International Journal of Scientific Research in Mathematical and Statistical Sciences
Research Paper
Vol.6, Issue.1, pp.105-117, February (2019)
E-ISSN: 2348-4519
DOI: https://doi.org/10.26438/ijsrmss/v6i1.105117

On Weighted Sushila Distribution with Properties and its Applications

A. A. Rather[1*] and C. Subramanian[2]

[1,2]Department of Statistics, Annamalai University, Annamalai nagar, Tamil Nadu, India

[1]*Corresponding Author: aafaq7741@gmail.com*

Available online at: www.isroset.org

Received: 31/Jan/2019, Accepted: 15/Feb/2019, Online: 28/Feb/2019

Abstract- In this paper, we have proposed a new version of Sushila distribution known as weighted Sushila distribution. The weighted Sushila distribution has three parameters. The different structural properties of the newly model have been studied. The maximum likelihood estimators of the parameters and the Fishers information matrix have been discussed. Finally, a real life data set has been analyzed, where it is observed that weighted Sushila distribution has a better fit compared to Sushila distribution.

Keywords: *Weighted distribution, Sushila distribution, Reliability analysis, Maximum likelihood estimator, Order statistics, Entropies, Likelihood ratio test.*

I. INTRODUCTION

The weighted distributions are applied in various research areas related to biomedicine, reliability, ecology and branching processes. In many applied sciences like engineering, medicine, behavioural science, finance, insurance and others, it is very crucial to modelling and analyzing lifetime data. For modelling this type of lifetime data, a number of continuous distributions are for modelling like weibull, lindley, exponential, lognormal and gamma. This concept of weighted distributions was given by Fisher (1934) to model the ascertainment bias. Later Rao (1965) developed this concept in a unified manner while modelling the statistical data when the standard distributions were not appropriate to record these observations with equal probabilities. As a result, weighted models were formulated in such situations to record the observations according to some weighted function. The weighted distribution reduces to length biased distribution when the weight function considers only the length of the units. The concept of length biased sampling was first introduced by Cox (1969) and Zelen (1974). More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, resulting distribution is called size-biased. There are various good sources which provide the detailed description of weighted distributions. Different authors have reviewed and studied the various weighted probability models and illustrated their applications in different fields. Weighted distributions are applied in various research areas related to reliability, biomedicine, ecology and branching processes. Dey *et al* (2015) discussed weighted exponential distribution with its properties and different methods of estimation. Kilany (2016) have obtained the weighted version of lomax distribution. Khan *et al.* (2018) discussed the weighted modified weibull distribution. Rather and Subramanian (2018) discussed the characterization and estimation of length biased weighted generalized uniform distribution. Recently, Rather *et al* (2018) obtained a new size biased Ailamujia distribution with applications in engineering and medical science which shows more flexibility than classical distributions.

Sushila distribution was introduced by Shanker *et al.* (2013) which is a mixed distribution between exponential distribution and gamma distribution. Sushila distribution is a two-parameter continuous distribution. The properties of Sushila distribution such as moments, mean residual life function, failure rate function, stochastic orderings, estimation of parameters by the method of maximum likelihood and the method of moments. An extension to the zero-inflated power series distributions or zero - one inflated models was introduced by Alshkaki (2016). Poisson distribution is one of power series distribution. An extension to the zero-inflated power series distributions is discrete distributions with number of frequencies with zero and one are inflated. Alshkaki (2016) studied its structure properties; its mean, variance, probability generating function. The Poisson-

Sushila distribution was introduced by Saratoon (2017) which is a two parameter discrete distribution. Various properties such as moments, mean, variance, skewness and kurtosis and estimating parameter by using maximum likelihood estimation have been studied and shown that the Poisson-Sushila distribution is more flexible than Poisson distribution in real data. Elgarhy and Shawki (2017) discussed about exponentiated sushila distribution. Again Elgarhy and Shawki (2017) obtained the various properties of kumaraswamy sushila distribution. Borah and Hazarika (2018) discussed about poisson-sushila distribution and its applications. Recently, Rather and Subramanian (2018) have discussed the statistical properties and applications of length biased sushila distribution.

II. WEIGHTED SUSHILA (WS) DISTRIBUTION

The probability density function (pdf) of Sushila distribution is given by

$$f(x;\alpha,\theta)=\frac{\theta^2}{\alpha(\theta+1)}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x} \;; \quad \alpha,\,\theta,x>0 \tag{1}$$

and its cumulative distribution function (cdf) is given by

$$F(x;\alpha,\theta)=1-\frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}x} \tag{2}$$

Suppose X is a non-negative random variable with probability density function $f(x)$. Let $w(x)$ be the non negative weight function, then the probability density function of the weighted random variable X_w is given by:

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))}\,, \quad x>0\,,$$

where $w(x)$ be a non-negative weight function and $E(w(x))=\int w(x)f(x)dx<\infty$.

In this paper, we will consider the weight function as $w(x)=x^c$ to obtain the weighted Sushila distribution. The probability density function of weighted Sushila distribution is given as:

$$f_w(x)=\frac{x^c f(x)}{E(x^c)} \;; \qquad x>0 \tag{3}$$

where $E(x^c)=\int_0^\infty x^c f(x;\alpha,\theta)\,dx$

$$E(x^c)=\frac{\alpha^c(\theta+c+1)\Gamma(c+1)}{\theta^c(\theta+1)} \tag{4}$$

Substitute (1) and (4) in equation (3), we will get the required pdf of weighted Sushila distribution as

$$f_w(x)=\frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}x^c\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x} \tag{5}$$

and the cumulative density function of WS distribution is obtained as

$$F_w(x)=\int_0^x f_w(x)dx$$

$$=\int_0^x \frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)} x^c\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}dx$$

After simplification, we will get the cumulative distribution function of WS distribution

$$F_w(x)=\frac{\theta}{(\theta+c+1)\Gamma(c+1)}\left(\gamma\left((c+1),\frac{\theta}{\alpha}x\right)+\frac{1}{\theta}\gamma\left((c+2),\frac{\theta}{\alpha}x\right)\right) \tag{6}$$

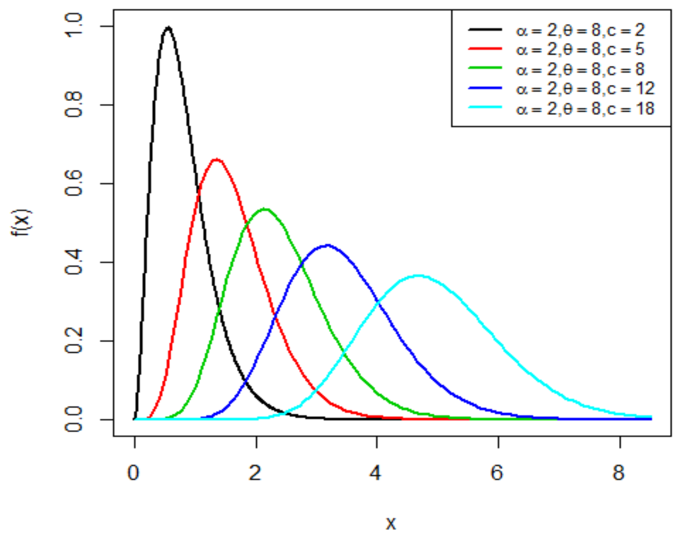

Fig.1: Pdf plot of Weighted Sushila distribution

III. RELIABILITY ANALYSIS

In this section, we will discuss about the survival function, failure rate, reverse hazard rate and Mills ratio of the WS distribution.

The survival function or the reliability function of the weighted Sushila distribution is given by

$$S(x)=1-\frac{\theta}{(\theta+c+1)\Gamma(c+1)}\left(\gamma\left((c+1),\frac{\theta}{\alpha}x\right)+\frac{1}{\theta}\gamma\left((c+2),\frac{\theta}{\alpha}x\right)\right)$$

The hazard function is also known as the hazard rate, instantaneous failure rate or force of mortality and is given by

$$h(x)=\frac{\theta^{c+2}x^c(\alpha+x)e^{-\frac{\theta}{\alpha}x}}{\alpha^{c+2}\left((\theta+c+1)\Gamma(c+1)-\left(\theta\gamma\left((c+1),\frac{\theta}{\alpha}x\right)+\gamma\left((c+2),\frac{\theta}{\alpha}x\right)\right)\right)}$$

The reverse hazard rate is given by

$$h_r(x)=\frac{\theta^{c+1}x^c(\alpha+x)e^{-\frac{\theta}{\alpha}x}}{\alpha^{c+2}\,\Gamma(c+1)\left(\gamma\left((c+1),\frac{\theta}{\alpha}x\right)+\frac{1}{\theta}\gamma\left((c+2),\frac{\theta}{\alpha}x\right)\right)}$$

and the Mills ratio of the WS distribution is

$$\text{Mills Ratio} = \frac{1}{h_r(x)} = \frac{\alpha^{c+2}\,\Gamma(c+1)\left(\gamma\left((c+1),\frac{\theta}{\alpha}x\right)+\frac{1}{\theta}\gamma\left((c+2),\frac{\theta}{\alpha}x\right)\right)}{\theta^{c+1}x^{c}(\alpha+x)e^{-\frac{\theta}{\alpha}x}}$$

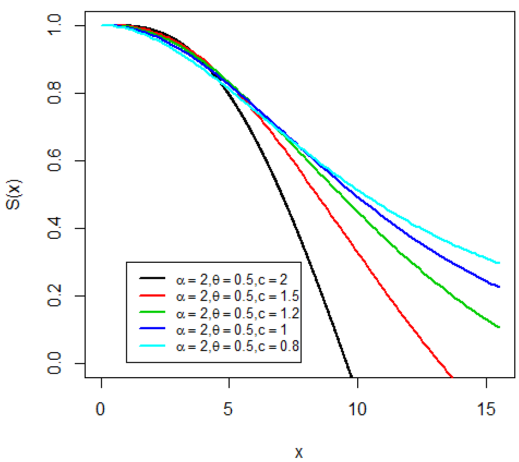

Fig.2: Showing the Survival function curves of WS distribution

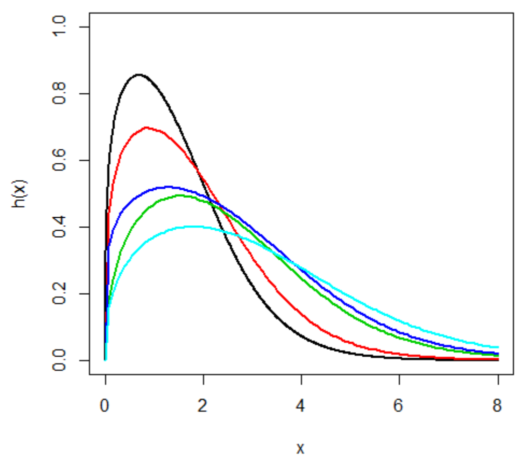

Fig.3: Showing the Hazard rate curves of WS distribution

IV. MOMENTS AND ASSOCIATED MEASURES

Let X denotes the random variable of WS distribution with parameters α, c and θ, then the r^{th} order moment $E(X^r)$ of WS distribution can be obtained as

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_w(x)dx$$

$$= \int_0^\infty \frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)} x^{c+r}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}dx$$

$$= \frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}\left(\int_0^\infty x^{(c+r+1)-1}e^{-\frac{\theta}{\alpha}x}dx + \frac{1}{\alpha}\int_0^\infty x^{(c+r+2)-1}e^{-\frac{\theta}{\alpha}x}dx\right)$$

$$\Rightarrow E(X^r) = \frac{\alpha^r(\theta\Gamma(c+r+1)+\Gamma(c+r+2))}{\theta^r(\theta+c+1)\Gamma(c+1)} \qquad (7)$$

Putting $r = 1$ in equation (7), we will get the mean of WS distribution which is given by

$$E(X)=\mu_1'=\frac{\alpha(\theta\Gamma(c+2)+\Gamma(c+3))}{\theta(\theta+c+1)\Gamma(c+1)}$$

and putting r = 2, we obtain the second moment as

$$E(X^2)=\frac{\alpha^2(\theta\Gamma(c+3)+\Gamma(c+4))}{\theta^2(\theta+c+1)\Gamma(c+1)}$$

Therefore,

$$\text{Variance }=\left(\frac{\alpha^2(\theta\Gamma(c+3)+\Gamma(c+4))}{\theta^2(\theta+c+1)\Gamma(c+1)}-\left(\frac{\alpha(\theta\Gamma(c+2)+\Gamma(c+3))}{\theta(\theta+c+1)\Gamma(c+1)}\right)^2\right)$$

$$\text{S. D.}(\sigma)=\frac{\alpha}{\theta}\sqrt{\left(\frac{\theta\Gamma(c+3)+\Gamma(c+4)}{(\theta+c+1)\Gamma(c+1)}-\frac{(\theta\Gamma(c+2)+\Gamma(c+3))^2}{((\theta+c+1)\Gamma(c+1))^2}\right)}$$

$$\mathit{Coefficient\ of\ Variation}\ (C.V.)=\frac{\sigma}{\mu_1'}=\frac{(\theta+c+1)\Gamma(c+1)}{(\theta\Gamma(c+2)+\Gamma(c+3))}\times\sqrt{\left(\frac{\theta\Gamma(c+3)+\Gamma(c+4)}{(\theta+c+1)\Gamma(c+1)}-\frac{(\theta\Gamma(c+2)+\Gamma(c+3))^2}{((\theta+c+1)\Gamma(c+1))^2}\right)}$$

$$\mathit{Coefficient\ of\ Dispersion}(\gamma)=\frac{\sigma^2}{\mu_1'}=\frac{\alpha}{\theta}\left(\frac{\theta\Gamma(c+3)+\Gamma(c+4)}{\theta\Gamma(c+2)+\Gamma(c+3)}-\frac{\theta\Gamma(c+2)+\Gamma(c+3)}{(\theta+c+1)\Gamma(c+1)}\right)$$

4.1 Harmonic mean

The Harmonic mean of the proposed model can be obtained as

$$H.M=E\left(\frac{1}{x}\right)=\int_0^\infty\frac{1}{x}f_w(x)\,dx$$

$$=\int_0^\infty\frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}x^{c-1}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}dx$$

$$=\frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}\left(\int_0^\infty x^{c-1}e^{-\frac{\theta}{\alpha}x}dx+\frac{1}{\alpha}\int_0^\infty x^{(c+1)-1}e^{-\frac{\theta}{\alpha}x}dx\right)$$

$$=\frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}\left(\frac{\alpha^c\Gamma c}{\theta^c}+\frac{\alpha^c\Gamma(c+1)}{\theta^{c+1}}\right)$$

$$\Rightarrow H.M = \frac{\theta(\theta+c)}{\alpha c(\theta+c+1)}$$

4.2 Moment generating function and Characteristic function

Let *X* have a weighted Sushila distribution, then the MGF of *X* is obtained as

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_w(x)dx$$

Using Taylor's series

$$M_X(t) = E(e^{tx}) = \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f_w(x)dx$$

$$= \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f_w(x)dx$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!} \mu_j'$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!} \frac{\alpha^j(\theta\Gamma(c+j+1)+\Gamma(c+j+2))}{\theta^j(\theta+c+1)\Gamma(c+1)}$$

$$\Rightarrow M_x(t) = \frac{1}{(\theta+c+1)\Gamma(c+1)} \sum_{j=0}^\infty \frac{t^j\alpha^j}{j!\theta^j}\left(\theta\Gamma(c+j+1)+\Gamma(c+j+2)\right)$$

Similarly, the characteristic function of WS distribution can be obtained as

$$\varphi_x(t) = M_x(it)$$

$$\Rightarrow M_x(it) = \frac{1}{(\theta+c+1)\Gamma(c+1)} \sum_{j=0}^\infty \frac{(it)^j\alpha^j}{j!\theta^j}\left(\theta\Gamma(c+j+1)+\Gamma(c+j+2)\right)$$

V. LIKELIHOOD RATIO TEST

Let $X_1, X_2, \ldots, X_n$ be a random sample from the weighted Sushila distribution. To test the hypothesis

$$H_o : f(x) = f(x;\alpha,\theta) \quad \text{against} \quad H_1 : f(x) = f_w(x;\alpha,c,\theta)$$

In order to test whether the random sample of size *n* comes from the Sushila distribution or WS distribution, the following test statistic is used

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^n \frac{f_w(x;\alpha,c,\theta)}{f(x;\alpha,\theta)}$$

$$=\prod_{i=1}^{n}\left(\frac{\theta^{c}(\theta+1)\,x_i^{\,c}}{\alpha^{c}(\theta+c+1)\Gamma(c+1)}\right)$$

$$=\left(\frac{\theta^{c}(\theta+1)}{\alpha^{c}(\theta+c+1)\Gamma(c+1)}\right)^{n}\prod_{i=1}^{n}x_i^{\,c}$$

We reject the null hypothesis if

$$\Delta=\left(\frac{\theta^{c}(\theta+1)}{\alpha^{c}(\theta+c+1)\Gamma(c+1)}\right)^{n}\prod_{i=1}^{n}x_i^{\,c}>k$$

$$\text{or, } \Delta^{*}=\prod_{i=1}^{n}x_i^{\,c}>k\left(\frac{\alpha^{c}(\theta+c+1)\Gamma(c+1)}{\theta^{c}(\theta+1)}\right)^{n}$$

$$\Delta^{*}=\prod_{i=1}^{n}x_i^{\,c}>k^{*}\text{, where } k^{*}=k\left(\frac{\alpha^{c}(\theta+c+1)\Gamma(c+1)}{\theta^{c}(\theta+1)}\right)^{n}$$

For large sample size n, $2\log\Delta$ is distributed as chi-square distribution with one degree of freedom and also p-value is obtained from the chi-square distribution. Thus we reject the null hypothesis, when the probability value is given by

$p\left(\Delta^{*}>\beta^{*}\right)$, where $\beta^{*}=\prod_{i=1}^{n}x_i^{\,c}$ is less than a specified level of significance and $\prod_{i=1}^{n}x_i^{\,c}$ is the observed value of the statistic Δ^{*}.

VI. ENTROPIES

The concept of entropy is important in different areas such as probability and statistics, physics, communication theory and economics. Entropies quantify the diversity, uncertainty, or randomness of a system. Entropy of a random variable X is a measure of variation of the uncertainty.

6.1: Renyi Entropy

The Renyi entropy is important in ecology and statistics as index of diversity. The Renyi entropy is also important in quantum information, where it can be used as a measure of entanglement. For a given probability distribution, Renyi entropy is given by

$$e(\beta)=\frac{1}{1-\beta}\log\left(\int f^{\beta}(x)dx\right)$$

where, $\beta>0$ and $\beta\neq 1$

$$e(\beta)=\frac{1}{1-\beta}\log\int_{0}^{\infty}\left(\frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}x^{c}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}\right)^{\beta}dx$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}\right)^{\beta}\int_{0}^{\infty}x^{\beta c}e^{-\frac{\beta\theta}{\alpha}x}\left(1+\frac{x}{\alpha}\right)^{\beta}dx\right) \qquad (8)$$

Using binomial expansion in equation (8), we get

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}\right)^{\beta}\sum_{i=0}^{\infty}\binom{\beta}{i}\left(\frac{1}{\alpha}\right)^{i}\int_{0}^{\infty}x_i^{\beta c+i}e^{-\frac{\beta\theta}{\alpha}x_i}dx\right)$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^{c+2}}{(\theta+c+1)\Gamma(c+1)}\right)^{\beta}\sum_{i=0}^{\infty}\binom{\beta}{i}\frac{\alpha^{1-\beta}\theta^{2\beta-i-1}\Gamma(\beta c+i+1)}{\beta^{\beta c+i+1}}\right)$$

6.2: Tsallis Entropy

A generalization of Boltzmann-Gibbs (B-G) statistical mechanics initiated by Tsallis has focussed a great deal to attention. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy (Tsallis, 1988) for a continuous random variable is defined as follows

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\int_{0}^{\infty}f^{\lambda}(x)dx\right)$$

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\int_{0}^{\infty}\left(\frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}x^{c}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}\right)^{\lambda}dx\right)$$

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\left(\frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}\right)^{\lambda}\int_{0}^{\infty}x^{\lambda c}e^{-\frac{\lambda\theta}{\alpha}x}\left(1+\frac{x}{\alpha}\right)^{\lambda}dx\right) \tag{9}$$

Using binomial expansion in equation (9), we get

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\left(\frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}\right)^{\lambda}\sum_{i=0}^{\infty}\binom{\lambda}{i}\left(\frac{1}{\alpha}\right)^{i}\int_{0}^{\infty}x_i^{\lambda c+i}e^{-\frac{\lambda\theta}{\alpha}x_i}dx\right)$$

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\left(\frac{\theta^{c+2}}{(\theta+c+1)\Gamma(c+1)}\right)^{\lambda}\sum_{i=0}^{\infty}\binom{\lambda}{i}\frac{\alpha^{1-\lambda}\theta^{2\lambda-i-1}\Gamma(\lambda c+i+1)}{\lambda^{\lambda c+i+1}}\right)$$

VII. ORDER STATISTICS

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ drawn from the continuous population with probability density function $f_x(x)$ and cumulative density function with $F_x(x)$, then the pdf of r^{th} order statistics $X_{(r)}$ is given by

$$f_{X(r)}(x)=\frac{n!}{(r-1)!(n-r)!}f_X(x)\left[F_X(x)\right]^{r-1}\left[1-F_X(x)\right]^{n-r} \tag{10}$$

Using the equations (5) and (6) in equation (10), the probability density function of r^{th} order statistics $X_{(r)}$ of WS distribution is given by

$$f_{X(r)}(x)=\frac{n!}{(r-1)!(n-1)!}\left(\frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}x^{c}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}\right)$$
$$\times\left(\frac{\theta}{(\theta+c+1)\Gamma(c+1)}\left(\gamma\left((c+1),\frac{\theta}{\alpha}x\right)+\frac{1}{\theta}\gamma\left((c+2),\frac{\theta}{\alpha}x\right)\right)\right)^{r-1}$$
$$\times\left(1-\frac{\theta}{(\theta+c+1)\Gamma(c+1)}\left(\gamma\left((c+1),\frac{\theta}{\alpha}x\right)+\frac{1}{\theta}\gamma\left((c+2),\frac{\theta}{\alpha}x\right)\right)\right)^{n-r}$$

Therefore, the probability density function of higher order statistics $X_{(n)}$ can be obtained as

$$f_{X(n)}(x)=\left(\frac{n\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}x^{c}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}\right)$$
$$\times\left(\frac{\theta}{(\theta+c+1)\Gamma(c+1)}\left(\gamma\left((c+1),\frac{\theta}{\alpha}x\right)+\frac{1}{\theta}\gamma\left((c+2),\frac{\theta}{\alpha}x\right)\right)\right)^{n-1}$$

and the pdf of 1^{st} order statistic $X_{(1)}$ can be obtained as

$$f_{X(1)}(x)=\left(\frac{n\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}x^{c}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}\right)$$
$$\times\left(1-\frac{\theta}{(\theta+c+1)\Gamma(c+1)}\left(\gamma\left((c+1),\frac{\theta}{\alpha}x\right)+\frac{1}{\theta}\gamma\left((c+2),\frac{\theta}{\alpha}x\right)\right)\right)^{n-1}$$

VIII. BONFERRONI AND LORENZ CURVES

The Bonferroni and the Lorenz curves are not only used in economics in order to study the income and poverty, but it is also being used in other fields like reliability, medicine, insurance and demography. The Bonferroni and Lorenz curves are given by

$$B(p)=\frac{1}{p\mu_1'}\int_0^q xf(x)dx$$

and $$L(p)=\frac{1}{\mu_1'}\int_0^q xf(x)dx$$

Where $\mu_1'=\dfrac{\alpha(\theta\Gamma(c+2)+\Gamma(c+3))}{\theta(\theta+c+1)\Gamma(c+1)}$ and $q=F^{-1}(p)$

$$B(p)=\frac{\theta(\theta+c+1)\Gamma(c+1)}{p\alpha(\theta\Gamma(c+2)+\Gamma(c+3))}\int_0^q\frac{\theta^{c+2}}{\alpha^{c+1}(\theta+c+1)\Gamma(c+1)}x^{c+1}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}dx$$

After simplification, we get

$$B(p)=\frac{\theta}{p\,(\theta\Gamma(c+2)+\Gamma(c+3))}\left(\gamma\left((c+2),\frac{\theta}{\alpha}q\right)+\frac{1}{\theta}\gamma\left((c+3),\frac{\theta}{\alpha}q\right)\right)$$

and

$$L(p)=p\,B(p)=\frac{\theta}{(\theta\Gamma(c+2)+\Gamma(c+3))}\left(\gamma\left((c+2),\frac{\theta}{\alpha}q\right)+\frac{1}{\theta}\gamma\left((c+3),\frac{\theta}{\alpha}q\right)\right)$$

IX. MAXIMUM LIKELIHOOD ESTIMATOR AND FISHER'S INFORMATION MATRIX

In this section, we will discuss the maximum likelihood estimators of the parameters of weighted Sushila distribution. Consider $X_1, X_2, \ldots, X_n$ be the random sample of size n from the WS distribution, then the likelihood function is given by

$$L(x;\alpha,c,\theta)=\frac{\theta^{n(c+2)}}{\alpha^{n(c+1)}((\theta+c+1)\Gamma(c+1))^n}\prod_{i=1}^{n}x_i^{\,c}\left(1+\frac{x_i}{\alpha}\right)e^{-\frac{\theta}{\alpha}x_i}$$

The log likelihood function is

$$\log L=n(c+2)\log\theta-n(c+1)\log\alpha-n\log(\theta+c+1)-n\log\Gamma(c+1)+c\sum_{i=1}^{n}\log x_i$$
$$+\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\alpha}\right)-\frac{\theta}{\alpha}\sum_{i=1}^{n}x_i \qquad (11)$$

The maximum likelihood estimates of α, c, θ can be obtained by differentiating equation (11) with respect to α, c, θ and must satisfy the normal equation

$$\frac{\partial\log L}{\partial\alpha}=-\frac{n(c+1)}{\alpha}-\sum_{i=1}^{n}\frac{x_i}{\alpha(\alpha+x_i)}+\frac{\theta}{\alpha^2}\sum_{i=1}^{n}x_i=0$$
$$\frac{\partial\log L}{\partial\theta}=\frac{n(c+2)}{\theta}-\frac{n}{(\theta+c+1)}+\frac{1}{\alpha}\sum_{i=1}^{n}x_i=0$$
$$\frac{\partial\log L}{\partial c}=n\log\theta-n\log\alpha-n\psi(c+1)-\frac{n}{(\theta+c+1)}+\sum_{i=1}^{n}\log x_i=0$$

Where $\psi(.)$ is the digamma function.

Because of the complicated form of likelihood equations, algebraically it is very difficult to solve the system of nonlinear equations. Therefore we use R and wolfram mathematics for estimating the required parameters.

To obtain confidence interval we use the asymptotic normality results. We have that, if $\hat{\lambda}=(\hat{\alpha},\hat{c},\hat{\theta})$ denotes the MLE of $\lambda=(\alpha,c,\theta)$, we can state the results as follows:

$$\sqrt{n}(\hat{\lambda}-\lambda)\rightarrow N_3(0, I^{-1}(\lambda))$$

Where $I(\lambda)$ is Fisher's Information Matrix, i.e.,

$$I(\lambda) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial c \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial c \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial c^2}\right) \end{pmatrix}$$

Where

$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = \frac{n(c+1)}{\alpha^2} + \sum_{i=1}^{n} E\left(\frac{x_i(2\alpha + x_i)}{(\alpha^2 + \alpha x_i)^2}\right) - \frac{2\theta}{\alpha^3}\sum_{i=1}^{n} E(x_i)$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{n(c+2)}{\theta^2} + \frac{n}{(\theta + c + 1)^2}$$

$$E\left(\frac{\partial^2 \log L}{\partial c^2}\right) = -n\psi'(c+1) + \frac{n}{(\theta + c + 1)^2}$$

Also, $E\left(\frac{\partial^2 \log L}{\partial \alpha \partial c}\right) = -\frac{n}{\alpha}$, $E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) = \sum_{i=1}^{n} \frac{E(x_i)}{\alpha^2}$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) = \frac{n}{\theta} + \frac{n}{(\theta + c + 1)^2}$$

Where $\psi(.)'$ is the first order derivative of digamma function.

Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidence intervals for α, c and θ.

X. DATA ANALYSIS

In this section, we have used a real lifetime data set in weighted Sushila distribution and the model has been compared with Sushila distribution.

The following data set is reported by Fuller *et al* (1994) which is related with strength data of window glass of the aircraft of 31 windows. The data are

18.83	20.80	21.657	23.03	23.23	24.05	24.321	25.50
25.52	25.80	26.69	26.77	26.78	27.05	27.67	29.90
31.11	33.20	33.73	33.76	33.89	34.76	35.75	35.91
36.98	37.08	37.09	39.58	44.045	45.29	45.431	

In order to compare the weighted Sushila distribution with Sushila distribution, we consider the criteria like Bayesian information criterion *(BIC)*, Akaike Information Criterion *(AIC)*, Akaike Information Criterion Corrected *(AICC)* and *-2 logL*.

The better distribution is which corresponds to lesser values of *AIC*, *BIC*, *AICC* and *– 2 log L*. For calculating *AIC, BIC, AICC* and *-2 logL* can be evaluated by using the formulas as follows:

$$AIC = 2K - 2logL, \qquad BIC = klogn - 2logL, \qquad AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$$

Where *k* is the number of parameters, *n* is the sample size and *-2 logL* is the maximized value of log likelihood function and are showed in table 1.

Table.1: Performance of distributions.

Distribution	Parameter Estimates			-2logL	AIC	AICC	BIC
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{c}$				
Sushila	0.137249849 (0.108963418)	0.008869415 (0.006921270)	-	252.493	256.493	257.3818	259.3609
Weighted Sushila	0.2567681 (0.8926121)	0.1574252 (0.547785)	16.9002969 (4.7500810)	208.2538	214.2538	215.1427	218.5558

From table 1, we can see that the weighted Sushila distribution have the lesser *AIC*, *BIC*, *AICC* and *-2 logL* values as compared to Sushila distribution. Hence, we can conclude that the weighted Sushila distribution leads to better fit than the Sushila distribution.

XI. CONCLUSION

In the present study, we have introduced a new generalization of the Sushila distribution namely as weighted Sushila distribution with three parameters (shape). The subject distribution is generated by using the weighted technique and the parameters have been obtained by using maximum likelihood estimator. Some mathematical properties along with reliability measures are discussed. The new distribution with its applications in real life-time data has been demonstrated. Finally the results are compared over Sushila distribution and have been found that weighted Sushila distribution provides better fit than Sushila distribution

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