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Research Article

A Novel Iterative Approach to Optimize the Banach Contraction Method for Solving Systems of Integro-Differential Equations

Thomas J.G.[1*], Okai J.O.[2], Isah Abdullahi[3]

[1,2,3]Dept. of Mathematics Abubakar Tafawa Balewa University, Bauchi, Nigeria

**Corresponding Author: fatimaumardz@gmail.com*

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Abstract— This paper presents an Optimized Banach Contraction Method (OBCM), which employs a novel iterative technique to solve integro-differential equations (IDEs) and their systems. The method offers a more efficient and faster approach than traditional methods, eliminating the need for discretization, linearization, or restrictive assumptions. It provides both analytical and approximate solutions for linear and nonlinear equations, without requiring the computation of polynomials or Lagrange multipliers. These advantages improve the reliability of the OBCM, with numerical results confirming its effectiveness.

Keywords— Integro-differential equations (IDEs), Banach contraction principle**,** Variational iteration method**,** discretization, linearization

1. Introduction

Solving integro-differential equations (IDEs) is a topic of significant interest among researchers. These equations play a crucial role in various physical processes, such as glass formation, nano hydrodynamics, dropwise condensation and desert wind ripple formation [1,2,3,4]. Various range of numerical and analytical methods have been developed to address these problems. For instance, see [5,6,7,8,9,10,11,12,13,14,15,16,17,18,19]. Each of these methods is generally applicable only to specific classes of IDEs. To solve highly nonlinear differential equations lacking explicit solutions, numerical and semi-analytical iterative techniques have been employed, relying on proposed initial conditions. These methods include the classic Picard approach, the HAM method, and the Banach Contraction technique, among others [20, 21]. The Banach Contraction method, introduced in 1922, is widely regarded as the most effective of these approaches [22], as the other methods often suffer from issues such as defects, errors, and error accumulation, complicating the solution process [21, 23]. In recent years, the Banach Contraction principle has been extended and generalized using fixed-point theory, leading to significant new results in various studies [23].

2. Related Work

In 2009, Varsha Daftardar-Gejji and Sachin Bhalekar proposed an iterative technique that can be applied to different types of nonlinear functional equations of the form $v = f + N(v)$. [21]. This iterative method, based on the Banach Contraction Principle, is abbreviated as BCPM. They demonstrated its validity by solving various types of equations through examples [21]. In this study, we aim to enhance and apply the Banach Contraction Method (BCM) to solve systems of integro-differential equations, analyze errors, assess solution accuracy, and showcase the method's high efficiency.

3. Basic Idea of the Banach Contraction Method

We examine a broad class of functional equations represented as discussed:

$$u(x) = N(u) + f(x) \qquad ...(1)$$

Where $N(u)$ is a nonlinear operator from a Banach space $B \to B$, (x) is a known Integrable function of x and $u(x)$ is an unknown function

We seek a solution of $u(x)$ of Eqn. (1) in series form as

$$u = \sum_{m=0}^{\infty} y_m \qquad ...(2)$$

$$N\left(\sum_{m=0}^{\infty} y_m\right) = N(u_0) + \sum_{m=1}^{\infty}\left\{N\left(\sum_{m=0}^{m} y_m\right) - N\left(\sum_{m=0}^{m-1} y_m\right)\right\} \qquad ...(3)$$

Combining Eqns. (2) and (3), Eqn. (1) is rewritten in the form

$$\sum_{m=0}^{\infty} y_m = f(x) + N(u_0) + \sum_{m=1}^{\infty}\left\{N\left(\sum_{m=0}^{m} y_m\right) - N\left(\sum_{m=0}^{m-1} y_m\right)\right\} \qquad ...(4)$$

Next, we define the recursive sequence of approximations as

$$u_0'(x) = f(x) \Rightarrow u_0(x) = \int_0^x f(x)dx \quad \ldots(5)$$

$$\left.\begin{aligned} &u_1(x) = u_0(x) + N(u_0) \\ &u_2(x) = u_0(x) + N(u_1) \\ &u_3(x) = u_0(x) + N(u_2) \\ &u_4(x) = u_0(x) + N(u_3) \\ &\vdots \\ &u_n(x) = u_0(x) + N(u_{n-1}), n \geq 1 \end{aligned}\right\} \quad \ldots(6)$$

Thus, the solution of Eqn. (1) is given by

$$u(x) = \lim_{n\to\infty} u_n(x) \quad \ldots(7)$$

3.1 The Enhanced Banach Contraction Approach for Solving Second-Order Single Volterra Integral-Differential Equations (VIDEs).

Let us examine the following Volterra Integral-Differential Equations (VIDEs) of k^{th} order. See [24]

$$u^k(x) = f(x) + \int_0^x k(x,t)F[u(t)]dt, \quad \ldots(8)$$

where $u^k(x) = \dfrac{d^k}{dx^k}$ and $F[u(x)]$ is a known non-linear function of $u(x)$ such as $u^2(x)$, $\sin(u(x))$ and $e^{u(x)}$, and $u(x)$ is unknown function. Because the Eqn. (8) combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0)$, $u'(0)$, . . . $u^{(k-1)}(0)$ for the determination of the particular solution $u(x)$ of Eqn. (8). Without loss of generality, we may assume a VIDE of the second kind given by

$$u''(x) = f(x) + \int_0^x k(x,t)F[u(t)]dt, \quad (9)$$

$$u(0) = c_0,\ u'(0) = c_1.$$

By integrating both sides of Eqn. (9) twice from 0 to x and use the initial conditions, we get

$$\left.\begin{aligned} &u(x) = c_0 + c_1 x + \int_0^x\int_0^x f(t)dt + \\ &\frac{1}{2!}\int_0^x (x-t)^2 k(x,t)F[u(t)]dt, \end{aligned}\right\} \quad (10)$$

Building on the fundamental concept of the Banach Contraction Method (BCM), we reformulate the recurrence relation for the Optimized Banach Contraction Method (OBCM) as follows:

$$u_0(x) = c_0 + c_1 x + \int_0^x\int_0^x f(t)dt$$

$$u_1(x) = \frac{1}{2!}\int_0^x (x-t)^2 k(x,t)F[u_0(t)]dt$$

$$u_2(x) = \frac{1}{2!}\int_0^x (x-t)^2 k(x,t)F[u_0(t)+u_1(t)]dt - u_1(x)$$

$$u_3(x) = \frac{1}{2!}\int_0^x (x-t)^2 k(x,t)F[u_0(t)+u_1(t)+u_2(t)]dt - (u_1(x)+u_2(x))$$

$$\vdots$$

$$u_{n+1}(x) = \frac{1}{2!}\int_0^x \begin{array}{c}(x-t)^2 k(x,t) \\ F\left[\left(\sum_{m=0}^{n} u_m(t)\right)\right]\end{array} dt - \left(\sum_{m=0}^{n-1} u_m(t)\right) \quad (11)$$

Notably, the above algorithm effectively addresses the challenges encountered in the traditional Banach Contraction Method (BCM).

3.2 The case of single VIDE of k^{th} order

applying the initial conditions, we obtain:

$$u(x) = \left\{\begin{aligned} &\sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} f(t)dt \\ &+ \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} k(x,t)F[u(t)]dt, \end{aligned}\right\} \quad (12)$$

Using the OBCM recurrence relation, we obtain:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} f(t)dt$$

$$u_1(x) = \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} k(x,t)F[u_0(t)]dt$$

$$u_2(x) = \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} k(x,t)F[u_0(t)+u_1(t)]dt - u_1(x)$$

$$u_3(x) = \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} k(x,t)F[u_0(t)+u_1(t)+u_2(t)]dt - (u_1(x)+u_2(x))$$

$$\vdots$$

$$\left.\begin{aligned} &u_{n+1}(x) = \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} k(x,t)F\left[\left(\sum_{m=0}^{n} u_m(t)\right)\right]dt \\ &\quad -\left(\sum_{m=0}^{n-1} u_m(t)\right) \end{aligned}\right\} (13)$$

2.3 The case of the non-linear system of VIDEs

In this section, the method will be applied to a system of nonlinear Volterra Integral-Differential Equations (VIDEs) of the second kind. The approach outlined here can also be extended to linear systems in a similar way. To illustrate the effectiveness of the proposed method, we apply the optimized Banach Contraction Method (BCM) to a system of nonlinear VIDEs for which an analytical solution is known. Consider the following system of nonlinear VIDEs of the second kind [25]:

$$\left.\begin{aligned} u_1''(x) &= f_1(x) + \int_0^x \begin{bmatrix} k_{11}(x,t)F_1(u_1(t)) + \\ k_{12}(x,t)F_2(u_2(t)) \end{bmatrix} dt \\ u_2''(x) &= f_2(x) + \int_0^x \begin{bmatrix} k_{21}(x,t)F_1(u_1(t)) + \\ k_{22}(x,t)F_2(u_2(t)) \end{bmatrix} dt \end{aligned}\right\} \quad (14)$$

With initial condition

$$u_1(0) = c_0, \; u_1'(0) = c_1.$$
$$u_2(0) = d_0, \; u_2'(0) = d_1. \quad \ldots(15)$$

By integrating both sides of Eqn. (14) twice from 0 to x and use the initial conditions, we get

$$u_1(x) = c_0 + c_1 x + \left[\frac{1}{2!}\int_0^x (x-t) f_1(t)dt + \frac{1}{2!}\int_0^x (x-t)\begin{bmatrix} k_{11}(x,t)F_1(u_1(t)) \\ + k_{12}(x,t)F_2(u_2(t)) \end{bmatrix} dt\right]$$
$$u_2(x) = d_0 + d_1 x + \left[\frac{1}{2!}\int_0^x (x-t) f_2(t)dt + \frac{1}{2!}\int_0^x (x-t)\begin{bmatrix} k_{21}(x,t)F_1(u_1(t)) \\ + k_{22}(x,t)F_2(u_2(t)) \end{bmatrix} dt\right] (16)$$

To use the optimize BCM; let

$$u_1(x) = \sum_{n=0}^{\infty} u_{1,n}(x), \, u_2(x) = \sum_{n=0}^{\infty} u_{2,n}(x) \quad (17)$$

Using the OBCM recurrence relation, we obtain

$$u_{1,0}(x) = c_0 + c_1 x + \frac{1}{2!}\int_0^x (x-t) f_1(t)dt$$

$$u_{2,0}(x) = d_0 + d_1 x + \frac{1}{2!}\int_0^x (x-t) f_2(t)dt$$

$$u_{1,1}(x) = \frac{1}{2!}\int_0^x (x-t)\left[k_{11}(x,t)F_1(u_{1,0}(t)) + k_{12}(x,t)F_2(u_{2,0}(t))\right]dt$$

$$u_{2,1}(x) = \frac{1}{2!}\int_0^x (x-t)\left[k_{21}(x,t)F_1(u_{1,0}(t)) + k_{22}(x,t)F_2(u_{2,0}(t))\right]dt$$

$$u_{1,2}(x) = \frac{1}{2!}\int_0^x (x-t)\left[k_{11}(x,t)F_1(u_{1,0}(t)+u_{1,1}(t)) + k_{12}(x,t)F_2(u_{2,0}(t)+u_{2,1}(t))\right]dt - \left(u_{1,0}(x)+u_{2,0}(x)\right)$$

$$u_{2,2}(x) = \frac{1}{2!}\int_0^x (x-t)\left[k_{21}(x,t)F_1(u_{1,0}(t)+u_{1,1}(t)) + k_{22}(x,t)F_2(u_{2,0}(t)+u_{2,1}(t))\right]dt - \left(u_{1,0}(x)+u_{2,0}(x)\right)$$

and so on. Continuing in this manner, the $(n+1)^{th}$ approximation of the exact solutions for the unknown functions $u_1(x)$ and $u_2(x)$ can be achieved as

$$u_{1,n+1}(x) = \frac{1}{2!}\int_0^x (x-t)^{k-1} k_{11}(x,t)F_1\left[\left(\sum_{m=0}^{n} u_{1,m}(t)\right) + k_{12}(x,t)F_2\left(\sum_{m=0}^{n} u_{2,m}(t)\right)\right]dt$$
$$-\frac{1}{2!}\int_0^x (x-t)^{k-1} k_{11}(x,t)F_1\left[\left(\sum_{m=0}^{n} u_{2,m}(t)\right) + k_{12}(x,t)F_2\left(\sum_{m=0}^{n} u_{2,m}(t)\right)\right]dt$$

$$u_{2,n+1}(x) = \frac{1}{2!}\int_0^x (x-t)^{k-1} k_{21}(x,t)F_1\left[\left(\sum_{m=0}^{n} u_{1,m}(t)\right) + k_{22}(x,t)F_2\left(\sum_{m=0}^{n} u_{2,m}(t)\right)\right]dt$$
$$-\frac{1}{2!}\int_0^x (x-t)^{k-1} k_{21}(x,t)F_1\left[\left(\sum_{m=0}^{n} u_{1,m}(t)\right) + k_{22}(x,t)F_2\left(\sum_{m=0}^{n} u_{2,m}(t)\right)\right]dt \, (18)$$

Therefore, the approximate solutions

$$u_1(x) = \sum_{m=0}^{n+1} u_{1,m}(x) \, ,$$
$$u_2(x) = \sum_{m=0}^{n+1} u_{2,m}(x). \quad 19)$$

The optimized Banach Contraction Method (BCM) will be illustrated through examples that involve both integro-differential equations and systems of integro-differential equations.

4. Results and Discussion

Example 1:
Consider the following system of nonlinear second-order IDEs (Hemeda, 2018)

$$u''(x) = x + u(x) + \int_0^x \left(-u^2(t) + v^2(t)\right)dt,$$
$$u(0) = 1, \qquad u'(0) = 0 \qquad \ldots(20a)$$
$$u''(x) = -x + v(x) + \int_0^x \left(u^2(t) - v^2(t)\right)dt,$$
$$v(0) = 0, \qquad v'(0) = 1 \qquad \ldots(20b)$$

With exact solution $u(x) = \cosh(x)$ and $v(x) = \sinh(x)$

Applying $L^{-1}(.) = \int_0^x\int_0^x (.)dtdt$ to both sides of Eqn. (20a) and Eqn. (20b), we get

$$u(x) = 1 + \frac{x^3}{6} + \int_0^x (x-t)u(t)dt + \frac{1}{2!}\int_0^x (x-t)^2\left(-u^2(t) + v^2(t)\right)dt, \quad \ldots(21a)$$

$$v(x) = x - \frac{x^3}{6} + \int_0^x (x-t)v(t)dt + \frac{1}{2!}\int_0^x (x-t)^2\left(u^2(t) - v^2(t)\right)dt, \quad \ldots(21b)$$

Thus, to evaluate the above system of equation, we go by applying the recurrence relation as defined in **section 2**.

$$u_0(x) = 1,$$

$$v_0(x) = x,$$

$$u_1(x) = \frac{x^3}{6} + \int_0^x (x-t)u_0(t)dt + \frac{1}{2!}\int_0^x (x-t)^2\left(-u_0{}^2(t) + v_0{}^2(t)\right)dt = \frac{1}{2}x^2 + \frac{1}{60}x^5$$

$$v_1(x) = -\frac{x^3}{6} + \int_0^x (x-t)v_0(t)dt + \frac{1}{2!}\int_0^x (x-t)^2\left(u_0{}^2(t) - v_0{}^2(t)\right)dt = -\frac{1}{60}x^5 + \frac{1}{6}x^3$$

Therefore, according to the algorithms in **section 2**, we have the other components of the OBCM for Eqn. (20a and 20b) as follows using the above recursive scheme:

$$u_{n+1}(x) = \frac{x^3}{6} + \int_0^x (x-t)\left(\sum_{m=1}^{n-1} u_m(t)\right)dt + \frac{1}{2!}\int_0^x (x-t)^2\left(-\left(\sum_{m=1}^{n-1} u_m(t)\right)^2 + \left(\sum_{m=1}^{n-1} v_m(t)\right)^2\right)dt - \left(\sum_{m=1}^{n-1} u_m(t)\right)$$

$$v_{n+1}(x) = -\frac{x^3}{6} + \int_0^x (x-t)\left(\sum_{m=1}^{n-1} v_m(t)\right)dt + \frac{1}{2!}\int_0^x (x-t)^2\left(\left(\sum_{m=1}^{n-1} u_m(t)\right)^2 - \left(\sum_{m=1}^{n-1} v_m(t)\right)^2\right)dt - \left(\sum_{m=1}^{n-1} v_m(t)\right)$$

For $n \geq 1$

$$u_2(x) = \frac{1}{1260}x^7 + \frac{1}{24}x^4 - \frac{1}{178200}x^{11} - \frac{1}{43200}x^{10} - \frac{1}{90720}x^9 - \frac{1}{10080}x^8 - \frac{1}{60}x^5$$

$$v_2(x) = -\frac{1}{1260}x^7 + \frac{1}{40}x^5 + \frac{1}{178200}x^{11} + \frac{1}{43200}x^{10} + \frac{1}{90720}x^9 + \frac{1}{10080}x^8$$

$$\begin{aligned} u_3(x) = &-\frac{1}{19219200}x^{13} - \frac{73}{119750400}x^{12} + \frac{1}{720}x^6 + \frac{1}{62163288000}x^{19} \\ &+ \frac{73}{418784256000}x^{18} + \frac{53}{54286848000}x^{17} + \frac{17}{3592512000}x^{16} + \frac{283}{23351328000}x^{15} \\ &+ \frac{59}{3632428800}x^{14} + \frac{103}{19958400}x^{11} + \frac{1}{50400}x^{10} + \frac{1}{36288}x^9 + \frac{1}{10080}x^8 \\ &- \frac{1}{1260}x^7 \end{aligned}$$

$$\begin{aligned} v_3(x) = &\frac{1}{19219200}x^{13} + \frac{73}{119750400}x^{12} - \frac{1}{62163288000}x^{19} - \frac{73}{418784256000}x^{18} \\ &- \frac{53}{54286848000}x^{17} - \frac{17}{3592512000}x^{16} - \frac{283}{23351328000}x^{15} - \frac{59}{3632428800}x^{14} \\ &- \frac{103}{19958400}x^{11} - \frac{1}{50400}x^{10} - \frac{1}{36288}x^9 - \frac{1}{10080}x^8 + \frac{1}{1008}x^7 \end{aligned}$$

The series solution is then obtained by summing the above iterations,

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \ldots$$

$$v(x) = v_0(x) + v_1(x) + v_2(x) + v_3(x) + \ldots$$

$$\begin{aligned} u(x) = &\frac{1}{24}x^4 - \frac{1}{2217600}x^{11} - \frac{1}{302400}x^{10} + \frac{1}{60480}x^9 + 1 + \frac{1}{2}x^2 - \frac{1}{19219200}x^{13} \\ &- \frac{73}{119750400}x^{12} + \frac{1}{720}x^6 + \frac{1}{62163288000}x^{19} + \frac{73}{418784256000}x^{18} \\ &+ \frac{53}{54286848000}x^{17} + \frac{17}{3592512000}x^{16} + \frac{283}{23351328000}x^{15} + \frac{59}{3632428800}x^{14} \end{aligned}$$

$$\begin{aligned} v(x) = &x + \frac{1}{120}x^5 + \frac{1}{6}x^3 + \frac{1}{5040}x^7 + \frac{1}{2217600}x^{11} + \frac{1}{302400}x^{10} - \frac{1}{60480}x^9 \\ &+ \frac{1}{19219200}x^{13} + \frac{73}{119750400}x^{12} - \frac{1}{62163288000}x^{19} - \frac{73}{418784256000}x^{18} \\ &- \frac{53}{54286848000}x^{17} - \frac{17}{3592512000}x^{16} - \frac{283}{23351328000}x^{15} - \frac{59}{3632428800}x^{14} \end{aligned}$$

Table 1: The comparison between exact solutions $u(x)$ and the approximate solution using OBCM

x	EXACT	OBCM	ABSOLUTE ERROR
0	1	1	0
0.1	1.0050042	1.0050042	2.32037E-13
0.2	1.0200668	1.0200668	5.5405E-11
0.3	1.0453385	1.0453385	1.32407E-09
0.4	1.0810724	1.0810724	1.23247E-08
0.5	1.127626	1.1276259	6.84604E-08
0.6	1.1854652	1.1854649	2.7462E-07
0.7	1.255169	1.2551681	8.81436E-07
0.8	1.3374349	1.3374325	2.4088E-06
0.9	1.4330864	1.4330805	5.83985E-06
1	1.5430806	1.5430677	1.29301E-05

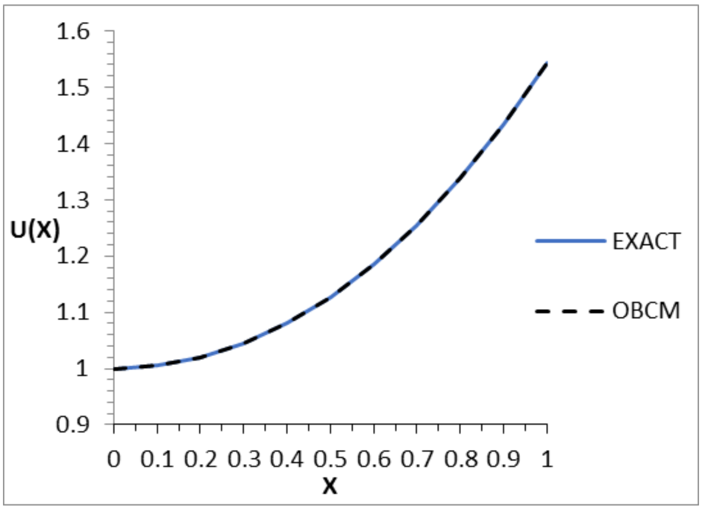

Figure 1: Graphs of the exact solution $u(x)$ and the approximate solution using OBCM.

Table 2: The comparison between exact solution $v(x)$ and the approximate solution using OBCM

x	EXACT (V(X))	OBCM (V(X))	ABSOLUTE ERROR
0	0	0	0
0.1	0.10016675	0.10016675	1.90264E-14
0.2	0.201336003	0.201336003	9.52666E-12
0.3	0.304520293	0.304520293	3.59075E-10
0.4	0.410752326	0.410752321	4.68166E-09
0.5	0.521095305	0.521095271	3.4085E-08
0.6	0.636653582	0.636653411	1.71485E-07
0.7	0.758583702	0.758583034	6.6784E-07
0.8	0.888105982	0.888103828	2.15398E-06
0.9	1.026516726	1.026510717	6.00863E-06
1	1.175201194	1.175186263	1.49301E-05

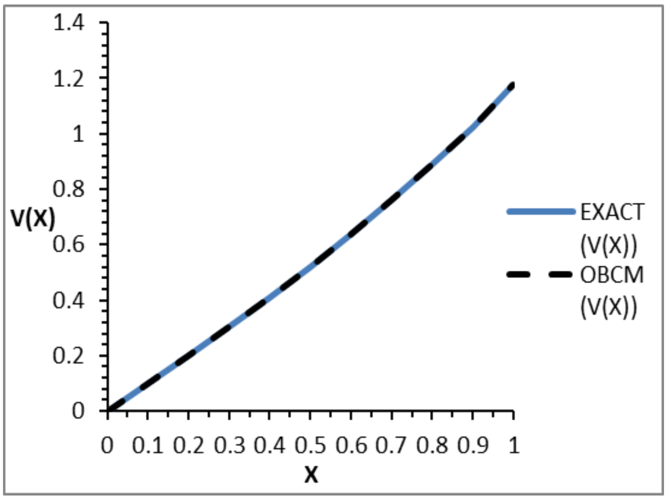

Figure 2: Graphs of the exact solution $v(x)$ and the approximate solution using OBCM.

Example 2:

Consider the system of nonlinear Fredholm integro-differential equation (Bakodah & Almuhalbedi, 2019)

$$u''(x) = 2 + \frac{12}{5}x - \int_0^1 x\left(u^2(t) + v^2(t)\right)dt,$$

$$u(0) = 1, \qquad u'(0) = 0 \qquad \ldots(22a)$$

$$v''(x) = -2 + \frac{4}{3}x - \int_0^1 x\left(u^2(t) - v^2(t)\right)dt,$$

$$v(0) = 1, \qquad v'(0) = 0 \qquad \ldots(22b)$$

With exact solution

$$u(x) = 1 + x^2$$
$$v(x) = 1 - x^2$$

Applying $L^{-1}(.) = \int_0^x\int_0^x (.)dtdt$ to both sides of Eqn. (22a) and Eqn. (22b), we get

$$u(x) = 1 + x^2 + \frac{12}{30}x^3 - \frac{1}{3!}x^3\int_0^1\left(u^2(t) + v^2(t)\right)dt, \quad \ldots(23a)$$

$$v(x) = 1 - x^2 + \frac{4}{18}x^3 - \frac{1}{3!}x^3\int_0^1\left(u^2(t) - v^2(t)\right)dt, \quad \ldots(23b)$$

Thus, to evaluate the above system of equation, we apply the recurrence relation as defined in **section 2**.

$$u_0(x) = 1,$$
$$v_0(x) = 1,$$

$$u_1(x) = x^2 + \frac{12}{30}x^3 - \frac{1}{3!}x^3\int_0^1\left(u_0^{\,2}(t) + v_0^{\,2}(t)\right)dt = x^2 + \frac{1}{15}x^3$$

$$v_1(x) = -x^2 + \frac{4}{18}x^3 - \frac{1}{3!}x^3\int_0^1\left(u_0^{\,2}(t) - v_0^{\,2}(t)\right)dt = -x^2 + \frac{2}{9}x^3$$

Therefore, according to **section 2**, we have the other components of the OADM for Eqn. (22a & 22b) as follows using the above recursive scheme:

$$u_{n+1}(x) = x^2 + \frac{12}{30}x^3 - \frac{1}{3!}x^3\int_0^1\left(\left(\sum_{m=1}^{n-1}u_m(t)\right)^2 + \left(\sum_{m=1}^{n-1}v_m(t)\right)^2\right)dt,$$

$$v_{n+1}(x) = -x^2 + \frac{4}{18}x^3 - \frac{1}{3!}x^3\int_0^1\left(\left(\sum_{m=1}^{n-1}u_m(t)\right)^2 - \left(\sum_{m=1}^{n-1}v_m(t)\right)^2\right)dt$$

For $n \geq 1$

$$u_2(x) = -\frac{14183}{170100}x^3$$
$$v_2(x) = -\frac{5449}{24300}x^3$$
$$u_3(x) = \frac{11595897701}{607614210000}x^3$$
$$v_3(x) = \frac{96211}{22504230}x^3$$
$$u_4(x) = -\frac{42841475331471350029801}{1550619118414481220000000}x^3$$
$$v_4(x) = -\frac{16308947364676667401}{6458222067532200000000}x^3$$

The series solution is then obtained by summing the above iterations,

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \ldots$$
$$v(x) = v_0(x) + v_1(x) + v_2(x) + v_3(x) + v_4(x) + \ldots$$
$$u(x) = 1 + x^2 - \frac{6082315921958885209801}{1550619118414481220000000}x^3$$

Table 3: The comparison between exact solutions $u(x)$ and the approximate solution using OBCM

x	EXACT U(X)	OBCM U(X)	ABSOLUTE ERROR
0	1	1	0
0.1	1.01	1.009999608	3.92251E-07
0.2	1.04	1.039996862	3.13801E-06
0.3	1.09	1.089989409	1.05908E-05
0.4	1.16	1.159974896	2.51041E-05
0.5	1.25	1.249950969	4.90314E-05
0.6	1.36	1.359915274	8.47262E-05
0.7	1.49	1.489865458	0.000134542
0.8	1.64	1.639799168	0.000200832
0.9	1.81	1.809714049	0.000285951
1	2	1.999607749	0.000392251

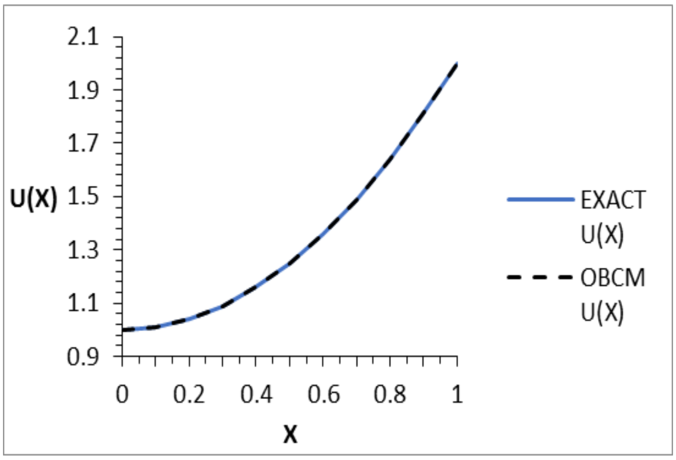

Figure 3: Graphs of the exact solution $u(x)$ and the approximate solution using OBCM.

Table 4: The comparison between exact solutions $u(x)$, the approximate solution using OBCM and ADM

x	EXACT	OBCM (n=4)	ADM (n=6)
0	1	1	1
0.1	1.01	1.009999608	1.009612278
0.2	1.04	1.039996862	1.036898224
0.3	1.09	1.089989409	1.079531505
0.4	1.16	1.159974896	1.135185791
0.5	1.25	1.249950969	1.201534748
0.6	1.36	1.359915274	1.276252044
0.7	1.49	1.489865458	1.357011347
0.8	1.64	1.639799168	1.441486326
0.9	1.81	1.809714049	1.527350647
1	2	1.999607749	1.61227798

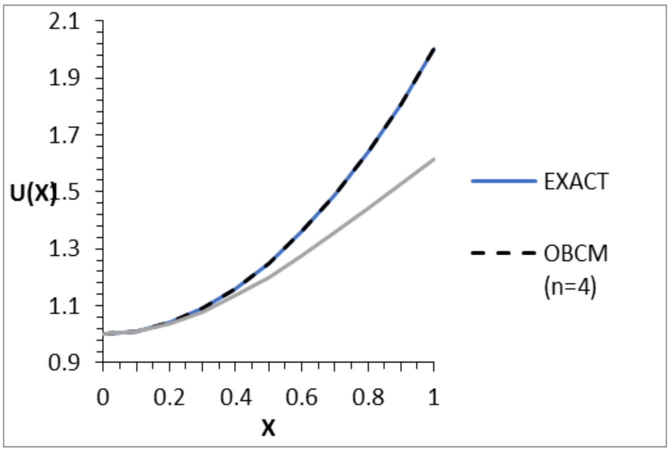

Figure 4: Graphs of the exact solution $u(x)$, the approximate solution Using OBCM and ADM

Table 5: The comparison between exact solutions $v(x)$ and the approximate solution using OBCM

x	EXACT	OBCM	ABSOLUTE ERROR
0	1	1	0
0.1	0.99	0.99	1.54688E-07
0.2	0.96	0.959999	1.2375E-06
0.3	0.91	0.909996	4.17658E-06
0.4	0.84	0.83999	9.90004E-06
0.5	0.75	0.749981	1.9336E-05
0.6	0.64	0.639967	3.34126E-05
0.7	0.51	0.509947	5.3058E-05
0.8	0.36	0.359921	7.92003E-05
0.9	0.19	0.189887	0.000112768
1	0	-0.00015	0.000154688

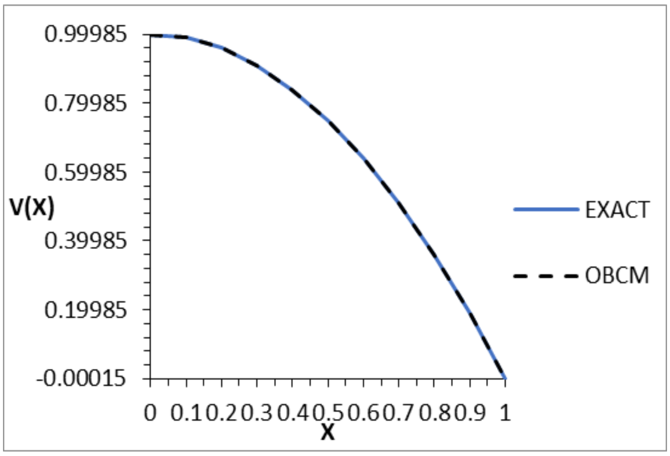

Figure 5: Graphs of the exact solution $v(x)$ and the approximate solution using OBCM

Table 6: The comparison between exact solutions $v(x)$, the approximate solution using OBCM and ADM

x	EXACT (V(X))	OBCM (V(X))	ADM (V(X))
0	1	1	1
0.1	0.99	0.989999845	0.990186726
0.2	0.96	0.959998762	0.961493812

0.3	0.91	0.909995823	0.915041615
0.4	0.84	0.8399901	0.851950495
0.5	0.75	0.749980664	0.773340811
0.6	0.64	0.639966587	0.680332922
0.7	0.51	0.509946942	0.574047186
0.8	0.36	0.3599208	0.455603963
0.9	0.19	0.189887232	0.326123611
1	0	-0.00015469	0.18672649

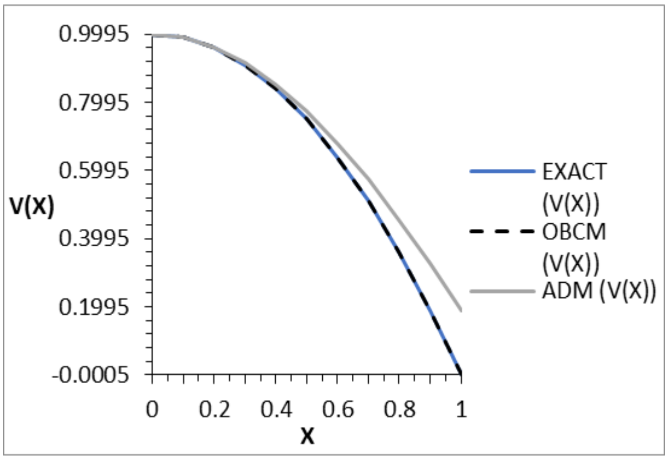

Figure 6: Graphs of the exact solution $v(x)$, the approximate solution using OBCM and ADM

Example 3:

Consider the following system of nonlinear Volterra integro-differential equation [(Wazwaz, 2011) and (Hemeda, 2018)]

$$u''(x)=1-\frac{1}{3}x^3-\frac{1}{2}v'^2(x)+\frac{1}{2}\int_0^x\left[u^2(t)+v^2(t)\right]dt,$$

$$u(0)=1,\ u'(0)=2, \qquad \ldots(24a)$$

$$v''(x)=-1+x^2-xu(x)+\frac{1}{4}\int_0^x\left[u^2(t)-v^2(t)\right]dt,$$

$$v(0)=-1, \quad v'(0)=0, \qquad \ldots(24b)$$

With exact solution

$$(u(x),v(x))=(x+e^x,x+e^x)$$

Applying $L^{-1}(.)=\int_0^x\int_0^x(.)dtdt$ to both sides of Eqn. (24a) and Eqn. (24b), we get

$$u(x)=1+2x+\frac{1}{2}x^2-\frac{1}{60}x^5-\frac{1}{2}\int_0^x(x-t)v'^2(t)dt+\frac{1}{4}\int_0^x(x-t)^2\left[u^2(t)+v^2(t)\right]dt, \ \ldots(25a)$$

$$v(x)=-1-\frac{1}{2}x^2+\frac{1}{12}x^4-\int_0^x t(x-t)u(t)dt+\frac{1}{8}\int_0^x(x-t)^2\left[u^2(t)-v^2(t)\right]dt, \qquad \ldots(25b)$$

Thus, to evaluate the above system of equation, we apply the recurrence relation as defined in **section 2**.

$$u_0(x)=1,$$

$$v_0(x)=-1,$$

$$u_1(x)=2x+\frac{1}{2}x^2-\frac{1}{60}x^5-\frac{1}{2}\int_0^x(x-t)v_0'^2(t)dt+\frac{1}{4}\int_0^x(x-t)^2\left[u_0^2(t)+v_0^2(t)\right]dt=$$

$$2x+\frac{1}{2}x^2-\frac{1}{60}x^5+\frac{1}{6}x^3$$

$$v_1(x)=-\frac{1}{2}x^2+\frac{1}{12}x^4-\int_0^x t(x-t)u_0(t)dt+\frac{1}{8}\int_0^x(x-t)^2\left[u_0^2(t)-v_0^2(t)\right]dt=$$

$$-\frac{1}{2}x^2+\frac{1}{12}x^4-\frac{1}{6}x^3$$

Therefore, according to **Section 2**, we have the other components of the OBCM for Eqn. (24a & 24b) as follows using the above recursive scheme:

$$u_{n+1}(x)=2x+\frac{1}{2}x^2-\frac{1}{60}x^5-\frac{1}{2}\int_0^x(x-t)\left(\sum_{m=1}^{n-1}v'_m(t)\right)^2dt$$

$$+\frac{1}{4}\int_0^x(x-t)^2\left[\left(\sum_{m=1}^{n-1}u_m(t)\right)^2+\left(\sum_{m=1}^{n-1}v_m(t)\right)^2\right]dt$$

$$v_{n+1}(x)=-\frac{1}{2}x^2+\frac{1}{12}x^4-\int_0^x t(x-t)\left(\sum_{m=1}^{n-1}u_m(t)\right)dt+\frac{1}{8}\int_0^x(x-t)^2\left[\left(\sum_{m=1}^{n-1}u_m(t)\right)^2-\left(\sum_{m=1}^{n-1}v_m(t)\right)^2\right]dt$$

For $n\geq1$

$$u_2(x)=\frac{1}{12355200}x^{13}+\frac{1}{1425600}x^{11}-\frac{1}{32400}x^{10}-\frac{17}{181440}x^9-\frac{11}{20160}x^8+\frac{2}{315}x^7$$
$$+\frac{13}{720}x^6+\frac{1}{24}x^4+\frac{1}{40}x^5$$

$$v_2(x)=\frac{1}{24710400}x^{13}-\frac{1}{316800}x^{11}+\frac{1}{259200}x^{10}+\frac{1}{120960}x^9+\frac{11}{40320}x^8$$
$$+\frac{1}{1008}x^7-\frac{1}{720}x^6-\frac{1}{8}x^4-\frac{1}{120}x^5$$

$$\ldots$$

$$u(x)=u_0(x)+u_1(x)+u_2(x)+$$

$$u(x)=1+2x+\frac{1}{2}x^2+\frac{1}{120}x^5+\frac{1}{6}x^3+\frac{1}{12355200}x^{13}+\frac{1}{1425600}x^{11}-\frac{1}{32400}x^{10}$$
$$-\frac{17}{181440}x^9-\frac{11}{20160}x^8+\frac{2}{315}x^7+\frac{13}{720}x^6+\frac{1}{24}x^4$$

$$v(x)=v_0(x)+v_1(x)+v_2(x)+$$

$$v(x)=-1-\frac{1}{2}x^2-\frac{1}{24}x^4-\frac{1}{6}x^3+\frac{1}{24710400}x^{13}-\frac{1}{316800}x^{11}+\frac{1}{259200}x^{10}$$
$$+\frac{1}{120960}x^9+\frac{11}{40320}x^8+\frac{1}{1008}x^7-\frac{1}{720}x^6-\frac{1}{120}x^5$$

Table 7: The comparison between exact solutions $u(x)$ And the approximate solution of OBCM

x	EXACT u(x)	OBCM u(x)	ABSOLUTE ERROR
0	1	1	0
0.1	1.2051709	1.20517094	1.72759E-08
0.2	1.4214028	1.4214039	1.14388E-06
0.3	1.6498588	1.64987226	1.34557E-05
0.4	1.8918247	1.89190264	7.79418E-05
0.5	2.1487213	2.14902729	0.000306023
0.6	2.4221188	2.42305784	0.000939044
0.7	2.7137527	2.71618243	0.002429719
0.8	3.0255409	3.03108798	0.005547052

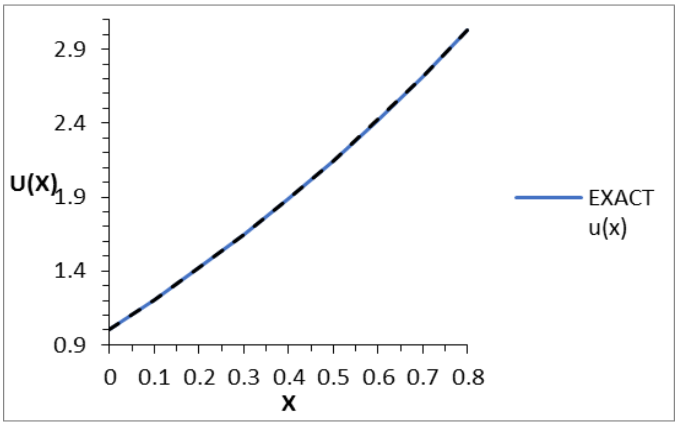

Figure 7: Graphs of the exact solution $u(x)$ and the approximate solution of OBCM for example 3

Table 8: The comparison between exact solutions $v(x)$ And the approximate solution of OBCM

x	EXACT v(x)	OBCM v(x)	ABSOLUTE ERROR
0	-1	-1	0
0.1	-1.0051709	-1.0051709	1.22036E-10
0.2	-1.0214028	-1.0214027	1.6006E-08
0.3	-1.0498588	-1.0498585	2.8012E-07
0.4	-1.0918247	-1.0918225	2.14872E-06
0.5	-1.1487213	-1.1487108	1.04872E-05
0.6	-1.2221188	-1.2220804	3.84493E-05
0.7	-1.3137527	-1.313637	0.000115698
0.8	-1.4255409	-1.4252397	0.00030125

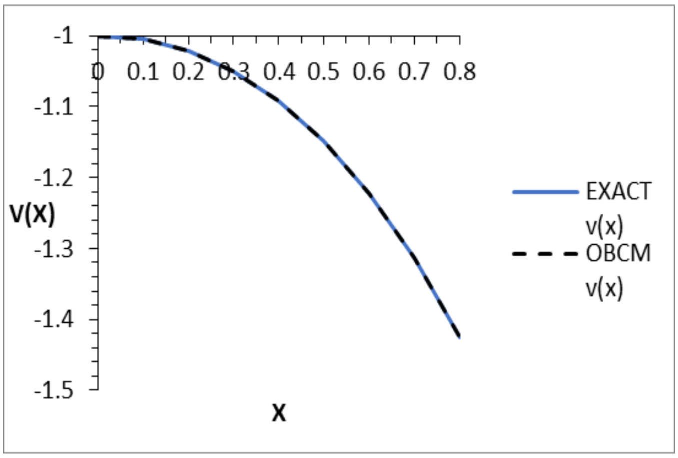

Figure 8: Graphs of the exact solution $v(x)$ and the approximate solution of OBCM for example 3

Table 9: The comparison between exact solutions $u(x)$, the approximate solution of OBCM and NIM (Hemeda, 2018)

x	EXACT u(x)	OBCM u(x)	NIM u(x)
0	1	1	1
0.1	1.2051709	1.2051709	1.205167
0.2	1.4214028	1.4214039	1.421328
0.3	1.6498588	1.6498723	1.64946
0.4	1.8918247	1.8919026	1.890496
0.5	2.1487213	2.1490273	2.145313
0.6	2.4221188	2.4230578	2.414704
0.7	2.7137527	2.7161824	2.699366
0.8	3.0255409	3.031088	2.999872

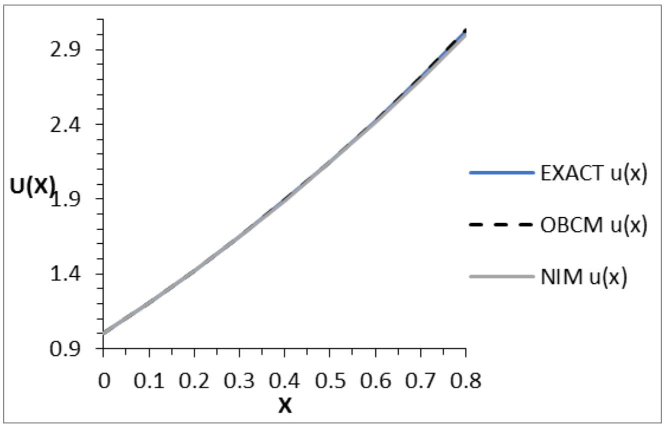

Figure 9: Graphs of the exact solution $u(x)$, the approximate solution of OBCM and NIM (Hemeda, 2018)

Table 10: The comparison between exact solutions $v(x)$, the approximate solution of OBCM and NIM (Hemeda, 2018)

x	EXACT v(x)	OBCM v(x)	NIM v(x)
0	-1	-1	-1
0.1	-1.0051709	-1.0051709	-1.00516
0.2	-1.0214028	-1.0214027	-1.0212
0.3	-1.0498588	-1.0498585	-1.04883
0.4	-1.0918247	-1.0918225	-1.08853
0.5	-1.1487213	-1.1487108	-1.14063
0.6	-1.2221188	-1.2220804	-1.2052
0.7	-1.3137527	-1.313637	-1.28216
0.8	-1.4255409	-1.4252397	-1.3712

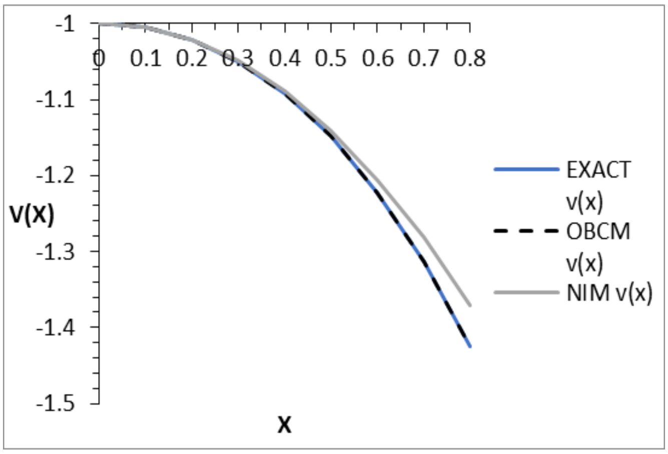

Figure 10: Graphs of the exact solution $u(x)$, the approximate solution of OBCM and NIM (Hemeda, 2018)

Example 4:

Consider the following nonlinear second-order IDE (Hemeda, 2018)

$$u''(x) = u(x) + \frac{1}{2}\left(1 - u^2(x)\right) + \int_0^x u^2(t)\,dt,$$

$$u(0) = u'(0) = 1. \qquad \ldots(26)$$

With exact solution $u(x) = e^x$

As per the above example, the IDE is equivalent to the integral equation

$$u(x) = 1 + x + \int_0^x (x-t)u(t)dt + \left[\begin{array}{l} \frac{1}{2}\int_0^x (x-t)\left(1-u^2(t)\right)dt \\ +\frac{1}{2}\int_0^x (x-t)^2 u^2(t)dt, \end{array}\right] \qquad (27)$$

Applying the recurrence relation as defined in **section 2.3**, we obtain

$$u_0(x) = 1 + x$$

$$u_1(x) = \int_0^x (x-t)u_0(t)dt + \frac{1}{2}\int_0^x (x-t)\left(1-u_0^{\,2}(t)\right)dt + \frac{1}{2}\int_0^x (x-t)^2 u_0^{\,2}(t)dt = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{60}x^5$$

Therefore, according to **section 2.3**, we have the other components of the OBCM for Eqn. (26) as follows using the above recursive scheme:

$$u_{n+1}(x) = \int_0^x (x-t)u_n(t)dt + \frac{1}{2}\int_0^x (x-t)\left(1-u_n^{\,2}(t)\right)dt + \frac{1}{2}\int_0^x (x-t)^2 u_n^{\,2}(t)dt - \left(\sum_{m=1}^{n-1} u_m(t)\right)$$

For $n \geq 1$

$$u_2(x) = \frac{1}{6177600}x^{13} + \frac{1}{950400}x^{11} + \frac{1}{518400}x^{10} - \frac{1}{120960}x^9 - \frac{1}{13440}x^8 + \frac{1}{5040}x^7 + \frac{1}{720}x^6 - \frac{1}{120}x^5$$

$$\begin{aligned} u_3(x) = & -\frac{23}{230630400}x^{13} - \frac{13}{39916800}x^{11} - \frac{1}{604800}x^{10} + \frac{1}{90720}x^9 + \frac{1}{10080}x^8 \\ & + \frac{1}{836679950346240000}x^{29} - \frac{61}{274771740672000000}x^{26} + \frac{1}{51519701376000000}x^{27} \\ & - \frac{1}{57702065541120000}x^{28} - \frac{83}{130882397184000000}x^{25} - \frac{31}{136117693071360000}x^{24} \\ & + \frac{401}{16334123168563200}x^{23} - \frac{263}{9942509754777600}x^{22} + \frac{2371}{45535599083520000}x^{21} \\ & - \frac{40081}{35777970708480000}x^{20} - \frac{6131}{460776895488000}x^{19} - \frac{105569}{1280474741145600}x^{18} \\ & - \frac{20731}{71137485619200}x^{17} - \frac{59}{64576512000}x^{16} - \frac{419}{653837184000}x^{15} \\ & + \frac{61}{12454041600}x^{14} + \frac{47}{159667200}x^{12} \end{aligned}$$

And so on. The solution in a series form is given by

$$\begin{aligned} u(x) = & 1 + x + \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{60}x^5 + \frac{1}{6177600}x^{13} + \frac{1}{950400}x^{11} \\ & + \frac{1}{518400}x^{10} - \frac{1}{120960}x^9 - \frac{1}{13440}x^8 + \frac{1}{5040}x^7 + \frac{1}{720}x^6 - \frac{1}{120}x^5 \\ & - \frac{23}{230630400}x^{13} - \frac{13}{39916800}x^{11} - \frac{1}{604800}x^{10} + \frac{1}{90720}x^9 + \frac{1}{10080}x^8 \\ & + \frac{1}{836679950346240000}x^{29} - \frac{61}{274771740672000000}x^{26} + \frac{1}{51519701376000000}x^{27} \\ & - \frac{1}{57702065541120000}x^{28} - \frac{83}{130882397184000000}x^{25} - \frac{31}{136117693071360000}x^{24} \\ & + \frac{401}{16334123168563200}x^{23} - \frac{263}{9942509754777600}x^{22} + \frac{2371}{45535599083520000}x^{21} \\ & - \frac{40081}{35777970708480000}x^{20} - \frac{6131}{460776895488000}x^{19} - \frac{105569}{1280474741145600}x^{18} \\ & - \frac{20731}{71137485619200}x^{17} - \frac{59}{64576512000}x^{16} - \frac{419}{653837184000}x^{15} \\ & + \frac{61}{12454041600}x^{14} + \frac{47}{159667200}x^{12} \end{aligned}$$

Table 11: The exact and approximate solution of Applying Algorithm in **section 2.3** for Eqn. (26)

x	EXACT	OBCM	ERROR
0	1	1	0
0.1	1.105171	1.105171	4.44E-16
0.2	1.221403	1.221403	1.53E-14
0.3	1.349859	1.349859	1.41E-12
0.4	1.491825	1.491825	3.48E-11
0.5	1.648721	1.648721	4.22E-10
0.6	1.822119	1.822119	3.27E-09
0.7	2.013753	2.013753	1.85E-08
0.8	2.225541	2.225541	8.39E-08
0.9	2.459603	2.459603	3.19E-07
1	2.718282	2.718283	1.06E-06

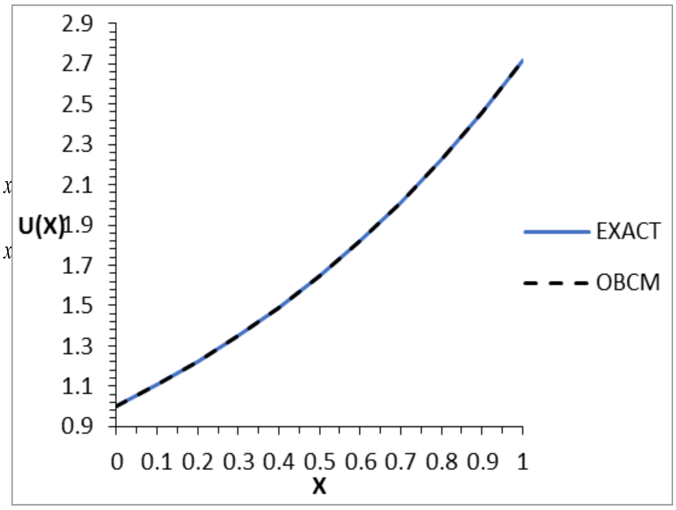

Figure 11: shows the exact solution $u(x) = e^x$ with the Approximate solution obtained from Eqn. (26)

The primary objective of this study was successfully achieved by solving a nonlinear system of integro-differential equations using the proposed method, specifically through the implementation of the optimized Banach Contraction Method (OBCM). A comparison of the results obtained from this method with the exact solutions and results from other methods is provided in the tables and figures.

The OBCM has demonstrated both efficiency and reliability in approximating solutions to nonlinear integro-differential equations, as shown in Tables 1-11. Unlike the Adomian Decomposition Method (ADM) and Differential Transform Method (DTM), which require specific Adomian polynomials for handling nonlinear terms, the OBCM operates without such assumptions.

When comparing the OBCM's results with those from ADM and the Numerical Iteration Method (NIM), it is clear that the OBCM produces more accurate numerical solutions, as evidenced by the tables and figures. Moreover, the approximation error decreases with increasing iterations, as shown in the figures and tables.

Overall, the OBCM converges more quickly and achieves higher-order accuracy without the restrictive assumptions required by other methods. This is further confirmed by the comparison results in Tables 1-11 and Figures 1-11.

5. Conclusion and Future Scope

This study introduced a semi-analytical approach grounded in the Banach Contraction Method (BCM) to tackle nonlinear integro-differential equations and systems of such equations. The method's effectiveness was illustrated through four examples. The findings reveal that this approach provides a more straightforward and efficient computational process compared to alternative methods, making it a preferable and more practical solution for addressing nonlinear problems.

Data Availability
Not applicable.

Conflict of Interest
All authors declare that they do not have any conflict of interest.

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No funding source exists,

Authors' Contributions
All authors reviewed and edited the manuscript and approved the final version of the manuscript.

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