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Research Article

A Robust Numerical Method for Solving Linear Delay Differential Equations

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Abstract— This study introduces the Modified New Iterative Method (MNIM) for solving linear delay differential equations (LDDEs) and enhancing accuracy with a post-treatment iterative technique. Key concepts, including LDDEs and El-Kalla polynomials, are outlined. MNIM's validity is confirmed through test cases, showing precise approximations with minimal error. The method proves efficient and reliable.

Keywords— Linear delay differential equations, El-Kalla polynomials, Adomian polynomials

1. Introduction

Delay differential equations (DDEs) represent an important class of functional differential equations where the derivative of the unknown function depends on its past values. These equations are commonly encountered in various scientific and engineering fields, such as population dynamics, control systems, and signal processing [1,2]. Solving DDEs analytically is often difficult, particularly for complex systems, due to their dependence on historical data and inherent delays. Numerical methods have become essential for solving DDEs. Traditional approaches, such as finite difference methods and Runge-Kutta schemes, are frequently used but may face challenges in terms of accuracy and computational efficiency when handling delay terms [3]. Iterative methods have gained prominence because they typically provide more accurate approximations with lower computational costs [4].

In this context, the Modified New Iterative Method (MNIM) emerges as a robust and efficient numerical scheme for solving linear delay differential equations (LDDEs). By leveraging iterative refinement and post-treatment strategies, MNIM improves solution accuracy while maintaining computational simplicity. This study validates MNIM through a series of significant test cases, highlighting its potential to address real-world problems in fields where LDDEs are critical. The importance of this study lies in its ability to offer a reliable and computationally efficient method for solving complex LDDEs, making it a valuable tool for researchers and practitioners working with delay-differential systems in various applications [5].

2. Related Work

The study of solution methodologies for Delay Differential Equations (DDEs) and their more complex variants, Fractional Delay Differential Equations (FDDEs), has led to notable advancements. Among these, Srivastava introduced the New Variational Iteration Method (NVIM), a promising technique for deriving approximate analytical solutions to FDDEs [6]. This method has been successfully applied to both linear and nonlinear initial value problems, with results compared against exact solutions. The analysis underscores the capability of NVIM to produce accurate approximate solutions with relatively few iterations, highlighting its potential as an efficient and reliable approach for solving FDDEs [6].

Jhinga and Daftardar-Gejji introduced an innovative predictor-corrector technique designed specifically to address nonlinear fractional delay differential equations (FDDEs) [7]. They conducted a thorough error analysis of this method and demonstrated its effectiveness through various illustrative examples. The results underscored its superior accuracy and time efficiency compared to established numerical approaches for FDDEs, such as the Fractional Adams-Moulton (FAM) and the Three-Term Numerical Predictor-Corrector Method (NPCM) [7]. A key observation from their study was that the L1-PCM method maintained convergence for very small values of the parameter α\alphaα, whereas the FAM and NPCM methods diverged under similar conditions [7].

Nemah utilized the Mahgoub transform in combination with the Variational Iteration Method (VIM) to solve nonlinear

FDDEs, aiming to eliminate unnecessary assumptions inherent in other algorithms [8]. In the first example, presented in Table 2, their approach successfully replicated the exact solution and closely matched results obtained via the Adomian Decomposition Method (ADM) and the Linear Adomian Decomposition Method (LADM) [8]. The second example, shown in Table 3, yielded notable findings: for ttt values ranging from 0.1 to 0.5, the method closely mirrored exact solutions, with superior performance at t=0.4t = 0.4t=0.4 and t=0.5t = 0.5t=0.5 compared to the Modified Adomian Decomposition Method (MADM) [8]. For ttt values between 0.6 and 0.9, the method exhibited closer convergence to the exact solution than MADM [8]. In Example 3 (Table 4), the Mahgoub-Variational Iteration Method (MVIM) not only achieved the exact solution but also aligned with results from the Homotopy Analysis Method (HAM) and ADM, outperforming MADM [8]. These examples collectively highlighted the superior efficacy of Nemah's approach for solving FDDEs [8].

3. Theory/Calculation

3.1 Overview of the New Iterative Method (NIM)

To understand the core principles of the New Iterative Method (NIM), it is useful to examine a well-established functional equation, as explored in the works of Daftardar-Gejji & Bhalekar (2010), Ramadan & Al-Luhaibi (2015), Moltot & Deresse (2022), and Ashitha & Ranjini (2020). This approach starts with analyzing the nonlinear functional equation introduced by Daftardar-Gejji & Jafari (2006).

$$y(x) = g(x) + N[y(x)] \qquad \dots(1)$$

In this context, N represents the nonlinear operator, and f is a known function. The goal is to determine a solution, denoted as

$y(x)$, which possesses a series representation in the following format:

$$y = \sum_{i=0}^{\infty} y_i \,. \qquad \dots(2)$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\} \qquad (3)$$

From Eqns. (2) and (3), Eqn. (1) is equivalent to

$$\sum_{i=0}^{\infty} y_i = g + N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\} \qquad (4)$$

We define the recurrence relation:

$$\begin{cases} y_0 = g, \\ y_1 = N(y_0) \\ y_{m+1} = N(y_0 + \dots + y_m) - N(y_0 + \dots + y_{m-1}), m = 1,2,\dots \end{cases} \qquad (5)$$

Then

$$(y_1 + \dots + y_{m+1}) = N(y_0 + \dots + y_m), m = 1,2,\dots \qquad (6)$$

and

$$y = g + \sum_{i=0}^{\infty} y_i \,. \qquad \dots(7)$$

and the series $\sum_{i=0}^{\infty} y_i$ absolutely and uniformly converges to a solution of Eqn. (1).

3.2 The Proposed New Iterative Method (NIM)

In a prior study, the New Iterative Method (NIM) was used to approximate solutions for ordinary differential equations. In this section, we present new algorithms aimed at simplifying the resolution of Delay Differential Equations (DDEs). To ensure a clear understanding of these newly developed, generalized NIM algorithms, we will first explore the fundamental structure of Delay Differential Equations (DDEs).

$$y^{(n)}(x) + P[y(x)] + N[y(x-t)] = f(x), \quad n = 1,2,3,\dots \qquad \dots(11)$$

$$y^{(k)} = \delta_i, \qquad i = 0,1,2,\dots \qquad (12)$$

where $y^{(n)}(x)$ is the derivative of y order n , P is the linear bounded operator, N is a nonlinear bounded operator, $f(x)$ is a given continuous function, and $y = y(x)$.

In this section, the general form of the n^{th}-order DDE Eqn. (11) with the initial value Eqn. (12) is treated using the suggested method MNIM.

Next, by isolating the term associated with the derivative, we get

$$y^{(n)}(x) = f(x) - P[y(x)] - N[y(x-t)] \qquad (13)$$

Applying the J^n on both sides of Eqn. (13), we get

$$y(x) = J^n\left[f(x) - P(y(x)) - N(y(x-t))\right] + \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!} \qquad (14)$$

Let's consider dividing this equation into two separate parts as follows:

$$y(x) = N(y(x)) + g(x) \qquad \dots(15)$$

where

$$N(y(x)) = J^n\left[f(x) - P(y(x)) - N(y(x-t))\right] \qquad (16)$$

In typical cases, N serves as the nonlinear operator; however, when dealing with the DDE, it is employed with linear functions. Additionally, "g" represents a known function, defined as:

$$g(x)=\sum_{i=0}^{r} y^{(i)}(0)\frac{x^{i}}{i!}, \tag{17}$$

In our quest for a solution to Eqn. (11), we seek a representation in the form of a series:

$$y(x)=\sum_{i=0}^{\infty} y_i(x). \tag{18}$$

The operator N can be decomposed into the following

$$N\left(\sum_{i=0}^{\infty} y_i(x-t)\right)=N(y_0)+\sum_{i=1}^{\infty}\left\{\begin{matrix} N\left(\sum_{j=0}^{i} y_j\right) \\ -N\left(\sum_{j=0}^{i-1} y_j\right)\end{matrix}\right\} \tag{19}$$

.

From Eqns. (11), (17) and Eqn. (18)

$$\sum_{i=0}^{\infty} y_i = g(x)+N(y_0)+\sum_{i=1}^{\infty}\left\{\begin{matrix} N\left(\sum_{j=0}^{i} y_j\right) \\ -N\left(\sum_{j=0}^{i-1} y_j\right)\end{matrix}\right\} \tag{20}.$$

$$y_0=g(x)=\sum_{i=0}^{r} y^{(i)}(0)\frac{x^{i}}{i!}, \quad \ldots(21)$$

$$y_1=J^{n}\left[\begin{matrix} f(x)-P(y_0(x)) \\ -N(y_0(x-t))\end{matrix}\right] \quad (22)$$

$$\begin{cases} y_2=N(y_0+.y_1)-N(y_0)= \\ J^{n}[-P(y_0(x))-N(y_0(x-t))-P(y_1(x))-N(y_1(x-t))]-J^{n}[y_0(x-t)] \end{cases} \quad \ldots(23)$$

$$\begin{cases} y_3=N(y_0+.y_1+.y_2)-(y_0+.y_1)= \\ J^{n}[-P(y_0(x))-N(y_0(x-t))-P(y_1(x))-N(y_1(x-t)) \\ -P(y_2(x))-N(y_2(x-t))]-J^{n}[y_0(x-t)+y_1(x-t)] \end{cases} \quad ..(24)$$

We define the recurrence relation from the systems of Eqn. (20) as follows:

$$\begin{cases} y_0=g(x)=\sum_{i=0}^{r} y^{(i)}(0)\frac{x^{i}}{i!}, \\ y_1=J^{n}\left[f(x)-P(y_0(x))-N(y_0(x-t))\right] \\ y_2=J^{n}[-P(y_1(x))-N(y_1(x-t))] \\ y_3=J^{n}[-P(y_2(x))-N(y_2(x-t))] \\ . \\ . \\ . \\ y_{n+1}=J^{n}\sum_{i=3}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right)-N\left(\sum_{j=0}^{i-1} y_j\right)\right\}, \quad i\geq 3. \end{cases} \quad \ldots(25)$$

Thus, the approximate analytical solution of the DDE in Eqn. (11), expressed in truncated series form, is:

$$y(x)=\lim_{k\to\infty}\sum_{n=0}^{k} y_n = y_0+y_1+y_2+y_3+\ldots \tag{26}$$

4. Experimental Method/Procedure/Design

This section evaluates the effectiveness and accuracy of the proposed method for solving integer-order linear delay differential equations (LDDEs).

a) 4.1 Linear Delay Differential Equations (LDDEs)

Example 1 (Adapted from Mohyud-din & Yildirim, 2010): Consider the following second-order LDDE:

$$y''(x)=\frac{3}{4}y(x)+y\left(\frac{x}{2}\right)-x^{2}+2, \quad 0\leq x\leq 1, y(0)=0, y'(0)=0. \tag{27}$$

The analytical solution is given by $y(x)=x^{2}$

In view of Eqn. (14), the Eqn. (27) is approximately expressed as follows:

$$y(x)=J^{2}\left[\frac{3}{4}y(x)+y\left(\frac{x}{2}\right)\right]+x^{2} -\frac{x^{4}}{12}+\sum_{i=0}^{r} y^{(i)}(0)\frac{x^{i}}{i!} \tag{28}$$

We deduce the following recurrence relation

$$y_0(x)=\sum_{i=0}^{r} y^{(i)}(0)\frac{x^{i}}{i!}+x^{2}-\frac{x^{4}}{12}=x^{2}-\frac{x^{4}}{12}$$

$$y_0\left(\frac{x}{2}\right)=\left(\frac{x}{2}\right)^{2}-\frac{\left(\frac{x}{2}\right)^{4}}{12}=\frac{x^{2}}{4}-\frac{x^{4}}{192}$$

$$y_1(x)=J^{2}\left[\frac{3}{4}y_0(x)+y_0\left(\frac{x}{2}\right)\right]= -\frac{13}{5760}x^{6}+\frac{1}{12}x^{4}$$

$$y_1\left(\frac{x}{2}\right)=-\frac{13}{368640}x^{6}+\frac{1}{192}x^{4}$$

$$y_2(x)=-\frac{91}{2949120}x^{8}+\frac{13}{5760}x^{6}$$

$$y_2\left(\frac{x}{2}\right)=-\frac{91}{754974720}x^{8}+\frac{13}{368640}x^{6}$$

$$y_3=-\frac{17563}{67947724800}x^{10}+\frac{91}{2949120}x^{8}$$

Now, in vision of Eqn. (39), the solution of Example 1 is

$$y(x) = y_0 + y_1 + y_2 + y_3 + \ldots =$$

$$x^2 - \frac{1}{12}x^4 - \frac{13}{5760}x^6 + \frac{1}{12}x^4 - \frac{91}{2949120}x^8 + \frac{13}{5760}x^6 - \frac{17563}{67947724800}x^{10} + \frac{91}{2949120}x^8$$

$$y(x) = x^2 - \frac{17563}{67947724800}x^{10} \quad (29)$$

TABLE 1: Comparison of the three-term approximate solution obtained using MNIM with VIM, ADM, and the exact solution for Example 1 at various values of the time variable.

x	EXACT	MNIM (n=3)	VIM (n=8)	ADM (n=7)
0.01	0.0001	0.0001	9.9999E-05	1E-04
0.02	0.0004	0.0004	0.00039999	0.0004
0.03	0.0009	0.0009	0.00089993	0.0009
0.04	0.0016	0.0016	0.00159979	0.0016
0.05	0.0025	0.0025	0.00249948	0.0025
0.06	0.0036	0.0036	0.00359892	0.0036
0.07	0.0049	0.0049	0.004898	0.0049
0.08	0.0064	0.0064	0.00639659	0.0064
0.09	0.0081	0.0081	0.00809453	0.0081
0.1	0.01	0.01	0.00999167	0.01

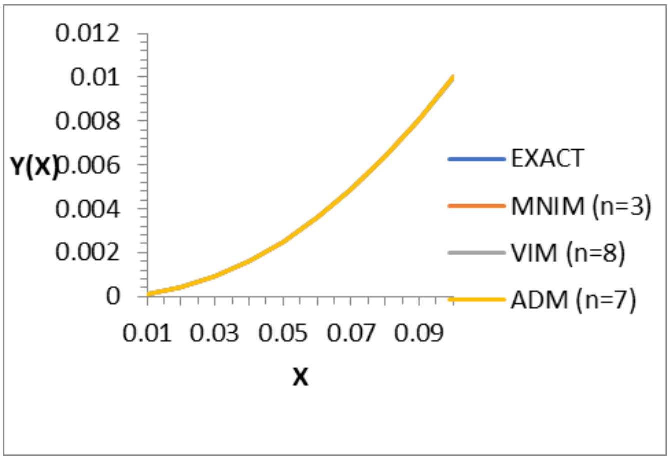

Figure 1: Solution plots for Example 1 comparing MNIM with ADM, VIM, and the exact solutions.

5. Results and Discussion

Graphs and tables play a crucial role in evaluating and comparing the accuracy of mathematical methods for solving practical problems. In this analysis, Figures 1 present a comparison of the approximate solutions obtained using the Modified New Iterative Method (MNIM) and the exact solutions for Example 1 at various time variable (xxx) values. These comparisons reveal the effectiveness of MNIM, as its solutions align closely with the exact results, demonstrating its reliability as a mathematical tool. The 2D graphs in Figures 1 also include solutions from the Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM), allowing for a comprehensive evaluation of MNIM's performance against established methods.

To further assess the precision of MNIM, the absolute error values are calculated and visualized alongside the solutions. These errors are notably minimal, highlighting the superior accuracy of MNIM. Table 1 complements the graphical analysis by providing a detailed numerical comparison of the approximate and exact solutions for Example 1, along with their absolute errors at different xxx values. The results confirm that MNIM consistently outperforms ADM and VIM, achieving higher accuracy with smaller errors. Together, these findings establish MNIM as an efficient and reliable method for solving differential equations with practical applications.

6. Conclusion and Future Scope

This study presents the Modified New Iterative Method (MNIM) as an innovative and robust approach for solving linear delay differential equations (DDEs). By integrating a post-treatment New Iterative Method (NIM), MNIM effectively addresses the inherent complexities of DDEs. Key theoretical foundations of DDEs are established, ensuring a clear framework for the application of MNIM. Through extensive testing on representative problems, the method's validity and consistency are confirmed. The graphical and numerical presentation of absolute errors across varying time values demonstrates MNIM's ability to deliver highly accurate solutions that closely align with exact results. These findings highlight MNIM's superiority in terms of accuracy and computational efficiency, positioning it as a reliable mathematical tool for tackling linear DDEs.

The success of MNIM in solving linear DDEs opens avenues for further exploration and application. Future research could extend MNIM to nonlinear delay differential equations and fractional delay differential equations, broadening its applicability to more complex systems. Additionally, integrating MNIM with advanced computational techniques, such as parallel computing or machine learning-based optimization, may enhance its efficiency for large-scale problems. Investigating its performance in real-world applications, such as control systems, biological modeling, and engineering simulations, can further validate its practical utility. Finally, exploring hybrid approaches that combine MNIM with other numerical or analytical methods could lead to even more powerful and versatile solution techniques for a wide range of differential equations.

Data Availability
Not applicable.

Conflict of Interest
All authors declare that they do not have any conflict of interest.

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