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Research Paper

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Asymptotic Behaviours of the Generalized Hypergeometric Polynomials Set $S_n(x, y)$ for Large Value of n

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Abstract: In this paper, we study the asymptotic behaviours of generalized hypergeometric polynomial Set S_n(x,y) for large value of n, the order of the polynomial sets. Generating function is generally used for the determination of the asymptotic behaviours of the polynomial set as the order of the polynomial set tends to infinity. Hence if the radius of convergence is finite, then the generating function has one or several singularities on the circle of convergence and the location and nature of these singularities determine the behaviours of the polynomial set when the order tends to infinity, But here we have obtained the asymptotic behaviours not from the generating function but by another method directly from the polynomial set. These behaviours for large n have been given in the form of Theorem. A number of well known results for orthogonal and non-orthogonal polynomials have been deduced as particular cases of these theorems.

AMS Subject Classification: Special function-33Cxx

Keywords: *Appell function, Hypergeometric polynomial, Orthogonal polynomial, Generating function, Asymptotic Behaviours.*

I. INTRODUCTION

Suman and Singh [1] defined the generalized hypergeometric polynomial set S_n (x, y) by means of the generating functions,

$$e^{\lambda yt}\; F\left[\begin{matrix}(G_r);\\ \qquad \lambda_1 y^{e_1} t^{e_1}\\ (H_s);\end{matrix}\right] \times F\left[\begin{matrix}(a_p);(A_h);(C_u)\\ \qquad \lambda_3 x^{e_3} t, \lambda_2 x^{e_2} y^{-e_2} t^{e_2}\\ (b_q);(B_k);(D_v)\end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} S^{\lambda;\lambda_1;\lambda_2;\lambda_3;(G_r);(a_p);(A_h);(C_u)}_{n,e_1;e_2;e_3;(H_s);(b_q);(B_k);(D_v)}(x,y)\,t^n \qquad \ldots (1.1)$$

Where λ, λ_1, λ_2, λ_3 are real and e_1, e_2, e_3 are positive integers.

The left hand side of (1.1) contains Appell function [2] of two variables in the notation of Burchnall and Chaundy[3]. The polynomial set contains a number of parameters, for simplicity, we shall denote.

$$S^{\lambda;\lambda_1;\lambda_2;\lambda_3;(G_r);(a_p);(A_h);(C_u)}_{n,e;e_1;e_2;e_3;(H_s)(b_q);(B_k);(D_v)}(x,y)$$

by S_n(x, y).

Where n denote the order of the polynomial set.

After little simplification (1.1) gives

$$S_n(x,y)=\sum_{m=0}^{n}\sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]}\frac{\left[(a_p)\right]_{n-m-e_1m_1-(e_2-1)m_2}}{\left[(b_q)\right]_{n-m-e_1m_1-(e_2-1)m_2}}$$

$$\times\frac{\left[(A_h)\right]_{n-m-e_1m_1-e_2m_2}\left[(G_r)\right]_{m_1}\left[(C_u)\right]_{m_2}\lambda^m\ \lambda_1^{m_1}}{\left[(B_k)\right]_{n-m-e_1m_1-e_2m_2}\left[(H_s)\right]_{m_1}\left[(D_v)\right]_{m_2}m!\ m_1!}$$

$$\times\frac{\left(\lambda_2x^{e_2}\right)^{m_2}\left(\lambda_3x^{e_3}\right)^{n-m-e_1m_1-e_2m_2}\ y^{m+e_1m_1-e_2m_2}}{m_2!\qquad (n-m-e_1m_1-e_2m_2)!} \quad \ldots (1.2)$$

The polynomial set S_n (x, y) happens to the generalization of as many as forty-one orthogonal and non- orthogonal polynomials.

II. NOTATIONS

(i) $(m) = 1, 2, 3, \ldots\ldots, m$.

(ii) $(A_p) = A_1, A_2, A_3, \ldots\ldots A_p$.

(iii) $[(A_p)] = A_1, A_2, A_3\ldots\ldots A_p$.

(iv) $[(A_p)]_n = (A_1)_n (A_2)_n, (A_3)_n\ldots\ldots (A_p)_n$.

(v) $\Delta(a,b)=\frac{b}{a},\frac{b+1}{a},\ldots\ldots\frac{b+a-1}{a}$.

(vi) $\Gamma(a\pm b)=\Gamma(a+b)\Gamma(a-b)$.

(vii) $\Gamma_*\Gamma_*(a+b)=\Gamma(a+b)\Gamma(a+b)$.

$$K=\frac{\left[(a_p)\right]_n\left[(A_h)\right]_n\left(\lambda_3x^{e_3}\right)^n}{\left[(b_q)\right]_n\left[(B_k)\right]_n n!}$$

$$K(m)=\frac{(-n)_m\left(1-(b_q)-n\right)_m\left(1-(B_k)-n\right)_m\ \lambda^m\ y^m(-1)^{(h-k+q-p)}}{\left(1-(a_p)-n\right)_m\left(1-(A_h)-n\right)_m\ \ m!}$$

$$K(m_1)=\frac{\Delta_{m_1}\left[e_1;-n+m\right]\Delta_{m_1}\left[e_1;1-(b_q)-n+m\right]}{\Delta_{m_1}\left[e_1;1-(a_p)-n+m\right]\Delta_{m_1}\left[e_1;1-(A_p)-n+m\right]}$$

$$\times\frac{\Delta_{m_1}\left[e_1;1-(B_k)-n+m\right]\left[(G_r)\right]_{m_1}\ y^{e_1m_1}\ \lambda_1^{m_1}(-e_1)^{e_1(k-h+q-p+1)m_1}}{\left[(H_s)\right]_{m_1}\ m_1!\ \left(\lambda_3x^{e_3}\right)^{e_1m_1}}$$

$$K(m_2)=\frac{\Delta_{m_2}[e_2;-n+m+e_1m_1]\Delta_{m_1}\left[e_2-1;1-(b_q)-n+m+e_1m_1\right]}{\Delta m_2\left[e_2-1;1-(a_p)-n+m+e_1m_1\right]}$$

$$\times\frac{\Delta_{m_2}\left[e_2;1-(B_k)-n+m+e_1m_1\right]\left[(C_u)_{m_2}\right]\left(\lambda_2 x^{e_2}\right)^{m_2}}{\Delta_{m_2}\left[e_2;1-(A_h)-n+m+e_1m_1\right]\left[(D_\nu)\right]_{m_2} m_2!}$$

$$\times\frac{\{-(e_2-1)\}^{(e_2-1)(q-p)m_2}(-e_2)^{e_2(k-h+1)m_2}}{\left(\lambda_3 x^{e_3} y\right)^{e_2m_2}}$$

III. BEHAVIOURS OF $S_n(x, y)$ FOR LARGE VALUE OF n

Theorem: 1(a) If $e_2 > 1$, them

$$\lim_{n\to\infty}\frac{\left[(b_q)\right]_n\left[(B_k)\right]_n n!}{\left[(a_p)\right]_n\left[(A_h)\right]_n\left(\lambda_3 x^{e_3}\right)^n n^{n(k-h+q-p+1)}} S_n\left(n^{k-g+q-p+1}x, n^{\frac{p-q}{e_2}} y\right)$$

$$=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\frac{\lambda^m\lambda_1^{m_1}y^{m+e_1m_1}\left[(G_r)\right]_{m_1}}{m!m_1!\left(\lambda_3x^{e_3}\right)^{m+e_1m_1}\left[(H_s)\right]_{m_1}}\ {}_uF_\nu\left[\begin{matrix}(C_u);\\ \dfrac{\lambda_2x^{e_2}}{\left(\lambda_3x^{e_3}y\right)^{e_2}}\\ (D_\nu);\end{matrix}\right] \quad \ldots (3.1)$$

Proof : We have from (1.2)

$$S_n\left(n^{k-g+q-p+1}x, n^{\frac{p-q}{e_2}} y\right)=\sum_{m=0}^{n}\sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]}\frac{\left[(a_p)\right]_{n-m-e_1m_1-(e_2-1)m_2}}{\left[(b_q)\right]_{n-m-e_1m_1-(e_2-1)m_2}}$$

$$\times\frac{\left[(A_h)\right]_{n-m-e_1m_1-e_2m_2}\left[(G_r)\right]_{m_1}\left[(C_u)\right]_{m_2}\lambda^m\lambda_1^{m_1}\lambda_2^{m_2}x^{e_2m_2}}{\left[(B_k)\right]_{n-m-e_1m_1-e_2m_2}\left[(H_s)\right]_{m_1}\left[(D_\nu)\right]_{m_2} m!\, m_1!\, m_2!}$$

$$\times\frac{\left(\lambda_3x^{e_3}\right)^{n-m-e_1m_1-e_2m_2} y^{m+e_1m_1-e_2m_2} n^{n(k-h+p-q+1)(n-m-e_1m_1-e_2m_2)}}{(n-m-e_1m_1-e_2m_2)!\qquad n^{(p-q)e_2m_2}}$$

$$=\frac{K\, n^{n(k-h+q-p+1)}}{1}\sum_{m=0}^{n}\sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]}\frac{K_{(m)}\quad K_{(m_1)}}{n^{n(k-h+q-p+1)m}}$$

$$\times\frac{\Delta_{m_2}[e_2;-n+m+e_1m_1]\Delta_{m_2}\left[e_2-1;1-(b_q)-n+m+e_1m_1\right]}{\Delta_{m_2}\left[e_2-1;1-(a_p)-n+m+e_1m_1\right]}$$

$$\times \frac{\Delta_{m_2}\left[e_2;1-\left(B_k\right)-n+m+e_1m_1\right]\left[\left(C_u\right)\right]_{m_2}\left(\lambda_2\ x^{e_2}\right)^{m_2}}{\Delta_{m_2}\left[e_2;1-\left(A_h\right)-n+m+e_1m_1\right]\left[\left(D_\nu\right)\right]_{m_2}\ m_2!}$$

$$\times \frac{\left(-e_2\right)^{e_2(k-h+q-p+1)m_2}\left\{-\left(e_2-1\right)\right\}^{(q-p)(e_2-1)m_2}}{n^{e_2(k-h+1)m_2}\, n^{(e_2-1)(q-p)e_2}} \qquad \ldots (3.2)$$

Therefore,

$$\lim_{n\to\infty} \frac{K}{n^{n(k-h+q-p+1)}} S_n\left(n^{k-h+q-p+1}x, n^{\frac{p-q}{e_2}}y\right)$$

$$= \lim_{n\to\infty} \sum_{m=0}^{n} \sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]} \sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]} \frac{K_{(m)}\quad K_{(m_1)}}{n^{n(k-h+q-p+1)m+m_1}}$$

$$\times \frac{\Delta_{m_2}\left[e_2;-n+m+e_1m_1\right]\Delta_{m_2}\left[e_2-1;1-\left(b_q\right)-n+m+e_1m_1\right]}{\Delta_{m_2}\left[e_2-1;1-\left(a_p\right)-n+m+e_1m_1\right]}$$

$$\times \frac{\Delta_{m_2}\left[e_2;1-\left(B_k\right)-n+m+e_1m_1\right]\left[\left(C_u\right)\right]_{m_2}\left\{-\left(e_2-1\right)\right\}^{(e_2-1)(q-p)}}{\Delta_{m_2}\left[e_2;1-\left(A_h\right)-n+m+e_1m_1\right]\left[\left(D_\nu\right)\right]_{m_2}\ m_2!}$$

$$\times \frac{\left(-e_2\right)^{e_2(k-h+1)m_2}\left(\lambda_2 x^{e_2}\right)^{m_2}}{\left(\lambda_3 x^{e_3} y\right)^{e_2m_2}\ n^{(e_2-1)(q-p)m_2}\ n^{(k-h+1)m_2}}$$

$$= \sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{\lambda^m \lambda_1^{m_1}\left(\lambda_2 x^{e_2}\right)^{m_2} y^{m+e_1m_1}\left[\left(G_r\right)\right]_{m_1}\left[\left(C_u\right)\right]_{m_2}}{\left(\lambda_3 x^{e_3}\right)^{m+e_1m_1}\left[\left(H_s\right)\right]_{m_1}\left[\left(D_\nu\right)\right]_{m_2}\left(\lambda_3 x^{e_3} y\right)^{e_2m_2}}$$

$$= \sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty} \frac{\lambda^m \lambda_1^{m_1} y^{m+e_1m_1}\left[\left(G_r\right)\right]_{m_1}}{m!\,m_1!\left(\lambda_3 x^{e_3}\right)^{m+e_1m_1}\left[\left(H_s\right)\right]_{m_1}}\ {}_uF_\nu\left[\begin{matrix}\left(C_u\right); \\ \dfrac{\lambda_2 x^{e_2}}{\left(\lambda_3 x^{e_3} y\right)^{e_2}} \\ \left(D_\nu\right);\end{matrix}\right] \qquad \ldots (3.3)$$

Hence the proof.

Particular Cases of (3.1) :

1. On Putting $p = 0 = q = h = k = u = \nu$; $m = 1 = m_1 = e_1 = e_3 = \lambda$; $\lambda_3 = 1 = e_2$; $\lambda_2 = -1$, $y = x$, , in (3.1),we get

$$\lim_{n\to\infty}\left\{\left(2nx\right)^{-n} H_n\left(nx\right)\right\} = e^{-\frac{1}{4x^2}}$$

2. If we take p =0 = q = h = k = u; v =1 = m = m_1 = e_1 = λ_2=e_3;D_1=1; λ_3 = 1; y = 2x,e_2 = 2, and $\frac{x}{\sqrt{x^2-1}}$ for

x, in (3.1),we get

$$\lim_{n\to\infty}(nx)^{-n}\left\{x^2\left(n^2-1\right)+1\right\}^{\frac{n}{2}}\rho_n\left\{\frac{nx}{x^2\left(n^2-1\right)+1}\right\}=I_0\left\{\frac{\sqrt{x^2-1}}{x}\right\}$$

where $I_0(x)$ is the modified Bessel function of the first kind of Index zero.

3. On putting h = 0 = u; k = 1 = v = e_3 = y; and writing for x and y in (3.1) , we get

$$\lim_{n\to\infty}\left[\frac{n!\left\{x^2\left(n^2-1\right)+1\right\}^{\frac{n}{2}}}{(nx)^n(2\lambda)_n}C_n^{(\lambda)}\left\{\frac{nx}{\sqrt{\left(n^2-1\right)\left(x^2+1\right)}}\right\}\right]$$

$$=\Gamma\left(\lambda+\frac{1}{2}\right)\left(\frac{2x}{\sqrt{x^2-1}}\right)^{\lambda-\frac{1}{2}}I_{\lambda-\frac{1}{2}}\left(\frac{\sqrt{x^2-1}}{x}\right)$$

where $C_n^{\lambda}(x)$ are the Gagenbauer polynomials.

Theorem: 1(b) If e_2 = 1, then

$$\lim_{n\to\infty}\frac{\left[\left(b_q\right)\right]_n\left[\left(B_k\right)\right]_n n!}{\left[\left(a_p\right)\right]_n\left[\left(A_h\right)\right]_n\left(\lambda_3x^{e_3}\right)^n n^{n(k-h+q-p+1)}}S_n\left(n^{k-h+q-p+1}x,n^{p-q}y\right)$$

$$=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\frac{\lambda^m\lambda_1^{m_1}y^{m+e_1m_1}\left[\left(G_r\right)\right]_{m_1}}{m!\,m_1!\left(\lambda_3x^{e_3}\right)^{m+e_1m_1}\left[\left(H_s\right)\right]_{m_1}}\;{}_uF_v\left[\begin{matrix}\left(C_u\right);\\ \\ \left(D_v\right);\end{matrix}\;\frac{\lambda_2x}{\lambda_3x^{e_3}y}\right]$$

… (3.4)

Proof : We have from (1.2)

$$S_n\left(n^{k-h+q-p+1}x,n^{p-q}y\right)=\sum_{m=0}^{n}\sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]}$$

$$\times\frac{\left[\left(a_p\right)\right]_{n-m-e_1m_1}\left[\left(A_h\right)\right]_{n-m-e_1m_1-m_2}\left[\left(G_r\right)\right]_{m_1}\left[\left(C_u\right)\right]_{m_2}}{\left[\left(b_q\right)\right]_{n-m-e_1m_1}\left[\left(B_k\right)\right]_{n-m-e_1m_1-m_2}\left[\left(H_s\right)\right]_{m_1}\left[\left(D_v\right)\right]_{m_2}}$$

$$\times\frac{\lambda^m\lambda_1^{m_1}\lambda_2^{m_2}x^{e_2m_2}\left(\lambda_3x^{e_3}\right)^{n-m-e_1m_1-m_2}y^{m+e_1m_1-m_2}}{m!\,m_1!\,m_2!\,\left(n-m-e_1m_1-m_1\right)!}$$

$$\times\frac{n^{n(h-k+q-p+1)(n-m-e_1m_1-e_2m_2)}}{n^{(p-q)n_{A_2}}}$$

$$= \frac{K\ n^{n(k-h+q-p+1)}}{1} \sum_{m=0}^{n} \sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]} \sum_{m_2=0}^{\left[\frac{n-m-e_1 m_1}{e_2}\right]} \frac{K_{(m)} \quad K_{(m_1)}}{n^{n(k-h+q-p+1)m+m_1}}$$

$$\times \frac{(-n+m+e_1 m_1)_{m_2} \left(1-(B_k)-n+m+e_1 m_1\right)_{m_2} \left[(C_u)\right]_{m_2}}{\left(1-(A_h)-n+m+e_1 m_1\right)_{m_2} \left[(D_v)\right]_{m_2} m_2! \left(\lambda_3 x^{e_3} y\right)^{m_2}}$$

$$\times \frac{(\lambda_2 x)^{m_2} (-1)^{(h-k+q-p+1)m_2}}{m_2!\ n^{(h-k+1)m_2}} \qquad \ldots (3.5)$$

Therefore

$$\lim_{n\to\infty} \frac{K}{n^{n(k-h+q+p-1)}} S_n\left(n^{k-h+q-p+1} x, n^{p-q} y\right)$$

$$= \lim_{n\to\infty} \sum_{m=0}^{n} \sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]} \sum_{m_2=0}^{\left[\frac{n-m-e_1 m_1}{e_2}\right]} \frac{K_{(m)} \quad K_{(m_1)}}{n^{n(k-h+q-p+1)m+e_1 m_1}}$$

$$\times \frac{(-n+m+e_1 m_1)_{m_2} \left(1-(B_k)-n+m+e_1 m_1\right)_{m_2} \left[(C_u)\right]_{m_2}}{\left(1-(A_h)-n+m+e_1 m_1\right)_{m_2} \left[(D_v)\right]_{m_2} m_2!}$$

$$\times \frac{(-1)^{(h-k+q-p+1)m_2} (\lambda_2 x)^{m_2}}{n^{(h-k+1)m_2}}$$

$$= \sum_{m=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{\lambda^m \lambda_1^m\ \lambda_2^{m_2} x^{m_2} y^{m+e_1 m_1} \left[(G_r)\right]_{m_1} \left[(C_u)\right]_{m_2}}{\left(\lambda_3 x^{e_3}\right)^{m+e_1 m_1} \left[(H_s)\right]_{m_1} \left[(D_v)\right]_{m_2} \left(\lambda_3 x^{e_3} y\right)^{m_2}}$$

$$= \sum_{m=0}^{\infty} \sum_{m_1=0}^{\infty} \frac{\lambda^m \lambda_1^{m_1} y^{m+e_1 m_1} \left[(G_r)\right]_{m_1}}{m!\ m_1! \left(\lambda_3 x^{e_3}\right)^{m+e_1 m_1} \left[(H_s)\right]_{m_1}}\ {}_uF_v\left[\begin{matrix}(C_u); & \\ & \dfrac{\lambda_2 x}{\lambda_3 x^{e_3} y} \\ (D_v); & \end{matrix}\right]$$

Hence the proof.

Particular Cases of (3.4) :

1. On Putting $h = 0 = k = u$; $v = 1 = n = \lambda_3$; $\lambda_2 = -1$, for y in (3.4), we achieve

$$\lim_{h\to\infty} \left\{ \frac{L_n^{(\alpha)}\left(\frac{y}{x}\right)}{(x+1)_\alpha} \right\} = y^{\frac{-\alpha}{2}} J_\alpha\left(2y^{\frac{1}{2}}\right)$$

where $J_\alpha(x)$ are the Bessel function of the first kind of index α.

2. On taking $h = 0 = u$; $k = 1 = \text{v} = e_3$; , $D_1 = 1 + \beta$ in (3.4), we get

$$\lim_{h\to\infty}\left\{\frac{n!\left[n^{-2}(x+1)-(x-1)\right]^n}{(x-1)^n(1+\beta)_n}P_n^{(\alpha,\beta)}\left(\frac{(x+1)+(x-1)n^2}{(x+1)-(x-1)n^2}\right)\right\}$$

$$=\Gamma(1+\beta)\left(\frac{x+1}{x-1}\right)^{\frac{-\beta}{2}}I_\beta\left(2\sqrt{\frac{x+1}{x-1}}\right)$$

where $I_n(x)$ are the modified Bessel function of the first kinds of index n.

3. On putting $h = 0 = u$; $k = 1 = \text{v} = e_3$; $B_1 = 1+\beta$; $D_1 = 1 + \alpha$ and instead of x and y in (3.4), we get

$$\lim_{h\to\infty}\left\{\frac{n!\left[(x+1)-n^2(x-1)\right]^n}{(x+1)^n(1+\alpha)_n}P_n^{(\alpha,\beta)}\left(\frac{n^2(x+1)+(x-1)}{n^2(x+1)-(x-1)}\right)\right\}$$

$$=\Gamma(1+\alpha)\left(\frac{x-1}{x+1}\right)^{\frac{-\alpha}{2}}I_\alpha\left(2\sqrt{\frac{x-1}{x+1}}\right)$$

4. On making the substitutions $h = 0 = u$; $k = 1 = v = e_3 = y$; and writing for x and y in (3.4), we get

$$\lim_{h\to\infty}\left\{\frac{n!\left[(x+1)-n^2(x-1)\right]^n}{(x+1)^n(2\lambda)_n}C_n^{(\lambda)}\left(\frac{n^2(x+1)+(x-1)}{n^2(x+1)-(x+1)}\right)\right\}$$

$$=\Gamma\left(\lambda+\frac{1}{2}\right)\left(\frac{x-1}{x+1}\right)^{\frac{-\lambda}{2}+\frac{1}{4}}I_{\lambda-\frac{1}{2}}\left(2\sqrt{\frac{x-1}{x+1}}\right)$$

where $C_n^\lambda(x)$ are the Gagenbauer polynomials[4].

Theorem: 2(a) If we take $e_2 > 1$, we have

$$\lim_{n\to\infty}\frac{n!\left[(b_q)\right]_n\left[(B_k)\right]_n\lambda_3^{-n}\left(\cos\frac{z}{n}\right)^{-ne_3}}{\left[(a_p)\right]_n\left[(A_h)\right]_n n^{n(h-k+q-p+1)}}S_n\left(n^{(h-k+q-p+1)}\cos\frac{z}{n}, n^{\frac{q-p}{e_2}}y\right)$$

$$=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\frac{\lambda^m\lambda_1^{m_1}y^{m+e_1m_1}\left[(G_r)\right]_{m_1}}{m!m_1!\left(\lambda_3x^{e_3}\right)^{m+e_1m_1}\left[(H_s)\right]_{m_1}}\,{}_uF_\text{v}\left[\begin{matrix}(C_u); \\ \dfrac{\lambda_2x^{e_2}}{\left(\lambda_3x^{e_3}y\right)^{e_2}} \\ (D_\text{v});\end{matrix}\right] \quad \text{… (3.6)}$$

Proof: We have from (3.4)

$$\lim_{n\to\infty} \frac{\left[\left(b_q\right)\right]_n \left[\left(B_k\right)\right]_n n!\lambda_3^{-n}\left(\cos\frac{z}{n}\right)^{-ne_3}}{\left[\left(a_p\right)\right]_n \left[\left(A_h\right)\right]_n \left(\lambda_3 x^{e_3}\right)^n n^{n(h-k+q-p+1)}} S_n\left(n^{(h-k+q-p+1)}\cos\frac{z}{n}, n^{\frac{h-k}{e_2}} y\right)$$

$$= \sum_{m=0}^{n} \sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]} \sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]} \frac{K_{(m)} \quad K_{(m_1)}}{\left(\frac{-n}{e_1-1}\right)^{(e_2-1)(q-p)m_2} \left(\frac{-n}{e_2}\right)^{e_2(p-q-1)m_2}}$$

$$\times \frac{\Delta_{m_2}\left[e_2;-n+m+e_1m_1\right]\Delta_{m_2}\left[e_2-1;1-\left(b_q\right)-n+m+e_1m_1\right]}{\Delta_{m_2}\left[e_2-1;1-\left(a_p\right)-n+m+e_1m_1\right] \quad m_2!}$$

$$\times \frac{\Delta_{m_2}\left[e_2;1-\left(B_k\right)-n+m+e_1m_1\right]\left[\left(C_u\right)\right]_{m_2}\left(\lambda_2 x^{e_2}\right)^{m_2}}{\Delta_{m_2}\left[e_2;1-\left(A_h\right)-n+m+e_1m_1\right]\left[\left(D_v\right)\right]_{m_2}\left(\lambda_3 x^{e_3} y\right)^{e_2m_2}}$$

After writing $\cos\frac{z}{n}$ for *n*, in the above result, we obtain.

$$\lim_{n\to\infty} \frac{n!\left[\left(b_q\right)\right]_n \left[\left(B_k\right)\right]_n \lambda_3^{-n}\left(\cos\frac{z}{n}\right)^{-ne_3}}{\left[\left(a_p\right)\right]_n \left[\left(A_h\right)\right]_n n^{n(h-k+q-p+1)}} S_n\left(n^{n(h-k+q-p+1)}\cos\frac{z}{n}, n^{\frac{p-q}{e_2}} y\right)$$

$$= \lim_{n\to\infty} \sum_{m=0}^{n} \sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]} \sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]} \frac{K_{(m)} \quad K_{(m_1)}\left[\left(C_u\right)\right]}{m_2!\left[\left(D_v\right)\right]_{m_2}\left(\cos\frac{z}{n}\right)^{e_2e_3m_2}}$$

$$\times \frac{\Delta_{m_2}\left[e_2;-n+m+e_1m_1\right]\Delta_{m_2}\left[e_2-1;1-\left(b_q\right)-n+m+e_1m_1\right]}{\Delta_{m_2}\left[e_2-1;1-\left(a_p\right)-n+m+e_1m_1\right]}$$

$$\times \frac{\Delta_{m_2}\left[e_2;1-\left(B_k\right)-n+m+e_1m_1\right]}{\Delta_{m_2}\left[e_2;1-\left(A_h\right)-n+m+e_1m_1\right]\left(\frac{-n}{e_2-1}\right)^{(e_2-1)(q-p)m_2}\left(\frac{-n}{e_2}\right)^{e_2(h-k+1)m_2}}$$

$$= \lim_{n\to\infty} \sum_{m=0}^{n} \sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]} \sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]} \frac{\lambda^m \lambda_1^{m_1} y^{m+e_1m_1}\left[\left(G_r\right)\right]_{m_1}}{m!\; m_1!\; \lambda_3^{m_1}\left[\left(H_s\right)\right]_{m_1}}$$

$$\times\frac{\left[(C_u)\right]_{m_1}\lambda_2^{m_2}\left(\cos\frac{z}{n}\right)^{e_2m_2}(-n)_{m_1}\left(1-(b_q)-n\right)_{m_1}\left(1-(B_k)-n\right)_{m_1}}{\left[(D_\nu)\right]_{m_1}\left(\cos\frac{z}{n}\right)^{ne_3+e_1e_3m_1}\left(\cos\frac{z}{n}\right)^{e_2e_3m_2}\left(1-(a_p)-n\right)_{m_1}}$$

$$\times\frac{\Delta_{m_1}[e_1;-n+m]\Delta_{m_1}\left[e_1;1-(b_q)-n+m\right]\Delta_{m_1}\left[e_1;1-(B_k)-n+m\right]}{\left(1-(A_h)-n\right)_{m_1}\Delta_{m_1}\left[e_1;1-(a_p)-n+m\right]\Delta_{m_1}\left[e_1;1-(A_h)-n+m\right]}$$

$$\times\frac{\Delta_{m_2}[e_2;-n+m+e_1m_1]\Delta_{m_2}\left[e_2-1;1-(b_q)-n+m+e_1m_1\right]}{\Delta_{m_2}\left[e_2-1;1-(a_p)-n+m+e_1m_1\right]\Delta_{m_2}\left[e_2;1-(A_h)-n+m+e_1m_1\right]}$$

$$\times\frac{\Delta_{m_2}\left[e_2;1-(B_k)-n+m+e_1m_1\right]}{\left(\frac{-n}{e_1}\right)^{e_1(h-k+q-p+1)m_1}\left(\frac{-n}{e_2}\right)^{e_2(h-k+1)m_2}\left(\frac{-n}{e_2-1}\right)^{(e_2-1)(q-p)m_2}m_2!}$$

$$\times\frac{1}{\left[\left(1-\sin^2\frac{z}{n}\right)^{\frac{1}{\sin^2\frac{z}{n}}}\right]^{\left(\frac{\sin^2\frac{z}{n}}{\frac{z}{n}}\right)^2\frac{z^2}{2n^2}e_1e_3m_1}}\times\frac{1}{\left[\left(1-\sin^2\frac{z}{n}\right)^{\frac{1}{\sin^2\frac{z}{n}}}\right]^{\left(\frac{\sin\frac{z}{n}}{\frac{z}{n}}\right)^2\frac{z^2}{2n^2}e_2m_2e_3}}$$

$$=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{\lambda^m\lambda_1^{m_1}y^{m+e_1m_1}\left[(G_r)\right]_{m_1}\left[(C_u)\right]_{m_2}}{\lambda_3^{m+e_1m_1}\left[(H_s)\right]_{m_1}\left[(D_\nu)\right]_{m_2}(\lambda_3ye_2)^{m_2}}$$

$$=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\frac{\lambda^m\lambda_1^{m_1}y^{m+e_1m_1}\left[(G_r)\right]_{m_1}}{\lambda_3^{m+e_1m_1}\left[(H_s)\right]_{m_1}}\ {}_uF_\nu\left[\begin{matrix}(C_u); & \\ & \frac{\lambda_2}{(\lambda_3y)^{e_2}}\\ (D_\nu); & \end{matrix}\right] \quad \ldots (3.7)$$

Hence the **proof .**
Similarly
Theorem: 2(b)

$$\lim_{n\to 0}\frac{n!\left[(b_q)\right]_n\left[(B_k)\right]_n\lambda_3^{-n}\left(\cos\frac{z}{n}\right)^{-ne_3}}{\left[(a_p)\right]_n\left[(A_h)\right]_n n^{n(h-k+q-p+1)}}S_n\left(n^{(h-k+p-q+1)}\cos\frac{z}{n},n^{p-q}\cos\frac{z}{n}\right)$$

$$=\lim_{n\to\infty}\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\frac{\lambda^m\lambda_1^{m_1}y^{m+e_1m_1}\left[(G_r)\right]_{m_1}}{\lambda_3^{m+e_1m_1}\left[(H_s)\right]_{m_1}}\ {}_uF_\nu\left[\begin{matrix}(C_u); & \\ & \frac{\lambda_2}{\lambda_3^{e_2}}\\ (D_\nu); & \end{matrix}\right] \quad \ldots (3.8)$$

Particular Cases of (3.6) :

I. On Putting p =0 = q = h = k = u = v; m = 1 = m_1 = e_1 = e_3 = λ; λ_3 = 1 = e_2; λ_2 = – 1, y = x in (3.6), we get

$$\lim_{h\to\infty}\left\{\left(2n\cos\frac{z}{n}\right)^{-n} H_n\left(\cos\frac{z}{n}\right)\right\} = e^{\frac{1}{4}}$$

II. If we take p =0 = q = h = k = u; v =1 = m = m_1 = e_1 = λ_2 = e_3;D_1 = 1; λ_3 = 1; y = 2x,

e_2 = 2, and $\frac{x}{\sqrt{x^2-1}}$ for x in (3.6), we get

$$\lim_{h\to\infty}\left\{\left(n\cos\frac{z}{n}\right)^{-n}\left(n^2\cos^2\frac{z}{n}-1\right)^{\frac{1}{2}n} P_n\left(\frac{n\cos\frac{z}{n}}{\sqrt{\left(n^2\cos^2\frac{z}{n}-1\right)}}\right)\right\} = I_0\,(I)$$

III. On making the substitutions h = 0 = u; k = 1 = v = e_3 = y; and writing for x and y in (3.6), we get

$$\lim_{h\to\infty}\left\{\frac{n}{(2\lambda)_n}\left(n\cos\frac{z}{n}\right)^{-n}\left(n^2\cos^2\frac{z}{n}-1\right)^{\frac{n}{2}} C_n^{(\lambda)}\left(\frac{n\cos\frac{z}{n}}{n^2\cos^2\frac{z}{n-1}}\right)\right\}$$

$$= {}_0F_1\left[\begin{matrix} —; \\ \quad\frac{1}{4} \\ \lambda+\frac{1}{2}; \end{matrix}\right] = 2^{\lambda-\frac{1}{2}}\Gamma\left(\lambda+\frac{1}{2}\right) I_{\lambda-\frac{1}{2}}(I)$$

where $C_n^{\lambda}(x)$ are the Gagenbauer polynomials.

IV. CONCLUSION

In this paper we studies the asymptotic behaviours of generalized hypergeometric polynomial set $S_n(x,y)$ for large value of n, where n is the order of the polynomial set. These behaviours for large n have been given in the form of theorems. A number of well known results for orthogonal and non-orthogonal polynomials have been deduced as particular cases of these theorems.

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