International Journal of Scientific Research in Mathematical and Statistical Sciences
Research Paper
Vol.6, Issue.4, pp.67-69, August (2019)
E-ISSN: 2348-4519
DOI: https://doi.org/10.26438/ijsrmss/v6i4.6769

Eigenvalues of Status Sum Adjacency Matrix of Graphs

S.Y. Talwar[1], H.S. Ramane[2*]

[1,2]Department of Mathematics, Karnatak University, Dharwad – 580003, India

[*]Corresponding Author: hsramane@yahoo.com, Tel.: +91-9945031752

Available online at: www.isroset.org

Received: 08/Jul/2019, Accepted: 12/Aug/2019, Online: 31/Aug/2019

***Abstract*—**The distance between two vertices is the length of a shortest path joining them. The status $\sigma(u)$ of vertex u in a connected graph G is defined as sum of distances between u and all other vertices in graph G. The status sum adjacency matrix of a graph G is $S_A(G) = [s_{ij}]$ in which $s_{ij} = \sigma(u) + \sigma(v)$ if u and v are adjacent vertices and $s_{ij} = 0$, otherwise. In this paper we initiate the study of eigenvalues of status sum adjacency matrix of graphs**.**

Keywords*—*Distance in graphs, status of a vertex, status sum adjacency matrix, eigenvalues

I. INTRODUCTION

Let G be a simple, connected graph with n vertices and m edges. Let the vertex set of G be $V(G) = \{v_1, v_2, \ldots, vn\}$ and edge set be $E(G)$. The edge joining the vertices u and v is denoted by uv. The adjacency matrix of a graph G is $n\times n$ matrix $A(G) = [a_{ij}]$, in which $a_{ij} = 1$, if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The eigenvalues of $A(G)$ are called the adjacency eigenvalues of G and can be ordered as $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ [1].

It is easy to observe that

$$\sum_{i=1}^{n} \lambda_i = 0$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = 2m \cdot$$

The adjacency eigenvalues of a complete graph K_n are $n - 1$ and -1 ($n - 1$ times). The eigenvalues of a complete bipartite graph $K_{p,q}$ are $\pm\sqrt{pq}$ and 0 ($p + q - 2$ times).

The distance between two vertices u and v denoted by $d(u,v)$, is the length of a shortest path joining them. Status of a vertex u in a graph G is defined as [2]

$$\sigma(u) = \sum_{v\in V(G)} d(u,v) \cdot$$

A graph is said to be r-distance balanced if all the vertices have same status equal to r.

The status sum adjacency matrix $S_A(G)$ of a connected graph G is an $n\times n$ matrix $S_A(G) = [s_{ij}]$, in which $s_{ij} = \sigma(v_i) + \sigma(v_j)$ if v_i and v_j are adjacent and $s_{ij} = 0$, otherwise. The eigenvalues of $S_A(G)$ are called as status sum adjacency eigenvalues and can be labeled as $x_1 \geq x_2 \geq \ldots \geq x_n$.

The status sum adjacency eigenvalues satisfies that [3]

$$\sum_{i=1}^{n} x_i = 0$$

and

$$\sum_{i=1}^{n} x_i^2 = 2 \sum_{uv\in E(G)} [\sigma(u) + \sigma(v)]^2 \,.$$

Lemma 1.1 [4] If $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are n-vectors, then Cauchy-Schwatz inequality is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

The paper has organized as follows. Section I gives the introduction. Section II gives the survey of previous work. In Section III we obtain the bounds for the status sum adjacency eigenvalues of a graph. Section IV gives the conclusion.

II. RELATED WORK

The vast study of eigenvalues of adjacency matrix is carried by many researchers [1]. Bounds for the adjacency eigenvalues and Laplacian eigenvalues of a graph are obtained in [5]. Bounds for the signless Laplacian eigenvalues are obtained in [6]. The work on status connectivity indices and coindices are carried out in [3]. Status connectivity indices and coindices of composite graphs are obtained in [7]. Zagreb radio indices are studied in [8].

III. BOUNDS FOR EIGENVALUES OF $S_A(G)$

We use the proof techniques of [5] to prove our results in this section..

Lemma 3.1. Let G be a graph with n vertices and m edges and with adjacency eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Let H be another connected graph with n vertices and status sum adjacency eigenvalues $x_1 \geq x_2 \geq \ldots \geq x_n$. Then

$$\sum_{i=1}^{n} \lambda_i x_i \leq 2\sqrt{m \sum_{uv\in E(H)} [\sigma(u)+\sigma(v)]^2}. \qquad (1)$$

Proof. By using Lemma 1.1 we have,

$$\left(\sum_{i=1}^{n} \lambda_i x_i\right)^2 \leq \left(\sum_{i=1}^{n} \lambda_i^2\right)\left(\sum_{i=1}^{n} x_i^2\right)$$

$$\left(\sum_{i=1}^{n} \lambda_i x_i\right)^2 \leq 2m\left(2\sum_{uv\in E(H)} [\sigma(u)+\sigma(v)]^2\right).$$

Therefore

$$\sum_{i=1}^{n} \lambda_i x_i \leq 2\sqrt{m \sum_{uv\in E(H)} [\sigma(u)+\sigma(v)]^2}.$$

Theorem 3.2. Let G be a connected graph with n vertices and m edges. Let x_1 be the maximum eigenvalue of $S_A(G)$. Then

$$x_1 \geq \frac{2}{n}\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]. \qquad (2)$$

Proof. By the definition of $S_A(G) = [s_{ij}]$, we observe that the sum of all the entries of $S_A(G)$ is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij} = 2\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)].$$

Let $y = [1, 1, \ldots 1]$ be the all one vector. Then by the Rayleigh's principal, we have

$$x_1 \geq \frac{yS_A(G)y^T}{yy^T} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij} = \frac{2}{n}\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)].$$

Corollary 3.3. Let G be a r-distance balanced graph with n vertices and m edges. Let x_1 be the largest eigenvalue of $S_A(G)$. Then

$$x_1 \geq \frac{4mr}{n}.$$

Proof. If G is an r-diatance balanced graph, then for each $u \in V(G)$, $\sigma(u) = r$. Hence by Eq. (2), we have

$$x_1 \geq \frac{2}{n}\sum_{uv\in E(G)} 2r = \frac{4mr}{n}$$

Theorem 3.4. Let G be a connected graph with n vertices and let $x_1 \geq x_2 \geq \ldots \geq x_n$ be the status sum adjacency eigenvalues of G. Then for $2 \leq p \leq n$,

$$x_1 \leq \sqrt{\frac{2p}{p-1}\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^2} + \frac{1}{p-1}\sum_{i=n-p+2}^{n} x_i. \qquad (3)$$

Proof. Let $H = K_p \cup \overline{K}_{n-p}$. The adjacency eigenvalues of H are $p - 1$, 0 ($n - p$ times) and $- 1$ ($p - 1$ times). The number of edges of H are $p(p - 1)/2$. Using Eq. (1), we have

$$(p-1)x_1 + (0)\sum_{i=2}^{n-p+1} x_i - \sum_{i=n-p+2}^{n} x_i \leq 2\sqrt{\frac{p(p-1)}{2}\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^2}$$

$$(p-1)x_1 \leq \sqrt{2p(p-1)\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^2} + \sum_{i=n-p+2}^{n} x_i$$

$$x_1 \leq \sqrt{\frac{2p}{p-1}\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^2} + \frac{1}{p-1}\sum_{i=n-p+2}^{n} x_i.$$

If we put $p = 2$ in Eq. (3), then we get following result.

Corollary 3.5. Let G be connected graph on n vertices. Then

$$x_1 - x_n \leq 2\sqrt{\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^2}. \qquad (4)$$

Corollary 3.6. Let G be a connected graph with n vertices. Then

$$x_1 \leq \sqrt{\frac{2(n-1)}{n}\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^2}. \qquad (5)$$

Proof. Substituting $p = n$ in Eq. (3) we get

$$x_1 \leq \sqrt{\frac{2n}{n-1}\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^2} + \frac{1}{n-1}\sum_{i=2}^{n} x_i$$

$$x_1 \leq \sqrt{\frac{2n}{n-1}\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^2} + \frac{1}{n-1}(-x_1)$$

Therefore

$$x_1 \leq \sqrt{\frac{2(n-1)}{n}\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^2}.$$

Theorem 3.7. Let G be a connected graph with n vertices and let $x_1 \geq x_2 \geq \ldots \geq x_n$ be the status sum adjacency eigenvalues of G. Then for $1 \leq k \leq n$,

$$\sum_{i=1}^{k} x_i \leq \sqrt{\frac{2k(n-k)}{n}\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^2}. \qquad (6)$$

Proof. Let $H = \cup_k K_p$, the union of k copies of the complete graph K_p. The adjacency eigenvalues of H are $p - 1$ (k times) and -1 ($n - k$ times). The number of vertices and edges of H are $n = pk$ and $kp(p - 1)/2$ respectively. Using Eq. (1), we have

$$(p-1)\sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \le 2\sqrt{\frac{kp(p-1)}{2}\sum_{uv\in E(G)}[\sigma(u)+\sigma(v)]^2}$$

$$p\sum_{i=1}^{k} x_i - \sum_{i=1}^{n} x_i \le \sqrt{2kp(p-1)\sum_{uv\in E(G)}[\sigma(u)+\sigma(v)]^2}$$

$$p\sum_{i=1}^{k} x_i \le \sqrt{2kp(p-1)\sum_{uv\in E(G)}[\sigma(u)+\sigma(v)]^2}$$

$$\sum_{i=1}^{k} x_i \le \sqrt{\frac{2k(p-1)}{p}\sum_{uv\in E(G)}[\sigma(u)+\sigma(v)]^2}$$

$$\sum_{i=1}^{k} x_i \le \sqrt{\frac{2k(n-k)}{n}\sum_{uv\in E(G)}[\sigma(u)+\sigma(v)]^2}\text{, since } n = pk.$$

If $k = 1$, then Eq. (6) reduces to Eq. (5).

Theorem 3.8. Let G be a connected graph with n vertices and let $x_1 \ge x_2 \ge \ldots \ge x_n$ be the status sum adjacency eigenvalues of G. Then for $1 \le k \le n$,

$$\sum_{i=1}^{k}\left(x_i - x_{n-k+i}\right) \le 2\sqrt{k\sum_{uv\in E(G)}[\sigma(u)+\sigma(v)]^2}\,. \qquad (7)$$

Proof. Let $H = \cup_k K_{p,q}$, the union of k copies of the complete bipartite graph $K_{p,q}$. The adjacency eigenvalues of H are $(pq)^{1/2}$ (k times), 0 ($n - 2k$ times) and $-(pq)^{1/2}$ (k times). The number of edges of H is kpq.

Using Eq. (1), we have

$$\sqrt{pq}\sum_{i=1}^{k} x_i + (0)\sum_{i=1}^{n-k} x_i - \sqrt{pq}\sum_{i=n-k+1}^{n} x_i \le 2\sqrt{kpq\sum_{uv\in E(G)}[\sigma(u)+\sigma(v)]^2}$$

$$\sqrt{pq}\left[\sum_{i=1}^{k} x_i - \sum_{i=1}^{k} x_{n-k+i}\right] \le 2\sqrt{kpq\sum_{uv\in E(G)}[\sigma(u)+\sigma(v)]^2}$$

$$\sum_{i=1}^{k}\left(x_i - x_{n-k+i}\right) \le 2\sqrt{k\sum_{uv\in E(G)}[\sigma(u)+\sigma(v)]^2}\,.$$

If $k = 1$, then Eq. (7) reduces to Eq. (4).

IV. CONCLUSION AND FUTURE SCOPE

The present work deals with the bounds for the eigenvalues of the status sum adjacency matrix of a graph. These bounds can be sharpened and graphs can be found for which these bounds are attainable.

ACKNOWLEDGMENT

S.Y.T. thanks Ministry of Tribal Affairs, Govt. of India, New Delhi for awarding National Fellowship for Higher Education No. 201718-NFST-KAR-01182. H.S.R. thanks University Grants Commission (UGC), New Delhi for the support through grant under UGC-SAP DRS-III, 2016-2021: F.510/3/DRS-III /2016 (SAP-I).

REFERENCES

[1] D. M. Cvetkovic, M. Doob, H. Sachs, "*Spectra of Graphs*", Academic Press, New York, **1980**.

[2] H. S. Ramane, A. S. Yalnaik, "Status connectivity indices of graphs and its applications to the boiling point of benzenoid hydrocarbons", Journal of Applied Mathematics and Computing, Vol.**55**, pp.**609-627**, **2017**.

[3] S. Y. Talwar, H. S. Ramane, "*Status sum adjacency energy of graphs*", Research Review International Journal of Multidisciplinary, (in press).

[4] S. Bernard, J. M. Child, "*Higher Algebra*", Macmillan India Limited, Delhi, India, **1994**.

[5] H. S. Ramane, H. B. Walikar, "*Bounds for the eigenvalues of graphs*", In Graphs, Combinatorics, Algorithms and Applications (Eds. S. Arumugam, B. D. Acharya, S. B. Rao), Narosa Publishing House, New Delhi, **India**, pp.**115-122**, **2005**.

[6] H. S. Ramane, S. S. Shinde, S. B. Gudimani, J. B. Patil, "*Bounds for the signless Laplacian eigenvalues of graphs*", International Journal of Mathematical Sciences and Engineering Applications, Vol.**6**, Issue **6**, pp.**129-135**, **2012**.

[7] K. Pattabiraman, A. Santhakumar, "*Computing status connectivity indices and its coindices of composite graphs*", In Applied Mathematics and Scientific Computing, (Eds. B. Rushi Kumar, R. Sivaraj, B. Prasad, M. Nalliah, A. Reddy), Trends in Mathematics, Birkhauser, Cham, pp.479-487. 2019.

[8] A. T. Raj, J. V. Kureethara, "*Zagreb radio indices of some graph with diameter two*", International Journal of Scientific Research in Mathematical and Statistical Sciences, Vol.**4**, Issue **6**, pp.**7-11**, **2017**.

AUTHORS PROFILE

Miss S. Y. Talwar obtained her B.Sc. and M.Sc. degrees from Karnatak University, Dharwad, India. Currently she is pursuing her Ph.D. in Mathematics at Karnatak University, Dharwad. Her area of interest includes Graph theory, Topological indices, Spectral graph theory. She has published 4 research articles.

Dr. H. S. Ramane obtained his B.Sc., M.Sc. and Ph.D. degree from Karnatak University, Dharwad, India. Currently he is a Professor of Mathematics at Karnatak University, Dharwad. He has 25 years of teaching and research experience. His area of interest includes Graph theory, Spectral graph theory, Chemical Graph Theory. He has published 110 research articles and guided 11 PhD. Students.

