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On The Equivalence of Categories of A^H −Semimodules and $A\#H$ −Semimodules

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Abstract- In this paper, we have introduced two functors between the category of $A^H -$ semimodules and the category of $A\#H -$ semimodules, where A is a H-semimodule semialgebra over a Hopf algebra. Further, assuming H as a finite dimensional semisimple Hopf algebra we have established a categorical equivalence between these two categories.

Keywords: Semialgebra, Smash product, Hopf algebra, Natural transformation, equivalent categories.

I. INTRODUCTION

In *[7]*, R.P. Sharma and others defined the tensor product of semimodules over a semiring. In our papers *[2]*, *[6]* we have extended the Hopf algebra action on a semialgebra A and studied some connection between Hopf semimodule semialgebra A and its smash product semialgebra $A\#H$.

In this paper we consider the study of categorical equivalence between the category of semimodules over $H -$fixed semialgebra A^H and the category of semimodules over the smash product semialgebra $A\#H$.

For, $M \in ob\left({}_{A^H}S\mathfrak{M}\right)$, define $F(M) = (A\#H) \otimes_{A^H} M$ and for a morphism $f: M_1 \to M_2$ in ${}_{A^H}S\mathfrak{M}$, let $F(f): (A\#H) \otimes_{A^H} M_1 \to (A\#H) \otimes_{A^H} M_2$ be defined by $F(f)(a \otimes m) = a \otimes f(m)$, for all $a \otimes m \in (A\#H) \otimes_{A^H} M_1$. For, $N \in ob({}_{A\#H}S\mathfrak{M})$ define $G(N) = N$, considered as left $A^H -$semimodule. Also, if $f: M_1 \to M_2$ is a morphism in ${}_{A\#H}S\mathfrak{M}$ then define $G(f): M_1{}' \to M_2{}'$ by $G(f)(m) = f'(m), \forall m \in M_1{}'$ where f' is $A^H -$morphism in ${}_{A^H}S\mathfrak{M}$ and $f'(m) = f(m)$. We have proved that F and G are functors and there is a natural isomorphism ζ between the functors $F \circ G$ and $I_{{}_{A\#H}S\mathfrak{M}}$. Also, we have proved that there is a natural transformation τ between the functors $G \circ F$ and $I_{{}_{A^H}S\mathfrak{M}}$. Under certain conditions, we have observed that the natural transformation τ is also a natural isomorphism.

This paper is organised as follows: The second section contains some basic concepts and results on semirings, Hopf algebras and category theory that are required for the development of this paper. In the third section, we introduce the Hopf algebra H action on a semialgebra A and define the semialgebra of $H -$invariants A^H and smash product semialgebra $A\#H$. In the fourth section, as a main result in this section we prove the categories of left $A^H -$semimodules and the categories of left $A\#H -$semimodules are equivalent. The fifth and the final section contain the conclusion of this paper.

II. PRELIMINARIES

In this section we present the necessary preliminaries on semirings, category theory and Hopf algebra that are required for the latter sections.

For definition and results in semirings we refer to *[3]*,

Definition 2.1.
A semiring R is a nonempty set R equipped with two binary operations '$+'$ and $'\cdot'$ called addition and multiplication such that, for $a, b, c \in R$,

1. $(R, +)$ is a commutative monoid with identity element 0.
2. $(R, \cdot)$ is a monoid with identity element 1.
3. Multiplication distributes over addition from either side.
$(i) a \cdot (b + c) = a \cdot b + a \cdot c$
$(ii)(a + b) \cdot c = a \cdot c + b \cdot c$
4. $a \cdot 0 = 0 \cdot a = 0$, for all $a \in R$.
5. $1 \neq 0$.

Definition 2.2.
Let R be a semiring. A left R −semimodule M is a commutative monoid $(M, +)$ with additive identity 0_M for which we have a function $R \times M \to M$, defined by
$(r, m) \mapsto rm$, which satisfies the following conditions:
$(i)\ (rr')m = r(r'm)$;
$(ii) r(m + m') = rm + rm'$;
$(iii)(r + r')m = rm + r'm$;
$(iv) 1_R m = m$;
$(v) r0_M = 0_M = 0_R m$, where $r, r' \in R$ and $m, m' \in M$.
A right R −semimodule is defined analogously.
For definition and results on category theory, we refer to [4],

Definition 2.3.
A category $\boldsymbol{C}$ consists of

1. A class $ob\ \boldsymbol{C}$ of objects.
2. For each ordered pair of objects (A, B), a set $\hom_C(A, B)$ whose elements are called morphisms with domain A and co-domain B.
3. For each ordered triple of objects (A, B, C), a map $(f, g) \longmapsto gf$ of the product set $\hom(A, B) \times \hom(B, C)$ into $\hom(A, C)$.

It is assumed that the objects and morphisms satisfy the following conditions:

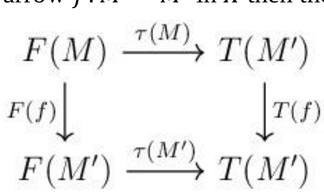
1. If $(A, B) \neq (C, D)$, then $\hom(A, B)$ and $\hom(C, D)$ are disjoint.
2. If $f \in \hom(A, B)$, $g \in \hom(B, C)$ and $h \in \hom(C, D)$, then $(hg)f = h(gf)$.
3. For every object A we have an element $1_A \in \hom(A, A)$ such that $f1_A = f$ for every $f \in \hom(A, B)$ and $1_A g = g$ for every $g \in \hom(B, A)$.

Definition 2.4.
If X and Y are categories, a (covariant) functor F from X to Y consists of

1. A map $M \to FM$ of $ob\ X$ into $ob\ Y$.
2. For every pair of objects (A, B) of X, a map $f \to F(f)$ of $\hom_X(A, B)$ into $\hom_Y(FA, FB)$.

We require that these satisfy the following conditions:
If $g \circ f$ is defined in X, then $F(g \circ f) = F(g) \circ F(f)$. Also, $F(1_A) = 1_{FA}$.

Definition 2.5.
Given two functors $F, T: X \to Y$ a natural transformation $\tau: F \to T$ is a function which assigns to each object M of X, an arrow $\tau_M: FM \to TM$ of T in such a way that for every arrow $f: M \to M'$ in X then the diagram



commutes.
A natural transformation τ with every componets τ_M invertible in Y is called natural isomorphism.

Definition 2.6.
We say that C and D are said to be equivalent categories if there exists functors $F: C \to D$ and $G: D \to C$ such that $G \circ F \cong 1_C$ and $F \circ G \cong 1_D$ where $\cong$ denotes the natural isomorphism of functors.

Definition 2.7.
Let K be semiring. Let P be a right K −semimodule, Q a left K −semimodule. We define a balanced product of P and Q to be a commutative monoid $(M, +)$ together with a map f of the product set $P \times Q$ into M satisfying the following conditions:

1. $f(x + x', y) = f(x, y) + f(x', y)$;
2. $f(x, y + y') = f(x, y) + f(x, y')$;
3. $f(xk, y) = f(x, ky)$

for all $x, x' \in P, y, y' \in Q, k \in K$.

Definition 2.8. ***[5]***
Let $\widetilde{M}$ be the category of all commutative monoids. Let $F, G \in \widetilde{M}$ and $F(F, G)$ denotes the free monoid generated by $F \times G$, ρ be the congruence on $F(F, G)$ generated by the pairs $\langle(a_1a_2, b), (a_1, b)(a_2, b)\rangle$ and $\langle(a, b_1b_2), (a, b_1)(a, b_2)\rangle$. Take $F \otimes G$ as $F(F, G)/\rho$. Let $F_1, G_1 \in \widetilde{M}$ and $\alpha \in \widetilde{M}(F, F_1)$ and $\beta \in \widetilde{M}(G, G_1)$, then assignment $(F, G) \mapsto F \otimes G$ and $(\alpha, \beta) \mapsto \alpha \otimes \beta$ determine the bifunctor $\_ \otimes \_ : \widetilde{M} \times \widetilde{M} \longrightarrow \widetilde{M}$. It is observed in *[5]*, that the bifunctor $\_ \otimes \_ : \widetilde{M} \times \widetilde{M} \longrightarrow \widetilde{M}$ is an internal tensor product.

Notation: Let $K\text{-}Smod$ and $Smod\text{-}K$ denote the categories of left and right K–semimodules, respectively, over a semiring$K$.

Now, we give the definition of tensor product of semimodules as given in *[7]*,

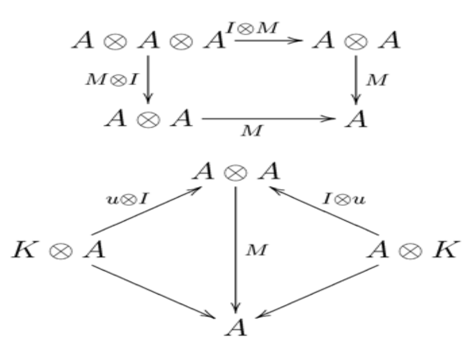
Definition 2.9.
Let $F \in Smod\text{-}K$ and $G \in K\text{-}Smod$, then both F and G are commutative monoids, so are in $\widetilde{M}$ and therefore has tensor product (considered as a commutative monoids). The tensor product $F \otimes_K G$ is defined as the factor monoid $(F \otimes_K G)/\sigma$, where σ is the congruence on $F \otimes_K G$ generated by the pairs $< ak \otimes b, a \otimes kb >$, *for* all $a \in F$, $b \in G$ and $k \in K$, such that for any balanced product (C, f) of F and G, there exists a unique morphism of monoids $\phi: F \otimes_K G \longrightarrow C$, satisfying $f = \phi \circ g$ where $g: F \times G \longrightarrow F \otimes_K G$ is given by $(m, n) \longmapsto m \otimes n$.

Remark:
Let K be a commutative semiring. Then every left K–semimodule is a right K–semimodule and vice-versa. Also, if $F, G \in K - Smod$, then $F \otimes_K G$ is a commutative monoid and it becomes a K −semimodule by defining $k(a \otimes b) = ka \otimes b = a \otimes kb$, for $a \in F$, $b \in G$ and $k \in K$.

Theorem 2.10.
Let K be commutative semiring. Then $(Smod - K, \otimes_K, K)$ is a monoidal category.

Definition 2.11.
The monoids in the monoidal category $(Smod - K, \otimes_K, K)$ are called K −semialgebras.

Therefore, a K–semialgebra can be defined as a triple (A, M, u) with A a K–semimodule, $M: A \otimes A \to A$, is called multiplication map, $u: K \to A$, a map called the unit map, and such that the following diagrams are commutative,



Now let us recall some definition and results in Hopf algebras *[8]*,

Definition 2.12.
A system$(H, M, u, \Delta, \varepsilon)$, where H has algebra structure over a commutative ring K with multiplication M and unit ε and H has co-algebra structure over K with co-multiplicaion Δ and co-unit ϵ satisfying:

(i) $M,\ u$ are co-algebra maps;

(ii) $\Delta,\ \varepsilon$ are algebra maps, is called a bialgebra.

Definition 2.13.
Let H be a bialgebra, the map $S: H \longrightarrow H$ satisfying

$$\sum_{(h)} S(h_1)\, h_2 = \varepsilon(h) 1_H = \sum_{(h)} h_1 S(h_2),$$

where $\Delta(h) = \sum_{(h)} h_1 \otimes h_2$, is called an antipode for H.

Definition 2.14.
A bialgebra with an antipode is Hopf algebra.

Lemma 2.15. ***[1]***
Let H be a finite dimentienal Hopf algebra, then

(i) If $e \in \int_l$, then $xH = Hx = kx$, and $\{x_1\}$ may be chosen to form a basis for H, where $\Delta(x) = \sum_{(x)}(x_1 \otimes x_2)$.

(ii) H is semisimple if and only if there exists $e \in \int_l (or \int_r)$ so that $e^2 = e$.

(iii) With e as in (ii) $\varepsilon(e) = 1, S(e) = e, eh = he = \varepsilon(h)e,$ hence
$\int_l = \int_r = eH = He = ke.$

III. HOPF ALGEBRA ACTIONS ON SEMIALGEBRAS

Definition 3.1.
Let A be a K-semialgebra with identity 1_A and H be Hopf algebra. We say A is called H-semimodule semialgebra if:

(i) A is an H-semimodule, where we denote the action of H of A by $h \cdot a$.

(ii) $h \cdot (ab) = \sum_{(h)}(h_1 \cdot a) \otimes (h_2 \cdot b)$, where $a, b \in A, h \in H$ and

$$\Delta(h) = \sum_{(h)} h_1 \otimes h_2$$

(iii) $h \cdot 1_A = \varepsilon(h) 1_A$, for all $h \in H$.

Now we introduce the H-invarients of an H-semimodule semialgebra A.

Definition 3.2. ***[2]***
Let A be K–semialgebra and H be a Hopf algebra acting on A, then the H–fixed subsemialgebra of A, denoted byA^H, is defined by

$$A^H = \{x \in A \mid h \cdot x = \varepsilon(h)x,\ \forall h \in H\}.$$

Definition 3.3. ***[2]***
Let H be Hopf algebra and A a H −semimodule semialgebra. The smash producta of A with$H$, written $A\#H$, is an H-semimodule semialgebra defined as follows

1. As a semimodule $A\#H$ is $A \otimes H$. Elements of $a \otimes h$ will be written $a\#h$.
2. Multiplication is defined by$(a\#g)(b\#h) = \sum_{(g)} a(g_1 \cdot b)\,\#(g_2 \cdot h)$.

IV. THE CATEGORIES OF A^H −SEMIMODULES AND $A\#H$-SEMIMODULES:

Notation:

1. ${}_{A^H}S\mathfrak{M}$ denotes category of all left A^H −semimodules.
2. ${}_{A\#H}S\mathfrak{M}$ denotes category of all left $A\#H$ −semimodules.

Remark:

1. If M is a left $A\#H$ −semimodule, then M is a left A^H −semimodule under the restriction of scalars.

2. If N is a left A^H −semimodule, then $(A\#H) \otimes_{A^H} N$ is a left $A\#H$ semimodule.

Lemma 4.1.
For $M \in {}_{A^H}S\mathfrak{M}$, define $F(M) = (A\#H) \otimes_{A^H} M$. Also if, $f: M_1 \to M_2$ is a morphism in ${}_{A^H}S\mathfrak{M}$ defined by $F(f): (A\#H) \otimes_{A^H} M_1 \to (A\#H) \otimes_{A^H} M_2$ by $F(f)(a \otimes m) =$
$a \otimes f(m), \forall a \otimes m \in (A\#H) \otimes_{R^H} M_1$. Then F is a functor from ${}_{A^H}S\mathfrak{M}$ into ${}_{A\#H}S\mathfrak{M}$.

Proof:
Let $C = {}_{A^H}S\mathfrak{M}$, $D = {}_{A\#H}S\mathfrak{M}$ and define $F: {}_{A^H}S\mathfrak{M} \to {}_{A\#H}S\mathfrak{M}$ by $F(M) = (A\#H) \otimes_{A^H} M$.
<u>Claim</u>: F is a functor:
Clearly $(A\#H) \otimes_{A^H} M$ is belonging to ${}_{A\#H}S\mathfrak{M}$. Therefore F is an object function. Let M_1, M_2 be in ${}_{A^H}S\mathfrak{M}$ and $f: M_1 \to M_2$ be morphism in ${}_{A^H}S\mathfrak{M}$, then define $F(f): (A\#H) \otimes_{A^H} M_1 \to (A\#H) \otimes_{A^H} M_2$ by $F(f)(a \otimes m) = a \otimes f(m)$, $\forall a \otimes m \in (A\#H) \otimes_{A^H} M_1$.
<u>Claim</u>: $F(f)$ is morphism in ${}_{A\#H}S\mathfrak{M}$:
Define, $\overline{F(f)}: (A\#H)\times M_1 \to (R\#H) \otimes_{A^H} M_2$ by $\overline{F(f)}(a, m) = a \otimes f(m)$

(i) $\overline{F(f)}(a_1 + a_2, m) = (a_1 + a_2) \otimes f(m)$
$= a_1 \otimes f(m) + a_2 \otimes f(m)$
$= \overline{F(f)}(a_1, m) + \overline{F(f)}(a_2, m)$

(ii) $\overline{F(f)}(a, m_1 + m_2) = a \otimes f(m_1 + m_2)$
$= a \otimes f(m_1) + a \otimes f(m_2)$
$= \overline{F(f)}\,(a, m_1) + \overline{F(f)}\,(a, m_2)$

(iii) Let $x \in A^H$, $\overline{F(f)}(ax, m) = ax \otimes f(m)$
$= a \otimes xf(m)$
$= a \otimes f(xm)$ ($\because f$ is morphism in ${}_{A^H}S\mathfrak{M}$)
$= \overline{F(f)}(a, xm)$

By universal property, there exists a morphism, $F(f): (A\#H) \otimes_{A^H} M_1 \to (A\#H) \otimes_{R^H} M_2$ given by $F(f)(a \otimes m) = a \otimes f(m)$, $\forall a \otimes m \in (A\#H) \otimes_{A^H} M_1$, therefore F is arrow function. Let $1_M: M \to M$, $M \epsilon {}_{A^H}S\mathfrak{M}$, be identity map(morphism) in ${}_{R^H}S\mathfrak{M}$. Then
$F(1_M)(a \otimes m) = a \otimes 1_M(m) = a \otimes m = 1_{(A\#H)\otimes_{A^H}M}(a \otimes m) = 1_{FM}(a \otimes m)$,
$\forall\, a \otimes m \in (A\#H) \otimes_{A^H} M$, which implies $F(1_M) = 1_{FM}$. Let $f: M_1 \to M_2$ and $g: M_2 \to M_3$ be two morphisms in ${}_{A^H}S\mathfrak{M}$, we have

$F(g \circ f)(a \otimes m) = a \otimes (g \circ f)(m)$
$= a \otimes g(f(m))$
$= F(g)(a \otimes f(m))$
$= F(g)\, F(f)(a \otimes m)$, $\forall a \otimes m \in (A\#H) \otimes_{A^H} M_1$
$\Rightarrow F(g \circ f) = F(g) \circ F(f)$.

Therefore, F is a functor from ${}_{A^H}S\mathfrak{M}$ to ${}_{A\#H}S\mathfrak{M}$.

Lemma 4.2.
For $M \in {}_{A\#H}S\mathfrak{M}$, define $G(M) = M'$, where $M' = M$ as a left A^H −semimodule. Also, if $f: M_1 \to M_2$ is a morphism in ${}_{A\#H}S\mathfrak{M}$ such that the map $G(f): M_1' \to M_2'$ defined by $G(f)(m) = f'(m), \forall m \in M_1$ where f' as a $A^H -$ morphism and $f'(m) = f(m)$. Then G is a functor from ${}_{A\#H}S\mathfrak{M}$ into ${}_{A^H}S\mathfrak{M}$.

Proof: Let $C = {}_{A^H}S\mathfrak{M}$, $D = {}_{A\#H}S\mathfrak{M}$ and define $G: {}_{A\#H}S\mathfrak{M} \to {}_{A^H}S\mathfrak{M}$ by $G(M) = M'$, where $M' = M$ as a A^H −semimodule.
<u>Claim</u>: G is a functor:
Clearly G is an object function. Let M_1, M_2 be in ${}_{A\#H}S\mathfrak{M}$ and $f: M_1 \to M_2$ be morphism in ${}_{A\#H}S\mathfrak{M}$, then define $G(f) \coloneqq f'$ where $f' = f: M_1' \to M_2'$ as a $A^H -$ morphism and $f'(m) = f(m)$. This is clearly arrow function. Let $1_M: M \to M$, $M \epsilon {}_{A\#H}S\mathfrak{M}$, be identity map(morphism) in ${}_{A\#H}S\mathfrak{M}$, then $G(1_M) = 1_M'$, where $1_M': M' \to M'$ as a $A^H -$ morphism and $1_M'(m') = 1_M(m') = 1_{M'}(m'), \forall m' \epsilon M'$, which implies $G(1_M) = 1'_M = 1_{M'} = 1_{GM}$.

Let $f_1: M_1 \to M_2$ and $g_1: M_2 \to M_3$ be two morphisms in ${}_{A\#H}S\mathfrak{M}$, then consider $g_1 \circ f_1: M_1 \to M_3$ a morphism in ${}_{A\#H}S\mathfrak{M}$, we have

$$\begin{aligned} G(g_1 \circ f_1)(m) &= (g_1 \circ f_1)'(m) \\ &= (g_1 \circ f_1)(m) \\ &= g_1(f_1(m)) \\ &= g_1'(f_1(m)) \\ &= G(g_1)(f_1(m)), \\ &= G(g_1)(G(f_1)(m) \\ &= (G(g_1) \circ G(f_1))(m), \quad \forall m \in M_1' \\ \Rightarrow G(g_1 \circ f_1) &= G(g_1) \circ G(f_1). \end{aligned}$$

∴G is functor from ${}_{A\#H}S\mathfrak{M}$ to ${}_{A^H}S\mathfrak{M}$.

Lemma 4.3.

There is a natural isomorphism between the functors $F \circ G$ and $I_{{}_{A\#H}S\mathfrak{M}}$, where $F \circ G$ is a functor from ${}_{A\#H}S\mathfrak{M}$ to ${}_{A\#H}S\mathfrak{M}$ and $I_{{}_{A\#H}S\mathfrak{M}}$ is identity functor on ${}_{A\#H}S\mathfrak{M}$.

Proof: We have two factors $F : {}_{A^H}S\mathfrak{M} \to {}_{A\#H}S\mathfrak{M}$, $G: {}_{A\#H}S\mathfrak{M} \to {}_{A^H}S\mathfrak{M}$, then $FG: {}_{A\#H}S\mathfrak{M} \to {}_{A\#H}S\mathfrak{M}$ is also functor with object function $(F \circ G)(N) = F(G(N))$ and arrow function$(F \circ G)(f) = F(G(f))$. Let$\bar{\zeta}: F \circ G \to I_{{}_{A\#H}S\mathfrak{M}}$. For each $N \in {}_{A\#H}S\mathfrak{M}$, define $\overline{\zeta_N}: (A\#H) \times N \to N$ by $\overline{\zeta_N}(a\#h, n) = (a\#h)n$. Since N is left $A\#H$ −semimodule, then $\overline{\zeta_N}$ is $A\#H$ −linear. Also, if $(b\#h) \epsilon A\#H$, $\overline{\zeta_N}\big((a\#h)(b\#h), n\big) = (a\#h)(b\#h)n = ((a\#h), (b\#h)n)$. Since $(b\#h) \in A\#H$ is arbitrary and therefore, by universal property, we have a morphism $\zeta_N: (A\#H) \otimes N \to N$ defined by

$$\overline{\zeta_N}(a\#h \otimes n) = (a\#h)n.$$

Let $f: N \to N'$ be morphism in ${}_{A\#H}S\mathfrak{M}$, we need to show the diagram

$$\begin{array}{ccc} (A\#H) \otimes_{A\#H} N & \xrightarrow{\zeta_N} & N \\ {\scriptstyle F\circ G(f)}\downarrow & & \downarrow{\scriptstyle Id(f)} \\ (A\#H) \otimes_{A\#H} N' & \xrightarrow{\zeta_{N'}} & N' \end{array}$$

commutes.

<u>**Claim:**</u> $Id(f) \circ \zeta_N = \zeta_{N'} \circ ((F \circ G)(f))$

We have, $(Id(f) \circ \zeta_N)(a\#h \otimes_{A\#H} n) = Id(f)\big(\zeta_N(a\#h \otimes_{A\#H} n)\big)$

$$\begin{aligned} &= Id(f)((a\#h)n) \\ &= (a\#h)f(n) \end{aligned}$$

Also, we have $(\zeta_{N'} \circ (F \circ G)(f))(a\#h \otimes_{A\#H} n) = \zeta_{N'}((F \circ G)(f)(a\#h \otimes_{A\#H} n))$

$$\begin{aligned} &= \zeta_{N'}(F(G(f)(a\#h \otimes_{A\#H} n))) \\ &= \zeta_{N'}(F(f(a\#h \otimes_{A\#H} n))) \\ &= \zeta_{N'}(a\#h \otimes_{A\#H} f(n)) \\ &= (a\#h)f(n) \end{aligned}$$

Since, $a\#h \otimes_{A\#H} n \,\epsilon\, (A\#H) \otimes N$ is arbitrary, which implies $Id(f) \circ \zeta_N = \zeta_{N'} \circ (F \circ G)(f)$. To prove ζ is natural isomorphism, it is enough to prove ζ_N is invertible in ${}_{A\#H}S\mathfrak{M}$. That is, if there exists $\zeta_N': N \to (A\#H) \otimes_{A\#H} N$ is morphism in ${}_{A\#H}S\mathfrak{M}$, such that $\zeta_N' \circ \zeta_N = I_{(A\#H)\otimes_{A\#H}N}$ and $\zeta_N \circ \zeta_N' = I_N$. Define $\zeta_N': N \to (A\#H) \otimes_{A\#H} N$ by $\zeta_N'(n) = (1_A\#1_H) \otimes_{A\#H} n, \forall\, n \in N$.

We have,

$$\begin{aligned} (\zeta_N \circ \zeta_N')(n) &= \zeta_N(\zeta_N'(n)) \\ &= \zeta_N((1_A\#1_H) \otimes_{A\#H} n) \\ &= (1_A\#1_H)n = n = I_N(n), \quad \forall n \in N. \end{aligned}$$

Since n is arbitrary in N, it follows that $\zeta_N \circ \zeta_N' = I_N$.

We have, $(\zeta_N' \circ \zeta_N)(a\#h \otimes_{A\#H} n) = \zeta_N'(\zeta_N(a\#h \otimes_{A\#H} n))$

$$\begin{aligned} &= \zeta_N'((a\#h)n) \\ &= (1_A\#1_H) \otimes_{A\#H} (a\#h)n \\ &= a\#h \otimes_{A\#H} n \\ &= I_{(A\#H)\otimes_{A\#H}N}(a\#h \otimes_{A\#H} n) \end{aligned}$$

Since, $a\#h \otimes_{A\#H} n \in (A\#H) \otimes N$ is arbitrary, which implies ${\zeta_N}' \circ \zeta_N = I_{(A\#H)\otimes_{A\#H}N}$. Hence there is a natural isomorphism between the functors $F \circ G$ and $I_{{}_{A\#H}S\mathfrak{M}}$.

Lemma 4.4.

There is a natural transformation between the functors $G \circ F$ and $I_{{}_{A^H}S\mathfrak{M}}$, where $G \circ F$ is a functor from ${}_{A^H}S\mathfrak{M}$ to ${}_{A^H}S\mathfrak{M}$ and $I_{{}_{A^H}S\mathfrak{M}}$ is identity functor on ${}_{A^H}S\mathfrak{M}$.

Proof:

We have two functors $F\colon {}_{A^H}S\mathfrak{M} \to {}_{A\#H}S\mathfrak{M}$, $\colon {}_{A\#H}S\mathfrak{M} \to {}_{A^H}S\mathfrak{M}$, then $G \circ F\colon {}_{A^H}S\mathfrak{M} \to {}_{A^H}S\mathfrak{M}$ is also a functor with object function $G \circ F(M) = G\big(F(M)\big)$ and arrow function $G \circ F(f) = G\big(F(f)\big)$. Let $\eta\colon GF \to I_{{}_{R^H}S\mathfrak{M}}$. For each $M \in {}_{A^H}S\mathfrak{M}$, define $\overline{\eta_M}\colon (A\#H) \times M \to M$ by $\overline{\eta_M}(a\#h, m) = \varepsilon(h)(e \cdot a)m$,

where $e \in \int_H$ as mentioned in Lemma(2.15).

(i) $\overline{\eta_M}((a\#h) + (b\#g), m) = \varepsilon(h)(e \cdot a)m + \varepsilon(g)(e \cdot a)m$
$= \overline{\eta_M}(a\#h, m) + \overline{\eta_M}(b\#g, m)$

(ii) $\overline{\eta_M}(a\#h,\ m_1 + m_2) = \varepsilon(h)(e \cdot a)(m_1 + m_2)$
$= \varepsilon(h)(e \cdot a)m_1 + \varepsilon(h)(e \cdot a)m_2$
$= \overline{\eta_M}(a\#h, m_1) + \overline{\eta_M}(a\#h, m_2)$

(iii) Let $b \in R^H$, $\overline{\eta_M}\big((a\#h)b, m\big) = \overline{\eta_M}(ab\#h, m)$
$= \varepsilon(h)(e \cdot ab)m$
$= \varepsilon(h)(e \cdot a)bm$
$= \overline{\eta_M}(a\#h, bm)$

By universal property, there exists a morphism $\eta_M\colon (A\#H) \otimes M \to M$ defined by

$$\eta_M(a\#h \otimes m) = \varepsilon(h)(e \cdot a)m.$$

For each morphism, $f\colon M \to M'$ in ${}_{A^H}S\mathfrak{M}$, we need to show, the diagram,

$$\begin{array}{ccc} (A\#H) \otimes_{A^H} M & \xrightarrow{\eta_M} & M \\ {\scriptstyle G\circ F(f)}\downarrow & & \downarrow{\scriptstyle Id(f)} \\ (A\#H) \otimes_{A^H} M' & \xrightarrow{\eta_{M'}} & M' \end{array}$$

commutes.

<u>Claim</u>: $Id(f) \circ \eta_M = \eta_{M'} \circ ((G \circ F)(f))$

We have, $(Id(f) \circ \eta_M)\big(a\#h \otimes_{A^H} m\big) = Id(f)(\ \varepsilon(h)(e \cdot a)m)$
$= \varepsilon(h)(e \cdot a)f(m)$

Also, we have $\eta_{M'} \circ (G \circ F)(f)\big(a\#h \otimes_{A^H} m\big) = \eta_{M'}(G(F(f)\big(a\#h \otimes_{A^H} m)\big)\big)$
$= \eta_{M'}(F(f)\big(a\#h \otimes_{A^H} m)\big)$
$= \eta_{M'}\big(a\#h \otimes_{A^H} f(m)\big)$
$= \varepsilon(h)(e \cdot a)f(m)$

Since, $a\#h \otimes_{A^H} m \in (A\#H) \otimes M$ is arbitrary, which implies $Id(f) \circ \eta_M = \eta_{M'} \circ ((G \circ F)(f))$.

Hence there is a natural transformation between the functors $G \circ F$ and $I_{{}_{A^H}S\mathfrak{M}}$.

Lemma 4.5.

For every $M \in ob\big({}_{A^H}S\mathfrak{M}\big)$, the map $\eta_M\colon (A\#H) \otimes_{A^H} M \to M$, given in the above lemma has a right inverse. i.e., there exist a map ${\eta_M}'\colon M \to (A\#H) \otimes_{A^H} M$ such that $\eta_M \circ {\eta_M}' = I_M$.

Proof:

Define ${\eta_M}'\colon M \to (A\#H) \otimes_{A^H} M$ by ${\eta_M}'(m) = (1_A\#1_H) \otimes_{A^H} m, \forall\, m \in M$.

We have,

$$(\eta_M \circ {\eta_M}')(m) = \eta_M({\eta_M}'(m))$$

$$= \eta_M((1_A \# 1_H) \otimes_{A^H} m)$$
$$= \varepsilon(1_H)(e \cdot 1_A)m$$
$$= 1_K \varepsilon(e) \cdot m$$
$$= m = I_M(m). (\because \varepsilon(e) = 1 \text{ as } H \text{ is semisimple})$$

Since m is arbitrary in M, implies that $\eta_M \circ \eta_M{}' = I_M$.

Remark:

In general, the map η_M may not have a left inverse. That is., $\eta_M{}' \circ \eta_M$ may not be equal to $I_{(A\#H)\otimes_{A^H}M}$. But, if the map $\phi: A\#H \to A^H$ given by $\phi(a\#h) = \varepsilon(h)(e \cdot a)$, preserve multiplication then any left A^H −semimodule M becomes a left $A\#H$ −semimodule under the action given by

$$(a\#h) \to m = \phi(a\#h)m = \varepsilon(h)(e \cdot a)m$$

Further, if the action of H on A is trivial or if $e \in Z_{A\#H}(A)$ then ϕ is multiplicative.

Theorem 4.6.

If H is a finite dimensional semisimple Hopf algebra acting on a semialgebra A and if $e \in Z_{A\#H}(A)$ then the categories ${}_{A^H}S\mathfrak{M}$ and ${}_{A\#H}S\mathfrak{M}$ are equivalent.

Proof: Follows from Lemma (4.1), (4.2), (4.3), (4.4), (4.5) and the above Remark.

V. CONCLUSION

In this paper, we have introduced two functors F: ${}_{A^H}S\mathfrak{M} \to {}_{A\#H}S\mathfrak{M}$, $G: {}_{A\#H}S\mathfrak{M} \to {}_{A^H}S\mathfrak{M}$ between the categories of A^H −semimodules and the categories of $A\#H$ −semimodules. Further we established a natural isomorphism ζ between the functors $F \circ G$ and $I_{{}_{A\#H}S\mathfrak{M}}$. Also we established a natural transformation τ between the functors $G \circ F$ and $I_{{}_{A^H}S\mathfrak{M}}$. Assuming H as a finite dimensional semisimple Hopf algebra, we established the equivalence between categories ${}_{A^H}S\mathfrak{M}$ and ${}_{A\#H}S\mathfrak{M}$.

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