zeta_1 - \zeta_2||.
\end{aligned}$$

Because $\Omega < 1$, Q is a contraction. Hence, Theorem (3.2) concludes the problem (17) and has a unique positive solution $\zeta \in C[0, 1]$, for more details see e.g. [16-18].

5. Illustrative Example

Consider the fractional BVP [20]:

$$-D_{0+}^{\frac{5}{3}, sint} \zeta(t) = 1 + \frac{\zeta(t)}{6 + \sin(\zeta(t))}, \qquad 0 < t < 1, \tag{32}$$

where $\upsilon = \frac{5}{3}, \vartheta(t) = \sin t$ *and* $\beta(t,\zeta) = 1 + \frac{\zeta(t)}{6+\sin(\zeta)}$ *it is easy to see that $\beta(t, \zeta)$ is a non-negative and continuous function for all $t \in [0, 1]$ and $\zeta \in [0, +\infty)$. It is clear that;*

$$|\beta(.,\zeta) - \beta(.,\upsilon)| \le \frac{1}{7}|\zeta - \upsilon| = \delta|\zeta - \upsilon|, \forall \zeta, \upsilon \in [0, +\infty).$$

Moreover, by some computations, we get;

$$\Omega = \frac{\delta[\vartheta(1) - \vartheta(0)]}{\Gamma(\upsilon + 1)} = \frac{\frac{1}{7}[\sin(1) - \sin(0)]}{\Gamma(\frac{5}{3} + 1)} = 0.08 < 1.$$

All suppositions of Theorem (4.1) hold. So, Theorem (4.1) guarantees that Eq. (32) has a unique positive solution $\zeta(t) \in C[0, 1]$. Observe that $\beta : [0, 1] \times \mathbb{R}^+ \to [1, +\infty)$ is continuous and

$$1 < \lim_{\zeta \to +\infty} \beta(t, \zeta) < 2$$

Thus, because all the suppositions in Corollary (4.2) are fulfilled with $a=1$, Corollary (4.2) can be applied to the problem (32).

6. Conclusion and Future Scope

The research of generalized FC has become a novel field of investigation. In this paper, positive solutions for the generalized RL-type problem were obtained. Firstly, we presented Green's function and showed its positivity; Next, using the fixed-point approach on a cone, and the upper (lower) solution method with control functions, we investigated some positive solutions. Finally, the uniqueness of the positive solution is proven via Banach's fixed-point approach. The proofs rely on reducing the proposed problem to the equivalent integral equation. One pertinent example is offered to illustrate the fundamental results. Many special cases for our problem arise with a certain function selection of ϑ. The epilog obtained in this work will be very advantageous in the applications. Also, we anticipate finding some applications in further nonlinear problems.

Data Availability

There is no data to provide other than that given in the article.

Conflict of Interest

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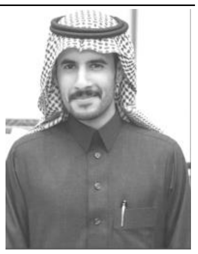

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