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Prime labeling of Union Graphs

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Abstract- A graph with n vertices is called prime graph if its vertices can be labeled with first n positive integers such that each pair of adjacent vertices have relatively prime labels. Such a labeling is called prime labeling. It is known that graphs like gear graphs, crown graphs, helm graphs, book graphs and $C_n(C_n)$ are prime. In this paper, we study the prime labeling of union of these graphs.

Keyword: Prime labeling, union of graphs, Corona product, Cartesian Product.
AMS Subject Classification(2010): 05C7

I. INTRODUCTION

All graphs considered in this paper are simple, finite, and undirected. For a graph G, $V(G)$ and $E(G)$ denotes the vertex set and edge set of G respectively whereas $|V(G)|$ and $|E(G)|$ denotes the cardinality of the respective sets. For terms not defined here, we refer the reader to [6].

Paths and cycles are the immediate examples of prime graphs but on the other hand the conjecture that trees are prime is not settled even after thirty-five years of research in this area. This makes the study of prime labeling a very interesting area of research. See [1] for a summary of prime labeling and prime graphs. The present paper is motivated from the existing examples of prime graphs like gear graphs, crown graphs, helms graphs and book graphs. The proof of these results are discussed in [2], [3] and [4]. Here we investigate the prime labeling of union of these graphs. For better understanding of the proofs, every theorem is supported with an appropriate example and a related figure. It is advisable to see these examples and figures while reading the proofs.

The following results regarding relatively prime integers are quite useful through out the paper. Since their proofs are elementary we only state them in the form of a Lemma.

Lemma 1.3 Let a and b be positive integers. Then

1. $gcd(a,b)=1$, whenever
   (i) $a=1$ and b is arbitrary
   (ii) $a=2^k$ and b is an odd integer
   (iii) $a-b=2^k$ whereas a and b are odd integers
   (iv) $a=mb+1$, where m is any integer.
2. $gcd(a,b)=gcd(a\pm b,b)$.
3. $gcd(a,b)=gcd(a,2^k b)$, where a is an odd integer.
4. For arbitrary integers m_1, m_2, n_1 *and* n_2, if $gcd(m_1a+m_2b, n_1a+n_2b)=1$, then $gcd(a,b)=1$.

Now we discuss some basic definions which will be used in main results.

Definition 1.1 *A graph with the vertex set* $V=\{u_0,u_1,u_2,\dots,u_n\}$ *for* $n\geq 3$ *and the edge set* $E=\{u_0u_i: 1\leq i\leq n\}\cup\{u_iu_{i+1}: 1\leq i\leq n-1\}\cup\{u_nu_1\}$ *is called wheel graph* W_n *of length* n.

Definition 1.2 *The helm* H_n *is the graph obtained from a wheel* W_n *by attaching a pendent edge at each vertex of the* n*-cycle.*

Definition 1.3 *A gear graph* G_n *is obtained from the wheel graph* W_n *by adding a vertex between every pair of adjacent vertices of the* n*-cycle.*

Definition 1.4 *For a graph* G, *if every edge of graph* G *is subdivided, then resulting graph is called barycentric subdivision of graph* G. *In other words barycentric subdivision is the graph obtained by inserting a vertex of degree 2 into every edge of original graph.*

Definition 1.5 *The corona product of graphs* G *and* H *is* *the graph $G \odot H$ obtained by taking one copy of G, called the center graph, $|V(G)|$ copies of H, called the outer graph, and making the i^{th} vertex of G adjacent to every vertex of the i^{th} copy of H, where $1 \le i \le |V(G)|$.*

Definition 1.6 *The Cartesian product of G and H is a graph, denoted by $G \times H$ whose vertex set is $V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent if $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$. Thus $V(G \times H) = \{(g,h): g \in V(G)$ and $h \in V(H)\}$, $E(G \times H) = \{(g,h)(g',h'): g = g', hh' \in E(H)$ or $gg' \in E(G)$, $h = h'\}$*

Definition 1.7 *Crown graph is corona product of n-cylce and complete graph K_1.*

Definition 1.8 *Book graph is cartesian product of star graph $K_{1,n}$ and path graph P_2.*

II. MAIN RESULTS

Theorem 2.1 *A union of two copies of crown graph $C_n \odot K_1$ is prime graph for all n.*

Proof. Suppose G is a union of two copies of crown graph $C_n \odot K_1$ with vertex set $V(G) = \{v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n\} \cup \{u_1, u_2, \dots, u_n, u'_1, u'_2, \dots, u'_n\}$ and $E(G) = \{v_i v'_i, v_n v'_n, v_i v_{i+1}, v_1 v_n, u_i u'_i, u_n u'_n, u_i u_{i+1}, u_1 u_n : i = 1,2,\dots,n-1\}$. We will discuss the prime labeling of G for the following two cases on n.

Case 1: $n \equiv 0(mod\ 3)$.

Here we define $f: V(G) \to \{1,2,\dots,4n\}$ by

$f(v_i) = 2i - 1,$
$f(v'_i) = f(v_i) + 1,$
$f(u_i) = 2n + 2i - 1,$
$f(u'_i) = f(u_i) + 1;$

where $i = 1,2,\dots,n$. Then for $1 \le i \le n-1$, $gcd\big(f(v_i), f(v_{i+1})\big) = 1 = gcd\big(f(u_i), f(u_{i+1})\big)$ are consecutive odd integers and for $1 \le i \le n$, $gcd(f(v_i), f(v'_i)) = 1 = gcd(f(u_i), f(u'_i))$ as they are consecutive integers. Also $gcd(f(v_1), f(v_n)) = gcd(1, f(v_n)) = 1$ and $gcd\big(f(u_1), f(u_n)\big) = gcd(2n+1, 4n-1) = \mathrm{gcd}(4n+2, 4n-1) = \mathrm{gcd}(3, 4n-1) = 1$ as $n \equiv 0(mod 3)$.

Case 2: $n \not\equiv 0(mod\ 3)$.

In this case define $f: V(G) \to \{1,2,\dots,4n\}$ by

$f(v_i) = 2i - 1, \quad i = 1,2,\dots,n$
$f(v'_i) = f(v_i) + 1, \quad i = 1,2,\dots,n$
$f(u_i) = 2n + 2i - 1, \quad i = 1,2,\dots,n-1$
$f(u'_i) = f(u_i) + 1, \quad i = 1,2,\dots,n-1$
$f(u_n) = 4n, \quad f(u'_n) = 4n - 1$

Once again, for $1 \le i \le (n-2)$, $gcd(f(v_i), f(v_{i+1})) = 1 = gcd(f(u_i), f(u_{i+1}))$ and also $gcd(f(v_{n-1}), f(v_n)) = 1$ because they are consecutive odd integers and for $1 \le i \le n$, $gcd(f(v_i), f(v'_i)) = 1 = gcd(f(u_i), f(u'_i))$ as they are consecutive integers. Moreover $gcd(f(v_1), f(v_n)) = 1$ because $f(v_1) = 1$. Further, since $n \not\equiv 0(mod\ 3)$, we have $gcd(f(u_{n-1}), f(u_n)) = gcd(4n-3, 4n) = 1$. Also $gcd\big(f(u_1), f(u_n)\big) = gcd(2n+1, 4n) = 1$ and this concludes that G is prime graph.

Example 2.1 *Prime labeling of $(C_6 \odot K_1) \cup (C_6 \odot K_1)$ and $(C_7 \odot K_1) \cup (C_7 \odot K_1)$ are shown in Figure 1 and Figure 2 respectively.*

Theorem 2.2 *A union of two copies of a gear graph G_n is a prime graph for all n.*

Proof. Suppose G is a union of two copies of a gear graph G_n with vertex set $V(G) = \{v_0, v_1, v_2, \dots, v_{2n}\} \cup \{u_0, u_1, u_2, \dots, u_{2n}\}$. Here v_0 is the apex vertex and $v_1, v_2, \dots, v_{2n}$ are the rim vertices of the first copy in which vertices with even suffix are of degree 2 and those with odd suffix are of degree 3. We name the vertices of the second copy of G_n in the same fashion using the set $\{u_0, u_1, u_2, \dots, u_{2n}\}$.

Case 1: $n \not\equiv 2(mod\ 3)$.

In this case define $f: V(G) \to \{1,2,\dots,4n+2\}$ by

$f(v_0) = 1, \quad f(u_0) = 2$
$f(v_i) = i + 2, \quad i = 1,2,\dots,2n$
$f(u_i) = 2n + i + 2, \quad i = 1,2,\dots,2n$

Since 3 does not divide $2n+2$ for $n \not\equiv 2(mod\ 3)$ and $gcd(2n+3, 4n+2) = 1$, it is easy to verify that f defines a prime labeling on G.

Case 2: $n \equiv 2(mod\ 3)$.

In this case define $f: V(G) \to \{1,2,\dots,4n+2\}$ by

$f(v_0) = 1, \quad f(u_0) = 2$
$f(v_1) = 2n + 3, \quad f(u_1) = 3$
$f(v_i) = i + 2, \quad i = 2,\dots,2n$
$f(u_i) = 2n + i + 2, \quad i = 2,\dots,2n$

Since 3 divides neither $4n+2$ nor $2n+4$ when $n \equiv 2(mod\ 3)$ and further $gcd(2n+3, 4) = 1$, it is easy to verify that f defines a prime labeling on G.

Example 2.2 *Prime labeling of $G_5 \cup G_5$ and $G_6 \cup G_6$ are shown in Figure 3 and Figure 4 respectively.*

Theorem 2.3 *The union of two copies of a helm graph H_n is a prime graph for all n.*

Proof. Suppose G is a union of two copies of a helm graph H_n with vertex set $V(G) = \{v_0, v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n\} \cup \{u_0, u_1, u_2, \dots, u_n, u'_1, u'_2, \dots, u'_n\}$; where v_0 is the apex vertex, $v_1, v_2, \dots, v_n$ are the rim vertices and $v'_1, v'_2, \dots, v'_n$ are the pendent vertices adjacent to $v_1, v_2, \dots, v_n$ respectively in the first copy of the helm graph H_n. We name the vertices of the second copy of H_n in the same fashion using the set $\{u_0, u_1, u_2, \dots, u_n, u'_1, u'_2, \dots, u'_n\}$.

Define $f: V(G) \to \{1,2,\dots,4n+2\}$ by

$f(v_0) = 1, \quad f(u_0) = 2$
$f(v_1) = 4, \quad f(v'_1) = 3,$
$f(v_i) = 2i+1, \quad f(v'_i) = f(v_i)+1,$ for $i = 2,3,\dots,n-1$
$f(v_n) = 2n+3, f(v'_n) = f(v_n)+1$
$f(u_j) = 2n+2j+1, \ f(u'_j) = f(u_j)+1,$ for $j = 2,3,\dots,n$
$f(u_1) = 2n+1, \quad f(u'_1) = f(u_1)+1$

Now using the results that $gcd(2n+3,4) = 1, gcd(2n+3,2n-1) = 1, gcd(2n+1,2n+5) = 1$ and $gcd(2n+1,4n+1) = 1$, along with few other elementary observations, it may easily be verified that $gcd(f(u), f(v)) = 1$ whenever u and v are adjacent vertices. Hence G is a prime graph.

Example 2.3 *Prime labeling of $H_6 \cup H_6$ is shown in Figure 5.*

Theorem 2.4 *Union of helm graph H_n and gear graph G_n is prime for all n.*

Proof. Let G be the union of helm graph H_n and gear graph G_n. Suppose the vertex sets of H_n and gear graph G_n are $\{v_0, v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n\}$ and $\{u_0, u_1, u_2, \dots, u_{2n}\}$ respectively.

Define $f: V(G) \to \{1,2,\dots,4n+2\}$ by

$f(v_0) = 1, \quad f(u_0) = 2$
$f(v_1) = 4, \quad f(v'_1) = 3,$
$f(v_i) = 2i+1, f(v'_i) = f(v_i)+1,$ for $i = 2,3,\dots,n$
$f(u_j) = 2n+j+2,$ for $j = 1,2,3,\dots,2n$

Then for $1 \le i \le n$, we have $gcd(f(v_0), f(v_i)) = 1$ as $f(v_0) = 1$ and $gcd(f(u_0), f(u_{2i+1})) = gcd(2,2n+2i+3) = 1$. Also $gcd(f(v_1), f(v_n)) = gcd(4, 2n+1) = 1$ and $gcd(f(u_1), f(u_{2n})) = gcd(2n+3,4n+2) = 1$. The rest of the adjacent vertices are labeled with either consecutive integers or consecutive odd integers, so they are relatively prime. Thus f is prime labeling on G.

Example 2.4 *Prime labeling of $H_6 \cup G_6$ is shown in Figure 6.*

Theorem 2.5 *Union of crown graph $C_n \odot K_1$ and gear graph G_n is prime for all n.*

Proof. Let G be the union of helm graph $C_n \odot K_1$ and gear graph G_n. Suppose the vertex sets of $C_n \odot K_1$ and gear graph G_n are $\{v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n\}$ and $\{u_0, u_1, u_2, \dots, u_{2n}\}$ respectively.

Define $f: V(G) \to \{1,2,\dots,4n+1\}$ by

$f(v_1) = 2, f(v'_1) = 3$
$f(v_i) = 2n+2i+1, f(v'_i) = f(v_i)-1,$ for $i = 2,3,\dots,n$
$f(u_0) = 1, \quad f(u_1) = 2n+3$
$f(u_i) = i+2,$ for $i = 2,3,\dots,2n$

Then for $1 \le i \le n$, we have,

$gcd(f(u_0), f(u_{2i+1})) = 1$ as $f(u_0) = 1$,
$gcd(f(u_1), f(u_2)) = gcd(2n+3,4) = 1,$
$gcd(f(v_1), f(v_2)) = gcd(2,2n+5) = 1,$
$gcd(f(v_1), f(v_n)) = gcd(2,4n+1) = 1.$

Rest of the adjacent vertices are labeled with either consecutive integers or consecutive odd integers, so they are relatively prime. Thus f is prime labeling on G.

Example 2.5 *Prime Labeling of $(C_5 \odot K_1) \cup G_5$ is shown in Figure 7.*

Theorem 2.6 *Union of crown graph $C_n \odot K_1$ and helm graph H_n is prime for all n.*

Proof. Let G be the union of crown graph $C_n \odot K_1$ with helm graph H_n and suppose the vertex sets of $C_n \odot K_1$ and H_n are $\{v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n\}$ and $\{u_0, u_1, u_2, \dots, u_n, u'_1, u'_2, \dots, u'_n\}$ respectively.

Define $f: V(G) \to \{1,2,\dots,4n+1\}$ by

$f(u_0) = 1$
$f(u_1) = 2, \quad f(u'_1) = 3,$
$f(u_i) = 2i+1, \ f(u'_i) = f(u_i)-1,$ for $i = 2,3,\dots,n-1$
$f(u_n) = 2n+3, f(u'_n) = f(u_n)-1,$
$f(v_1) = 2n+1, f(v'_1) = f(v_1)-1,$
$f(v_i) = 2n+2i+1, f(v'_i) = f(v_i)-1,$ for $i = 2,3,\dots,n-1$
$f(v_n) = 4n, f(v'_n) = f(v_n)+1.$

Since $gcd(f(u_n), f(u_{n-1})) = gcd(2n+3,2n-1) = 1,$
$gcd(f(v_1), f(v_2)) = gcd(2n+1,2n+5) = 1,$
$gcd(f(v_1), f(v_n)) = gcd(2n+1,4n) = 1,$
once again it is not difficult to verify that f is a prime labeling on G.

Example 2.6 *Prime labeling of $(C_4 \odot K_1) \cup H_4$ is shown in Figure 8.*

Our next result is about the union of book graphs.

Theorem 2.7 *Union of two copies of book graph $B_n = K_{1,n} \times P_2$ is prime labeling for all n.*

Proof. Let G be a graph which is a union of two copies of B_n where $V(G) = \{u_0, u_1, u_2, \dots, u_n, u'_0, u'_1, u'_2, \dots, u'_n\} \cup \{v_0, v_1, v_2, \dots, v_n, v'_0, v'_1, v'_2, \dots, v'_n\}$ and $E(G) = \{u_0u'_0, u_0u_i, u'_0u'_i, u_iu'_i : i = 1,2,\dots,n\} \cup \{v_0v'_0, v_0v_i, v'_0v'_i, v_iv'_i : i = 1,2,\dots,n\}$. Thus $|V(G)| = 4(n+1)$ and $|E(G)| = 6n+2$. We construct a prime labeling $f: V(G) \to \{1,2,\dots,4(n+1)\}$ as follows.

Suppose $A = \{5,6,\dots,4(n+1)\}$ and B is the set of all even integers in A which are not divisible by 6. Then it is to see that $|B| > n$. Label $v_1, v_2, \dots, v_n$ using the first n consecutive integers of B and call them $f(v_1), f(v_2), \dots, f(v_n)$ respectively. Now label

$u_1, u_2, \dots, u_n$ randomly using all the n integers of the set $\{6,8,10,\dots,4n+2,4n+4\}\backslash\{f(v_1), f(v_2), \dots, f(v_n)\}$ and call them $f(u_1), f(u_2), \dots, f(u_n)$ respectively. Further, define $f(v_{i'}) = f(v_i) - 1, f(u'_i) = f(u_i) - 1$ for $i = 1,2,\dots,n$ where as $f(u_0) = 1, f(u'_0) = 2, f(v_0) = 3$ and $f(v'_0) = 4$. We claim that the bijection $f: V(G) \rightarrow \{1,2,\dots,4(n+1)\}$ defined above is a prime labeling. As $f(u_0) = 1$, $f(u_0)$ and $f(u_i)$ are relatively prime. Since $f(v_0) = 3$ and $f(v_i)$ are even integers different from the multiples of 6, $gcd(f(v_i), f(v_0)) = 1$. Also $f(u'_0) = 2, f(v'_0) = 4$ whereas $f(u'_i)$ and $f(v'_i)$ are odd integers for all i , and hence $gcd(f(u'_i), f(u'_0)) = 1 = gcd(f(v'_i), f(v'_0))$. The remaining edges of the graph G are of the form $u_i u'_i$ or $v_i v'_i$ whose end points are labeled with consecutive integers and hence this proves our claim that f is a prime labeling on G.

Example 2.7 *Prime labeling of $B_8 \cup B_8$ is shown in Figure 9.*

Note that by removing the edges $v_i v'_i$ and $u_i u'_i$ from the graph $G = B_n \cup B_n$, we obtain a graph which is the union of two copies of a bistar graph. Hence Theorem 10 immediately gives the following corollary.

Corollary 2.8 *A graph, which is union of two copies of bistar graph, is prime graph.*

Our final result is about the graph $C_n(C_n)$ which was introduced by S. K. Vaidya et al [5]. It is obtained by taking the barycentric subdivision of a cycle C_n and joining newly inserted vertices of incident edges by an edge. It looks like C_n inscribed in C_n. See for instance, Figure 10.

Theorem 2.9 *Union of two copies of $C_n(C_n)$ is a prime graph for $n \not\equiv 1(mod\ 3)$.*

Proof. Let G be a graph which is a union of two copies of $C_n(C_n)$ with $V(G) = \{v_{i,j}, u_{i,j}: i = 1,2,\dots,n; j = 1,2\}$where for $j = 1,2$; $v_{1,j}, v_{2,j}, \dots, v_{n,j}$ denote the vertices of the (outer) cycle in the j^{th} copy of $C_n(C_n)$ and $u_{1,j}$, $u_{2,j}, \dots, u_{n,j}$ denote the newly inserted vertices on the edges $v_{n,j}v_{1,j}$, $v_{1,j}v_{2,j}, \dots, v_{n-1,j}v_{n,j}$ of this (outer) cycle respectively. Thus $E(G) = \{v_{i,j}u_{i+1,j}, v_{n,j}u_{1,j}, u_{i,j}u_{i+1,j}, u_{1,j}u_{n,j}, u_{i,j}v_{i,j}, u_{n,}v_{n,j}: i = 1,2,\dots,n-1; j = 1,2\}$ whereas $|V(G)| = 4n$ and $|E(G)| = 6n$.

Define $f: V(G) \rightarrow \{1,2,\dots,4n\}$ by

$f(v_{i,j}) = 2i + 2n(j-1)$,

$f(u_{i,j}) = f(v_{i,j}) - 1$.

Note that for the edges $u_{1,1}u_{n,1}$ and $u_{1,1}v_{n,1}$; $gcd(f(u_{1,1}), f(u_{n,1})) = 1 = gcd(f(u_{1,1}), f(v_{n,1}))$, since $f(u_{1,1}) = 1$. Further, for the edge $u_{1,2}v_{n,2}$; $gcd(f(u_{1,2}), f(u_{n,2})) = gcd(2n+1, 4n-1) = 1$, since $n \not\equiv 1(mod\ 3)$ and for the edge $u_{1,2}v_{n,2}$ $gcd(f(u_{1,2}), f(v_{n,2})) = gcd(2n+1, 4n) = gcd(2n+1, 2) = 1$. The end points of all the remaining edges of the graph G are labeled with either consecutive integers or consecutive odd integers. So f is a prime labeling on graph G.

Example 2.8 *Prime labeling of $C_n(C_n) \cup C_n(C_n)$ is shown in Figure 10.*

III. CONCLUSION

In this paper we have mainly derived results about prime labeling of union of crown, helm, gear and book graphs. It may be noted that while considering the union of graphs we have assumed (in some sense) that both the graphs are of same order. We are not sure if these results hold when these graphs are of different orders. So, this may be considered as a future scope of study in this direction. Ofcourse, one may also think about union of many other graph families which are known to be prime but nothing is known about their union graphs.

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AUTHORS PROFILE

S. K. Patel is a Professor of Mathematics at the Government Engineering College-Bhuj affiliated to the Gujarat Technological University. He has a teaching experience of more than 20 years and his research areas are Harmonic analysis and Graph theory. He is also a good problem solver especially in Real analysis. He has solved many problems posted in American Mathematical Monthly, Mathematics Magazine and College Mathematics Journal which are all published by the Mathematical Association of America and all his solutions are acknowledged in these journals. Recently, he was honored by GUDMAA (Gujarat University Department of Mathematics Alumni Association) for his achievements and contributions to the Mathematics community of Gujarat.

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Figures

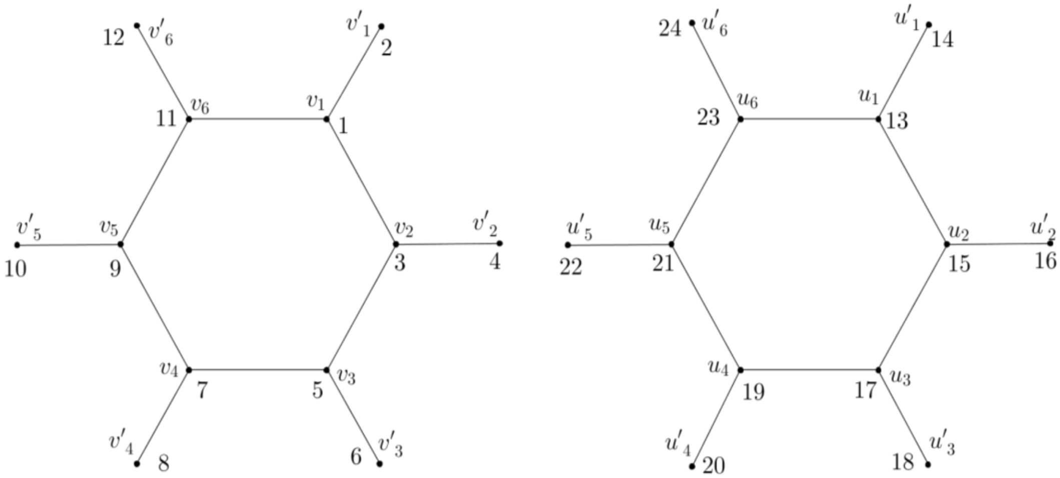

Figure 1 : Prime Labeling of $(C_6 \odot K_1) \cup (C_6 \odot K_1)$

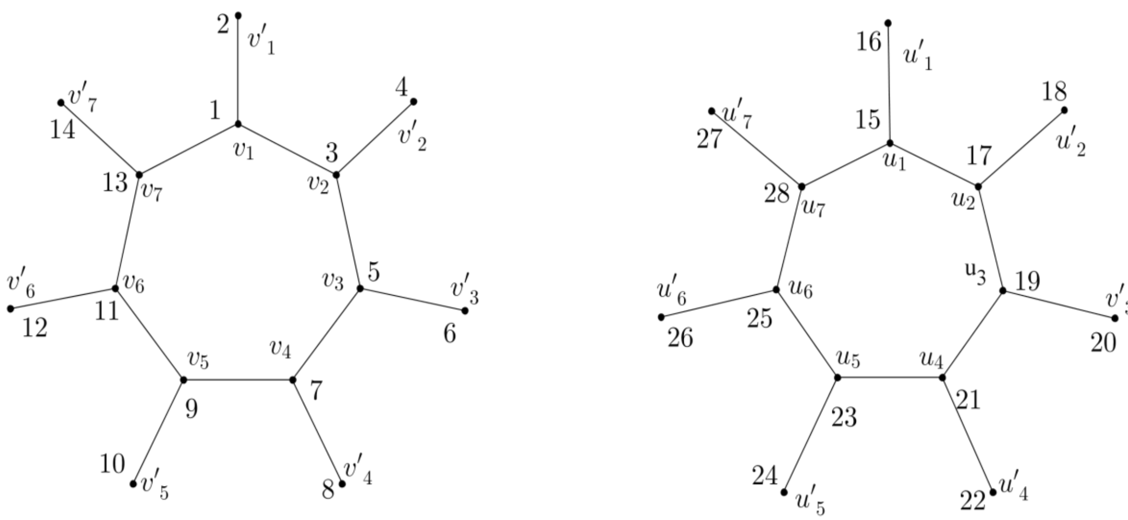

Figure 2 : Prime Labeling of $(C_7 \odot K_1) \cup (C_7 \odot K_1)$

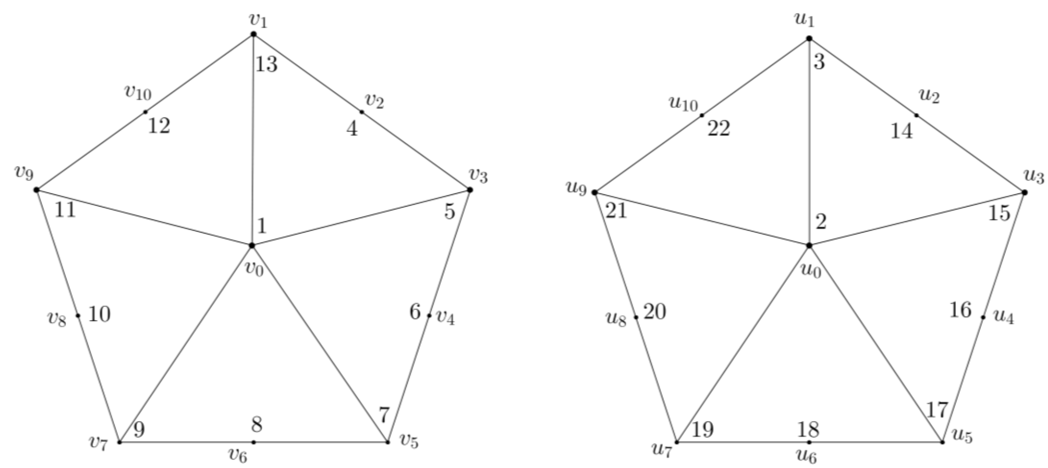

Figure 3: Prime Labeling of $G_5 \cup G_5$

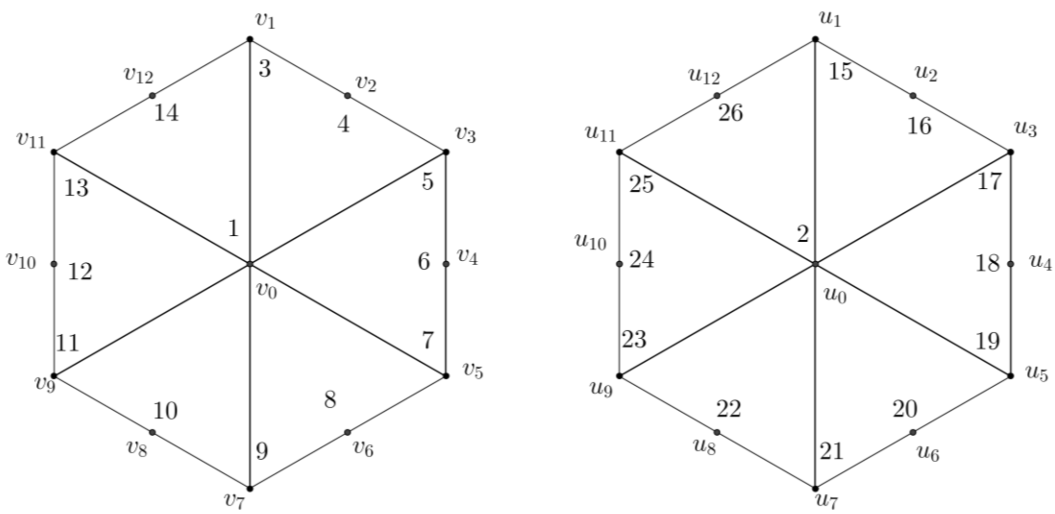

Figure 4: Prime Labeling of $G_6 \cup G_6$

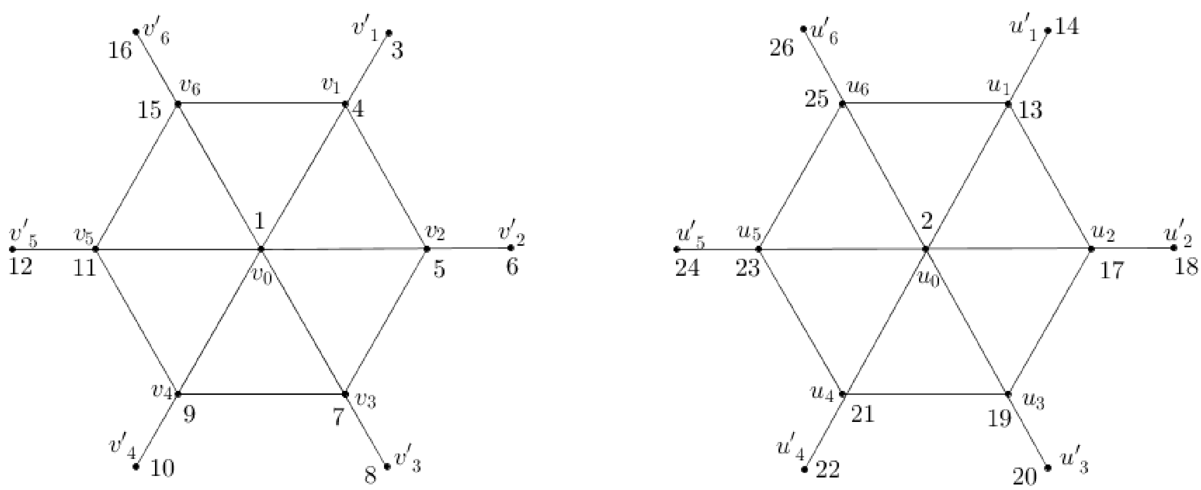

Figure 5: Prime Labeling of $H_6 \cup H_6$

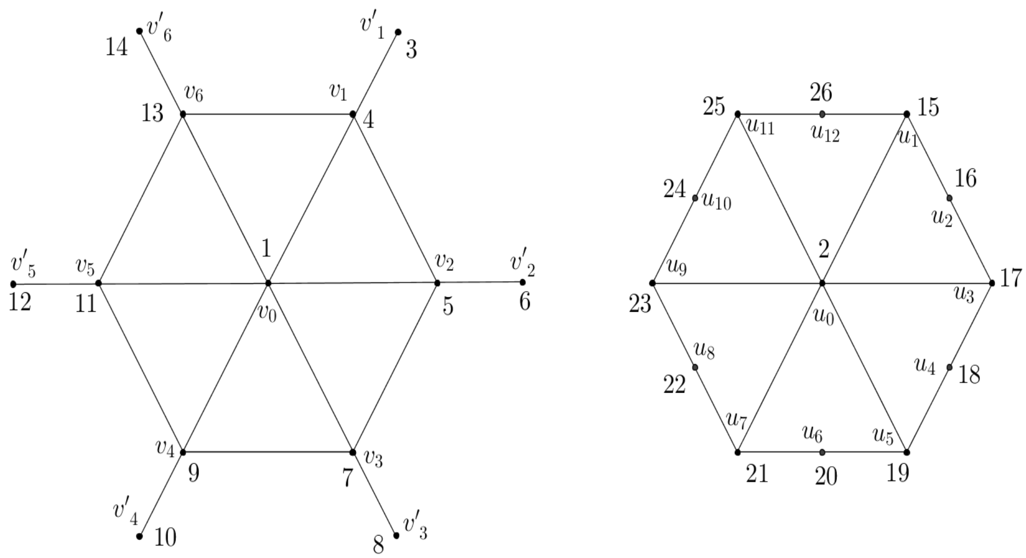

Figure 6: Prime Labeling of $H_6 \cup G_6$

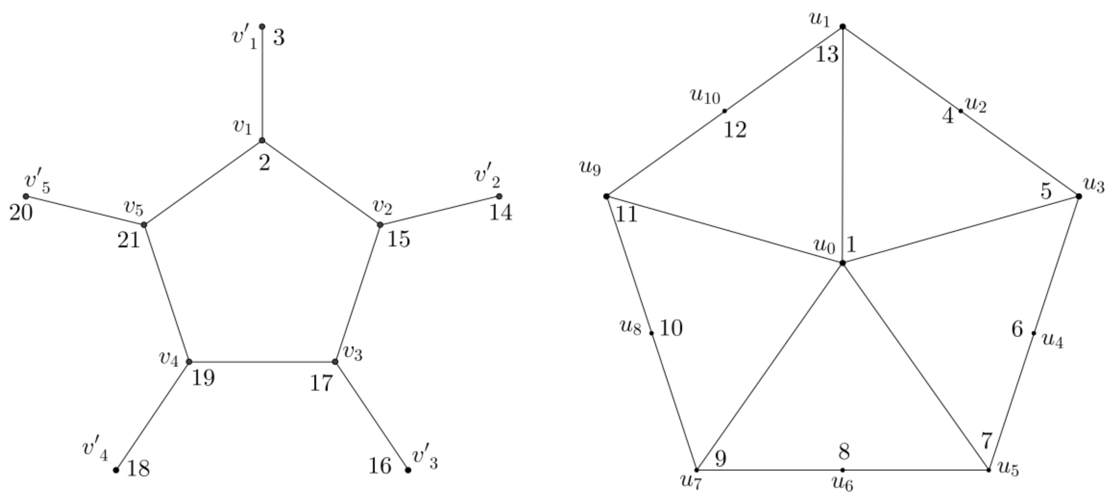

Figure 7: Prime Labeling of $(C_5 \odot K_1) \cup G_5$

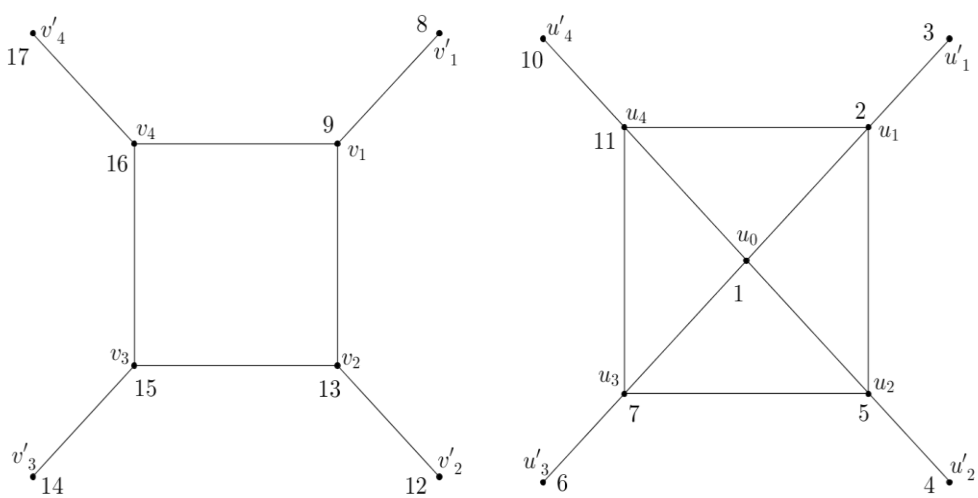

Figure 8: Prime Labeling of $(C_4 \odot K_1) \cup H_4$

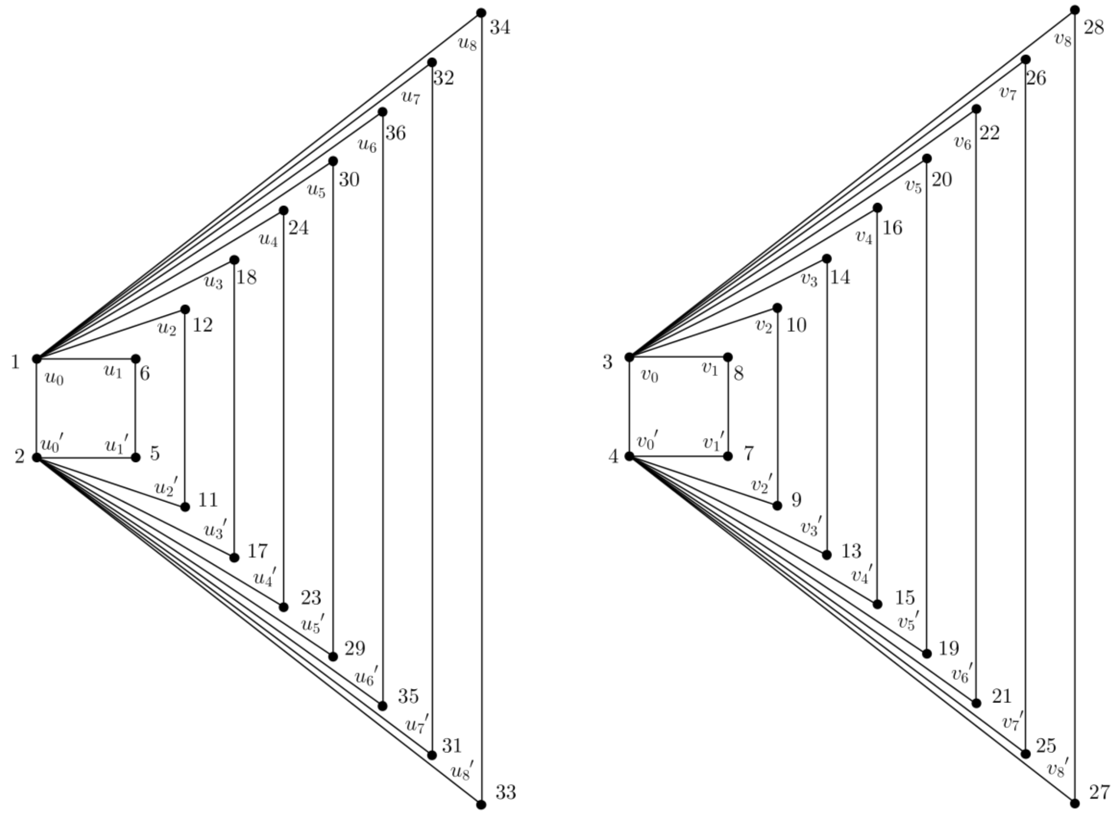

Figure 9: Prime Labeling of $B_8 \cup B_8$

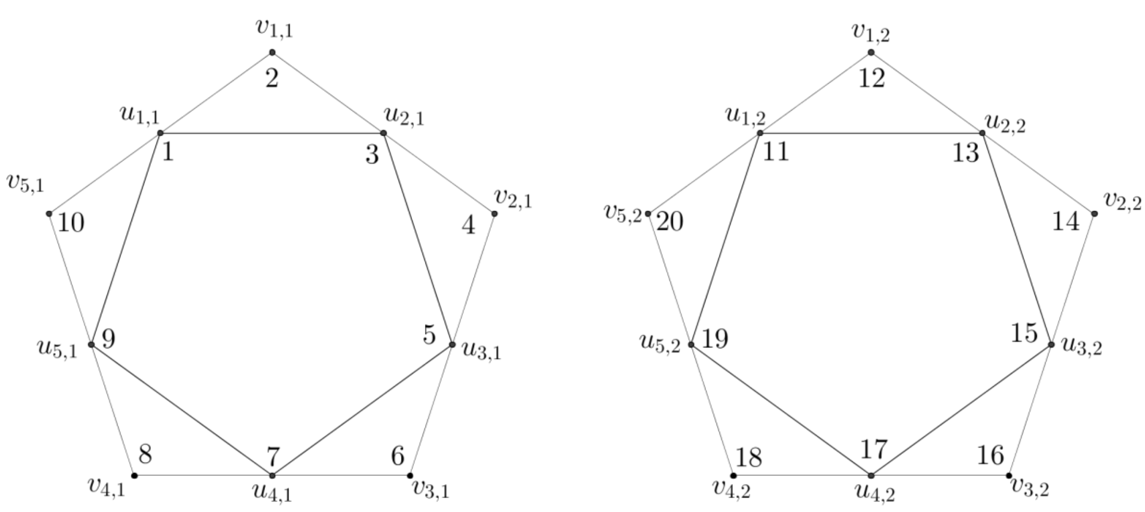

Figure 10: Prime Labeling of $C_5(C_5) \cup C_5(C_5)$

