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Common Fixed Point Theorems for Weakly Compatible Mappings on Cone Banach Space

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Abstract: The aim of this paper is to obtain coincidence points and common fixed points for eight self-mappings under weakly compatible condition on Cone Banach Space. In theorem 2 and theorem 3 we prove the coincidence points and common fixed points for six and four self-mappings respectively. In this literature our results is a generalization of many existing results.

Mathematics subject classification: 47H10, 54H25

Key wards: Coincidence Point, Fixed Point, Compatible Mappings, Weakly Compatible Mappings, Cone Banach Space.

I. INTRODUCTION

In 2007, Huang and Zhang [4] introduced the notion of Cone Metric Space, replacing the set of real numbers by ordered Banach Space and proved some fixed point theorems for functions satisfying Contractive conditions in this space. In the last decade many researchers fined fixed point theorems in this space and generalized many results such as M.Abbas and G.Junck[1], R.K.Gujetiya et al.[3],etc. Very recently some results on fixed point theorems have been extended to Cone Banach Space.
E.Karapinar[5] proved some fixed point theorems for self mappings satisfying some contractive condition on a Cone Banach Space.
T.Abdeljawad et al.[2] generalize the results of E. Karapinar [5] .
R.Tiwary et al.[6] generalize the results of T. Abdeljawad et al.[2].
In this paper we generalize the results of R.Tiwary et al.[6] and prove coincidence points and common fixed point theorem for eight self mappings with the help of weakly compatible mappings.

II. DEFINITIONS

Definition 1:[7] Let E be a real Banach spaces and K be a subset of E. K is called a cone if and only if
(a). K is closed, nonempty and $K \neq \{0\}$.
(b). ax+by in K for all x, y in K and $a, b \geq 0$.
(c). $x \in K$ and $-x \in K \Rightarrow x=0 \Leftrightarrow K \cap (-K) = \{0\}$.

Consider a cone $K \subset E$. We define a partial ordering " $\leq$ " with respect to K by $x \leq y$ if and only if $y-x \in K$, we write $x < y$ to indicate that $x \leq y$ but $x \neq y$ and $x << y$ to indicate that $y-x \in \text{int}K$. The intK denotes the interior of K.
Let X be a non empty set and $K \subset E$ be a real Banach space. Suppose the metric mapping $d: X\times X\rightarrow E$ is satisfies the following conditions:
1. $d(x,y) \geq 0$ and $d(x,y) =0$ iff x=y, for all x, y in X.
2. $d(x,y) = d(y,x)$, for all x ,y in X.
3 .$d(x,z)\leq d(x,y) + d(y,z)$; for all x,y,z in X.
Then d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 2:[7] Let, X be a vector space over R. Suppose the mapping $\|.\| : X\rightarrow E$ satisfies

1. $\|x\| > 0$, for all $x \in X$.
2. $\|x\| =0$ if and only if x=0.
3. $\|x+y\| \leq \|x\| + \|y\|$, for all $x,y \in X$.
4. $\|kx\| = |k|\ \|x\|$, for all $k \in R$.

Then $\|.\|$ is called a norm on X, and $(X, \|.\|)$ is called a cone normed space. Clearly each cone normed space is a cone metric space with defined by $d(x,y)= \|x-y\|$.

Definition 3:[7] Let, (X, $\|.\|$) be a cone normed space, x $\in$ X and $\{x_n\}$ is a sequence in X. then,

1. $\{x_n\}$ converges to x if for every c $\in$ E with 0<<c there is a natural number N such that $\|x_n - x\| \leq$ c for all n $\geq$ N. we shall denote it by $\lim_{n\to\infty} x_n$=x or, $x_n \to$x.
2. $\{x_n\}$ is a Cauchy sequence , if for every c $\in$ E with 0<<c there is a natural number N such that $\|x_n - x_m\| \leq$ c for all n,m $\geq$ N.
3. (X, $\|.\|$) is a complete cone normed space if every Cauchy sequence is convergent.

A complete cone normed space is called a Cone Banach space.

Definition 4: Two self maps A and B of a cone normed space (X, $\|.\|$) are said to be compatible if $\lim_{n\to\infty} \|ABx_n - BAx_n\|$ = 0 for all a in X, where $\{x_n\}$ is a sequence in X such that if $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n$= x for some x in X.

Definition 5: Maps f and g are said to be commuting if fgx = gfx, for all x $\in$ X.

Definition 6: Let, f and g be two self maps on a set X, if fx=gx for some x in X then x is called coincidence point of f and g.

Definition 7: Let A and B be mappings from a cone normed space (X, $\|.\|$) into itself. A and B are said to be weakly compatible if they commute at their coincidence point i.e, Ax=Bx for some x in X implies ABx=BAx.

III. MAIN RESULTS

Theorem 1: Let, (X,$\|.\|$) be a Cone Banach Space and d: X×X→E with d(x,y)= $\|x - y\|$. Let, A, B, C, D, K, M, P and V be eight self mappings on X that satisfy the conditions:

(a) V(X) $\subseteq$ ABC(X) and P(X) $\subseteq$ DKM(X).

(b) a $\|Px - Vy\|$ + b { $\|ABCx - Px\|$ + $\|DKMy - Vy\|$ } + c{ $\|DKMy - Px\|$ + $\|ABCx - Vy\|$ } $\leq$ r $\|ABCx - DKMy\|$; for all x,y $\in$ X, 0 $\leq$ r < a+2b+3c, a+b+c≠0, r≠a+2c. …(1)

(c) (P,ABC) and (V,DKM) are weakly compatible.

(d) If one of P(X), ABC(X), V(X), DKM(X) is a complete subspace of X then,
1. P and ABC have a coincidence point and
2. V and DKM have a coincidence point in X.

Then, A, B, C, D, K, M, P and V have a unique common fixed point in X.

Proof: Let, $x_0 \in$ X be arbitrary, then $V(x_0) \in$ X. Since, V(X) $\subseteq$ ABC(X) there exists $x_1 \in$ X such that $ABC(x_1)$ = $V(x_0)$ and for x_1 there exists $x_2 \in$ X such that $DKM(x_2)$ = $P(x_1)$ and so on, continuing this process we can define a sequence $\{y_n\}$ in X such that $y_n = Vx_n = ABCx_{n+1}$ and $y_{n+1}=Px_{n+1}=DKMx_{n+2}$.

Now, put $x=x_n$ and $y=x_{n+1}$ in (1) and get,

a $\|Px_n - Vx_{n+1}\|$ + b { $\|ABCx_n - Px_n\|$ + $\|DKMx_{n+1} - Vx_{n+1}\|$ } + c{ $\|DKMx_{n+1} - Px_n\|$ +$\|ABCx_n - Vx_{n+1}\|$} $\leq$ r $\|ABCx_n - DKMx_{n+1}\|$.

Or, a $\|y_n - y_{n+1}\|$ + b { $\|y_{n-1} - y_n\|$ + $\|y_n - y_{n+1}\|$ } + c{ $\|y_n - y_n\|$ +$\|y_{n-1} - y_{n+1}\|$ } $\leq$ r $\|y_{n-1} - y_n\|$.

Or, a $\|y_n - y_{n+1}\|$ + b { $\|y_{n-1} - y_n\|$ + $\|y_n - y_{n+1}\|$} $\leq$ r $\|y_{n-1} - y_n\|$ - c$\|y_{n-1} - y_{n+1}\|$ $\leq$ r $\|y_{n-1} - y_n\|$ –c{$\|y_{n-1} - y_n\|$ + $\|y_n - y_{n+1}\|$}.

Or, a $\|y_n - y_{n+1}\|$ + b{ $\|y_{n-1} - y_n\|$ + $\|y_n - y_{n+1}\|$} + c{$\|y_{n-1} - y_n\|$ + $\|y_n - y_{n+1}\|$} $\leq$ r $\|y_{n-1} - y_n\|$.

Or, $\|y_n - y_{n+1}\| \leq \frac{r-b-c}{a+b+c} \|y_{n-1} - y_n\|$.

Or, $\|y_n - y_{n+1}\| \leq$ k $\|y_{n-1} - y_n\|$; where k=$\frac{r-b-c}{a+b+c}$, k< 1 as r < a+2b+3c ……….(2)

Proceeding as above we will get,

$\|y_n - y_{n+1}\| \leq$ k $\|y_{n-1} - y_n\| \leq$ k^2 $\|y_{n-2} - y_{n-1}\| \leq \ldots \leq$ k^n $\|y_0 - y_1\|$ where k< 1 ….(3)

Now, let m > n , then,

$\|y_n - y_m\|$ $\leq$ $\|y_n - y_{n+1}\|$ + $\|y_{n+1} - y_{n+2}\|$ +….+$\|y_{m-1} - y_m\|$

$\leq$($k^n + k^{n+1} + \ldots + k^{m-1}$) $\|y_0 - y_1\|$ (by (3))

$= \frac{k^n(1-k^m)}{1-k} \|y_0 - y_1\| \le \frac{k^n}{1-k} \|y_0 - y_1\|$. …………..(4)

Let, c > 0, then there is a δ >0 such that $c+N_\delta(0) \subseteq H$ where $N_\delta(0)=\{y \in X: \|y\| \le \delta\}$. Since k <1 there exists a positive integer N such that $\frac{k^n}{1-k}\|y_0 - y_1\| \le \delta$ for every n ≥ N. Hence $\frac{k^n}{1-k}\|y_0 - y_1\| \in N_\delta(0)$, which implies $-\frac{k^n}{1-k}\|y_0 - y_1\| \in N_\delta(0)$. Therefore, $c - \frac{k^n}{1-k}\|y_0 - y_1\| \in c+ N_\delta(0) \subseteq H$ implies $\frac{k^n}{1-k}\|y_0 - y_1\| \le c$ for n ≥N. So, by definition $\{y_n\}$ is a Cauchy sequence in X.

Since X is complete there exists a z in X such that $\lim_{n\to\infty} y_n=z$, and $\lim_{n\to\infty} Vx_n= \lim_{n\to\infty} ABCx_{n+1} =\lim_{n\to\infty} Px_{n+1}=\lim_{n\to\infty} DKMx_{n+2}=z$.

Now, suppose that ABC(X) is complete.

Then there exists a point p in X such that ABCp=z. ……….(5)

Now, put x=p and y=x_n in (1) and get,

$a\|Pp - Vx_n\| + b\{\|ABCp - Pp\| + \|DKMx_n - Vx_n\|\} + c\{\|DKMx_n - Pp\| + \|ABCp - Vx_n\|\} \le r\|ABCp - DKMx_n\|$.

Taking $\lim_{n\to\infty}$ and using (5) in the above inequality we get,

$a\|Pp - z\| + b\{\|z - Pp\| + \|z - z\|\} + c\{\|z - Pp\| + \|z - z\|\} \le r\|z - z\|$.

i.e., (a+b+c) $\|Pp - z\| \le 0$.

i.e., $\|Pp - z\| = 0$ (as $a+b+c \neq 0$).

So,Pp=z. ……..(6)

From (5) and (6) we get, ABCp=z=Pp.

That is p is a coincidence point of ABC and P.

As $P(X) \subseteq DKM(X)$, Pp=z implies $z \in DKM(X)$. let, u in X then DKMu=z. ………….(7)

Now, put x=x_{n+1} and y=u in (1) and get,

$a\|Px_{n+1} - Vu\| + b\{\|ABCx_{n+1} - Px_{n+1}\| + \|DKMu - Vu\|\} + c\{\|DKMu - Px_{n+1}\| + \|ABCx_{n+1} - Vu\|\} \le r\|ABCx_{n+1} - DKMu\|$

Taking $\lim_{n\to\infty}$ and using (7) in the above inequality we get,

$a\|z - Vu\| + b\{\|z - z\| + \|z - Vu\|\} + c\{\|z - z\| + \|z - Vu\|\} \le r\|z - z\|$.

Or, (a+b+c) $\|z - Vu\| \le 0$.

i.e., $\|z - Vu\| = 0$ (as $a+b+c \neq 0$).

So,Vu=z. ……..(8)

From (7) and (8) we get, DKMu=z=Vu.

That is u is a coincidence point of V and DKM.

Since, (P,ABC) and (V, DKM) are weakly compatible mapping in X.

So, P.ABCp=ABC.Pp i.e., Pz=ABCz (by(5),(6)) ……….(9)

And V.DKMu=DKM.Vu i.e., Vz=DKMz (by(7)and(8)) ………..(10)

Now, put x=z and y=x_n in (1) and get,

$a\|Pz - Vx_n\| + b\{\|ABCz - Pp\| + \|DKMx_n - Vx_n\|\} + c\{\|DKMx_n - Pz\| + \|ABCz - Vx_n\|\} \le r\|ABCz - DKMx_n\|$.

Taking $\lim_{n\to\infty}$ and using (9) in the above inequality we get,

$a\|Pp - z\| + b\{\|Pz - Pz\| + \|z - z\|\} + c\{\|z - Pz\| + \|Pz - z\|\} \le r\|Pz - z\|$.

i.e., (a+2c-r) $\|Pz - z\| \le 0$.

i.e., $\|Pz - z\| = 0$ (as $a+2c-r \neq 0$).

So, Pz=z.

So, from (9) we get, Pz=ABCz=z. …………..(11)

Now, put x=x_{n+1} and y=z in (1) and get,

$a\|Px_{n+1} - Vz\| + b\{\|ABCx_{n+1} - Px_{n+1}\| + \|DKMz - Vz\|\} + c\{\|DKMz - Px_{n+1}\| + \|ABCx_{n+1} - Vz\|\} \le r\|ABCx_{n+1} - DKMz\|$.

Taking $\lim_{n\to\infty}$ and using (10) in the above inequality we get,

$a\|z - Vz\| + b\{\|z - z\| + \|Vz - Vz\|\} + c\{\|Vz - z\| + \|z - Vz\|\} \le r\|z - Vz\|$.

Or, (a+2c-r) $\|z - Vz\| \le 0$.

i.e., $\|z - Vz\| = 0$ (as $a+2c-r \neq 0$).

So, Vz=z.

From (10) we get, Vz=DKMz=z. …………….(12)

Now, put x=Cz and y=z in (1),

$a\|P(Cz) - Vz\| + b\{\|ABC(Cz) - P(Cz)\| + \|DKMz - Vz\|\} + c\{\|DKMz - P(Cz)\| + \|ABC(Cz) - Vz\|\} \le r\|ABC(Cz) - DKMz\|$.

Or, $a\|Cz - z\| + b\{\|Cz - Cz\| + \|z - z\|\} + c\{\|z - Cz\| + \|Cz - z\|\} \le r\|Cz - z\|$. (by (11)& (12))

Or,(a+2c-r) $\|Cz - z\| \le 0$.

i.e., $\|Cz - z\| = 0$ (as $a+2c-r \neq 0$).

i.e.,Cz=z. ....(13)

Now, put x=Bz and y=z in (1),

$a \|P(Bz) - Vz\| + b \{ \|ABC(Bz) - P(Bz)\| + \|DKMz - Vz\| \} + c\{ \|DKMz - P(Bz)\| + \|ABC(Bz) - Vz\|\} \leq r \|ABC(Bz) - DKMz\|$.

Or, $a\|Bz - z\| + b \{\|Bz - Bz\| + \|z - z\| \} + c\{ \|z - Bz\| + \|Bz - z\| \} \leq r \|Bz - z\|$. (by (11)& (12))

Or,$(a+2c-r) \|Bz - z\| \leq 0$.

i.e., $\|Bz - z\| = 0$ (as $a+2c-r \neq 0$).

i.e.,Bz=z. ....(14)

From (11) ABCz=z i.e., ABz=z (by (13)) i.e., Az=z. (by (14)) ....(15)

Now, put x=z and y=Mz in (1),

$a \|Pz - V(Mz)\| + b \{ \|ABCz - Pz\| + \|DKM(Mz) - V(Mz)\| \} + c\{ \|DKM(Mz) - Pz\| + \|ABCz - V(Mz)\|\} \leq r \|ABCz - DKM(Mz)\|$.

Or, $a\|z - Mz\| + b \{\|z - z\| + \|Mz - Mz\| \} + c\{ \|Mz - z\| + \|z - Mz\| \} \leq r \|z - Mz\|$. (by (11)& (12))

Or, $(a+2c-r) \|z - Mz\| \leq 0$.

i.e., $\|z - Mz\|=0$ (as $a+2c-r \neq 0$).

i.e., Mz=z. ....(16)

Now, put x=z and y=Kz in (1),

$a \|Pz - V(Kz)\| + b \{ \|ABCz - Pz\| + \|DKM(Kz) - V(Kz)\| \} + c\{ \|DKM(Kz) - Pz\| + \|ABCz - V(Kz)\|\} \leq r \|ABCz - DKM(Kz)\|$.

Or, $a \|z - Kz\| + b \{ \|z - z\| + \|Kz - Kz\| \} + c\{\|Kz - z\| + \|z - Kz\|\} \leq r \|z - Kz\|$. (by (11)& (12))

Or, $(a+2c-r) \|z - Kz\| \leq 0$.

i.e., $\|z - Kz\|=0$ (as $a+2c-r \neq 0$).

i.e., Kz=z. ....(17)

From (12) we get, DKMz=z i.e., DKz=z (by (16)) i.e., Dz=z. (by (17)) ....(18)

From (11) to (18) we get z is a fixed point of A, B, C, D, K, M, P and V.

Now, we will prove that z is a unique fixed point.

If possible let there exists another fixed point w(≠z).

Put, x=z and y=w in (1), then we get,

$a \|Pz - Vw\| + b \{ \|ABCz - Pz\| + \|DKMw - Vw\| \} + c\{ \|DKMw - Pz\| + \|ABCz - Vw\|\} \leq r \|ABCz - DKMw\|$.

Or, $a \|z - w\| + b \{ \|z - z\| + \|w - w\| \} + c\{ \|w - z\| + \|z - w\| \} \leq r \|z - w\|$.

Or, $(a+2c-r) \|z - w\| \leq 0$.

i.e., $\|z - w\| = 0$ (as $a+2c-r \neq 0$).

i.e., z=w.

So, the fixed point z is unique.

Thus z is a unique fixed pint of A, B, C, D, K, M, P and V.

Theorem 2: Let, $(X,\|.\|)$ be a Cone Banach Space and d: $X \times X \to E$ with $d(x,y)= \|x - y\|$. Let, A, B, D, K, P and V be six self mappings on X that satisfy the conditions:

(e) $V(X) \subseteq AB(X)$ and $P(X) \subseteq DK(X)$.

(f) $a \|Px - Vy\| + b \{ \|ABx - Px\| + \|DKy - Vy\| \} + c\{ \|DKy - Px\| + \|ABx - Vy\| \} \leq r \|ABx - DKy\|$; for all $x,y \in X$, $a,b,c > 0$, $0 \leq r < a+2b+3c$, $a+b+c \neq 0, r \neq a+2c$. ...(19)

(g) (P,AB) and (V,DK) are weakly compatible.

(h) If one of P(X), AB(X), V(X), DK(X) is a complete subspace of X then,

3. P and AB have a coincidence point and
4. V and DK have a coincidence point in X.Then, A, B, D, K, P and V have a unique common fixed point in X.

Theorem 3: Let, $(X,\|.\|)$ be a Cone Banach Space and d: $X \times X \to E$ with $d(x,y)= \|x - y\|$. Let, A, D, P and V be four self mappings on X that satisfy the conditions:

(i) $V(X) \subseteq A(X)$ and $P(X) \subseteq D(X)$.

(j) $a \|Px - Vy\| + b \{ \|Ax - Px\| + \|Dy - Vy\| \} + c\{ \|Dy - Px\| + \|Ax - Vy\| \} \leq r \|Ax - Dy\|$; for all$x,y \in X$, $a,b,c>0$, $0 \leq r < a+2b+3c$, $a+b+c \neq 0$, $r \neq a+2c$. .... (20)

(k) (P,A) and (V,D) are weakly compatible.

(l) If one of P(X), A(X), V(X), D(X) is a complete subspace of X then,

5. P and A have a coincidence point and
6. V and D have a coincidence point in X.

Then, A, D, P and V have a unique common fixed point in X.

IV. CONCLUSIONS

In this paper we prove coincidence point and common fixed points by the help of weakly compatible mappings in Cone Banach Space. Our result is a generalization of R. Tiwary et al.[6], T. Abdeljawad et al.[2], E. Karapinar[5]. The main result in this paper is a generalization of many existing results in this literature. We also prove coincidence point and common fixed point for six mappings in theorem 2 and for four mappings in theorem 3.

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