International Journal of Scientific Research in Mathematical and Statistical Sciences
Research Paper
Volume-5, Issue-6, pp.94-101, December (2018)
E-ISSN: 2348-4519

Marichev-Saigo Maeda Fractional Operators representations of the Generalized Miller-Ross Function

[1*]Javid Majid, [2] Imtiyaz Majid

[1]School of Mathematics and Allied Sciences, Jiwaji University, Gwalior (M.P.), 474011, INDIA
[2]B.U.Bhopal,(M.P.),462026, INDIA

*Corresponding Author: javidmajidmaths@gmail.com, Tel.: +91-7006780181

Available online at: www.isroset.org

Received: 04/Nov/2018, Accepted: 07/Dec/2018, Online: 31/Dec/2018

Abstract- In this paper we will implement the generalized fractional operators induced by Saego-Maeda using the Appell's $F_3(.)$ function and set up the image formulas associated with the generalized Miller-Ross function in terms of the generalized Fox- Wright function. We will also employ certain integral transforms on the results obtained from the differentials and the integrals and present some more image formulas.

Keywords- Generalized Miller-Ross function, Saego-Maeda fractional operator, Fox Wright function.

I. INTRODUCTION

The fractional calculus is the special branch of applied mathematics which is rapidly developing with large number of applications in the real world. It is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. This branch of calculus has gained considerable importance and applications in various sub-fields of applicable mathematical analysis. The computation of fractional derivatives and the fractional integrals of the special functions of one and more variables is very important from the point of view of the usefulness of these results in the evaluation of generalized integrals and the solution of differential and integral equations.

Several authors and researchers such as Baleanu et al. [1][2], Kilbas [3], Miller and Ross[4], Saigo M. [5] Mittag - Leffler Function [6], Prabhakar [7], Kumar et al. [8] so forth have studied in depth the properties, applications and different extensions of the operators of fractional order. They have used the fractional order integral and differential models in various fields of the real world problem (also see [9],[10],[11],[12],[13],[14],[15],[16],[17]). Here the author's aim is to establish the various image formulas for the generalized Miller-Ross function involving the generalized fractional calculus operators introduced by Saigo-Meada [18].

Section **I** of the paper contains the brief introduction of the topic and the work done by various authors in this field, section **II** contains the mathematical preliminaries, section **III** contains the main theorems and the results and section **IV** concludes the work.

II. MATHEMATICAL PRELIMINARIES

Saigo and Maeda in 1998 introduced the following generalized fractional and differential operators of the complex order with Appell $F_3(.)$ function in the kernel, as follows:

Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ and $x > 0$, then the generalized fractional calculus operators (the Marichev-Saigo Maeda operators) involving the Appell function, or Horn's F_3 – function introduced by Appell and Kampe de Feriet [19] are defined by the following equations:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)dt \tag{2.1}$$

$$(\Re(\gamma) > 0)$$

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(\frac{d}{dx}\right)^k \left(I_{0+}^{\alpha,\alpha',\beta+k,\beta',\gamma+k} f\right)(x)$$

$$(\Re(\gamma) \le 0);\ k = [-\Re(\gamma)+1] \tag{2.2}$$

$$\left(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)dt \tag{2.3}$$

$$(\Re(\gamma) > 0)$$

$$(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = \left(-\frac{d}{dx}\right)^{k} (I_{0-}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k} f)(x)$$
$$(\Re(\gamma) \le 0); \; k = [-\Re(\gamma)+1] \tag{2.4}$$

and

$$(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = (I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f)(x) \tag{2.5}$$
$$= \left(\frac{d}{dx}\right)^{k} (I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k} f)(x)$$
$$(\Re(\gamma) > 0); \; k = [\Re(\gamma)+1])$$

$$(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = (I_{0-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f)(x) \tag{2.6}$$
$$= \left(-\frac{d}{dx}\right)^{k} (I_{0-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k} f)(x)$$
$$(\Re(\gamma) > 0); \; k = [\Re(\gamma)+1]) \tag{2.7}$$

Following Saigo and Maeda [18], the image formulas for a power function, under operators (2.1) and (2.3), are given by:

$$(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1})(x) = x^{\rho-\alpha-\alpha'+\gamma-1}$$
$$\left[\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)}\right] \tag{2.8}$$

where $\Re(\rho) > max\{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\}$ and $\Re(\gamma) > 0$.

$$(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1})(x) = x^{\rho+\gamma-\alpha-\alpha'-1}$$
$$\left[\frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\gamma+\alpha+\alpha')\Gamma(1-\rho+\beta'+\alpha-\gamma)}{\Gamma(1-\rho)\Gamma(1-\rho+\beta'-\gamma+\alpha+\alpha')\Gamma(1-\rho+\alpha-\beta)}\right] \tag{2.9}$$

Where $\Re(\rho) < 1 + min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}$ and $\Re(\gamma) > 0$

The generalized Fox- Wright function ${}_p\psi_q$ was introduced by Wright [20] and has been given by the series

$${}_p\psi_q = {}_p\psi_q \left\{\begin{matrix}(a_1,A_1), \ldots (a_p,A_p); \\ (b_1,B_1), \ldots (b_q,B_q);\end{matrix} x\right\}$$
$$= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i+nA_i)}{\prod_{j=1}^{q} \Gamma(b_j+nB_j)} \frac{x^n}{n!} \tag{2.10}$$

Where $\Gamma(x)$ is the Euler gamma function.
Where $x, a_i, b_j \in \mathbb{C}$; $A_i, B_j \in \Re$; $A_i \neq 0, B_j \neq 0; i = 1, \ldots, p;$ $j = 1, \ldots, q$

This function is known as generalized Wright function for all values of x, the conditions for its existence are as follows:

$$1 + \left(\sum_{j=1}^{q} B_j\right) - \left(\sum_{i=1}^{p} A_i\right) \ge 0 \tag{2.11}$$

The generalized hyper-geometric function ${}_pF_q$ is defined as follows [21]

$${}_pF_q \left\{\begin{matrix}a_1, \ldots, a_p; \\ b_1, \ldots, b_q;\end{matrix} x\right\} = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{x^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{(a_1)_n, (a_2)_n, \ldots, (a_p)_n}{(b_1)_n, (b_2)_n, \ldots, (b_q)_n} \frac{x^n}{n!} \tag{2.12}$$

Where $(\propto)_n$ is the Pochhammer symbol, which is defined (for $\propto \in \mathbb{C}$) by:

$$(\propto)_n = \left\{\begin{matrix} 0 & n = 0 \\ \propto(\propto+1)(\propto+2)\ldots(\propto+n-1) & n > 0 \end{matrix}\right\} \tag{2.13}$$

$(\propto)_n = \frac{\Gamma(\propto+n)}{\Gamma(\propto)} (\propto \in \mathbb{C}\backslash\mathbb{Z}_0^-)$

$\mathbb{Z}_0^-$ denotes the set of non-positive integers.

Miller and Ross [4] introduced the function in 1993 known as the Miller-Ross function that arises as the basis for the solution of the fractional order initial value problem, which they defined as

$$E_t(v,a) = \frac{d^{-v}e^{at}}{dt^{-v}} = t^v e^{at} \gamma^*(v,at) = \sum_{k=0}^{\infty} \frac{a^k t^{k+v}}{\Gamma(k+v+1)} \tag{2.14}$$

Where $\gamma^*(v,at)$ is the incomplete gamma function.

Farhan, Sharma and Jain [22] introduced the generalized Miller-Ross function in 2014 as

$${}_0^aN_{p,q}^{\lambda,\xi}(a_1, \ldots, a_p; b_1, \ldots, b_q; x)$$
$$= \sum_{n=0}^{\infty} \frac{(a_1)_n, (a_2)_n, \ldots, (a_p)_n a^n x^{n+\xi}}{(b_1)_n, (b_2)_n, \ldots, (b_q)_n \Gamma(\lambda n+\xi+1)} \tag{2.15}$$

Where $a_1, a_2, \ldots a_p, b_1, b_2, \ldots, b_q, \lambda, \xi \in \mathbb{C}, \Re(\lambda) > 0$, $\Re(\xi) > 0$ and $(a_i)_k, (b_j)_k$ are Pochhammer symbols.

The gamma function, Γx was introduced by Leonard Euler [21] as generalization of the factorial function on the set $\Re$ of all real numbers and $\mathbb{C}$ for all complex numbers defined by

$$\Gamma x = \int_0^{\infty} t^{(x-1)} e^{-t} dt, x \epsilon \Re^+ \tag{2.16}$$

$$\Gamma 1 = 1, \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

The integral formulae involving the Whittaker function (Mathai at al. [23]) is used in finding the image formulae:

$$\int_0^{\infty} z^{\delta-1} e^{-\frac{z}{2}} W_{\sigma,\eta}(z) dz = \frac{\Gamma(\delta+\eta+\frac{1}{2})\Gamma(\delta-\eta+\frac{1}{2})}{\Gamma(\delta-\sigma+\frac{1}{2})} \tag{2.17}$$

$$\left(\sigma \in \mathbb{C}, \Re(\delta \pm \eta) > -\frac{1}{2}\right)$$

The Whittaker function (Mathai et al [23]) is defined by

$$W_{\sigma,\eta}(z) = \frac{\Gamma(-2\eta)}{\Gamma(\frac{1}{2}-\sigma-\eta}M_{\sigma,\eta}(z) + \frac{\Gamma(2\eta)}{\Gamma(\frac{1}{2}-\sigma+\eta}M_{\sigma,-\eta}(z) \quad (2.18)$$

$$=W_{\sigma,-\eta}(z)$$

$$\left(\sigma \in \mathbb{C}, \Re\left(\frac{1}{2}+\eta \pm \delta\right) > 0\right)$$

where

$$M_{\sigma,\eta}(z) = z^{\eta+\frac{1}{2}}e^{-\frac{z}{2}}\,{}_1F_1\left(\frac{1}{2}-\sigma+\eta; 2\eta+1; z\right) \quad (2.19)$$

$$\Re\left(\frac{1}{2}+\eta \pm \delta\right) > 0, \quad |\arg z| < \pi$$

III. MAIN RESULTS

Image formulas associated with fractional operators

In this section, we will establish the image formulas for the generalized Miller-Ross function involving the Saigo-Maeda fractional operators (2.1), (2.3), (2.5) and (2.6) in terms of the generalized Fox-Wright function.

Throughout this paper, we will assume that $x > 0, \alpha, \alpha', \beta, \beta', \gamma, \rho, \lambda, \xi, \upsilon, \eta, \delta \in \mathbb{C}, \Re(\lambda) > 0, \Re(\xi) > 0$ and we will also assume that the constants satisfy the conditions $a_i, b_j \in \mathbb{C}, A_i, B_j \in \Re$ $(A_i, B_j \neq 0,\ i = 1,2,\dots,p,$ $j = 1,2,\dots,q.)$ such that the condition (1.11) is also satisfied.

Theorem 3.1. Let $\Re(\xi) > 0, \Re(\upsilon) > 0$, then the fractional integral $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the generalized Miller-Ross function under the conditions $\Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\rho + \upsilon\xi) > max\{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\}$ exists and is given by

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\, t^{\rho-1}\, {}_0^aN_{p,q}^{\lambda,\xi}\,(t^{\upsilon})\right)(x) = x^{A-\alpha-\alpha'+\gamma-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times\,{}_{p+4}\psi_{q+4}\left[\begin{matrix}(a_1,1),\dots,(a_p,1),(A,\upsilon),(A+\beta'-\alpha',\upsilon)\\(b_1,1),\dots,(b_q,1),(\xi+1,\lambda),(A+\beta',\upsilon)\end{matrix}\right.$$
$$\left.\begin{matrix}(A+\gamma-\alpha-\alpha'-\beta,\upsilon),(1,1)\\(A+\gamma-\alpha-\alpha',\upsilon),(A+\gamma-\beta-\alpha',\upsilon)\end{matrix};\ ax^{\upsilon}\right] \quad (3.1)$$

where $A = \rho + \upsilon\xi$

Proof. Taking the LHS of (3.1) as $\mathcal{J}$, and using (2.15) we get

$$\mathcal{J} = \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\,(t^{\rho-1}\right.$$

$$\left.\times\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}(t^{\upsilon})^{n+\xi})\right)(x)$$

$$\mathcal{J} = \sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}$$

$$\times\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\,(t^{\rho+\upsilon n+\upsilon\xi-1})\right)(x)$$

Applying (2.8) with $\rho = (\rho + \upsilon n + \upsilon\xi)$, we get

$$\mathcal{J} = x^{A-\alpha-\alpha'+\gamma-1}\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n\Gamma(n+1)}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}\frac{(ax^{\upsilon})^n}{n!}$$

$$\times\left[\frac{\Gamma(A+\upsilon n)\ \Gamma(A+\gamma-\alpha-\alpha'-\beta+\upsilon n)\,\Gamma(A+\beta'-\alpha'+\upsilon n)}{\Gamma(A+\beta'+\upsilon n)\ \Gamma(A+\gamma-\alpha-\alpha'+\upsilon n)\ \Gamma(A+\gamma-\beta-\alpha'+\upsilon n)}\right] \quad (3.2)$$

where $A = \rho + \upsilon\xi$

Interpreting the right hand side of (3.2) in view of (2.10) we arrive at the required result.

Theorem 3.2. Let $\Re(\xi) > 0, \Re(\upsilon) > 0$, then the fractional integral $I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the generalized Miller-Ross function under the conditions $\Re(\gamma) > 0, \Re(\lambda) > 0, \Re(1-\gamma-\rho-\upsilon\xi) < 1 + min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}$ and is given by

$$\left(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}\, t^{-\gamma-\rho}\, {}_0^aN_{p,q}^{\lambda,\xi}\,(t^{-\upsilon})\right)(x) = x^{-A-\alpha-\alpha'}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times\,{}_{p+4}\psi_{q+4}\left[\begin{matrix}(a_1,1),\dots,(a_p,1),(1,1),(A+\alpha+\beta',\upsilon)\\(b_1,1),\dots,(b_q,1),(\xi+1,\lambda),(A+\alpha+\alpha'+\beta',\upsilon)\end{matrix}\right.$$
$$\left.\begin{matrix}(A+\alpha+\alpha',\upsilon),(A-\beta+\gamma,\upsilon)\\(A+\gamma,\upsilon),(A+\gamma-\beta+\alpha,\upsilon)\end{matrix};\ ax^{-\upsilon}\right] \quad (3.3)$$

where $A = \rho + \upsilon\xi$

Proof. Taking the LHS of (3.3) as $\mathcal{J}$, and using (2.15) we get

$$\mathcal{J} = \left(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}\,(t^{-\gamma-\rho}\,(x)\right.$$

$$\left.\times\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}(t^{-\upsilon})^{n+\xi})\right)(x)$$

$$\mathcal{J} = \sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}$$

$$\times\left(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}\,(t^{-\gamma-\rho-\upsilon n-\upsilon\xi})\right)(x)$$

Applying (2.9) with $\rho = (1-\gamma-\rho-\upsilon n-\upsilon\xi)$, we get

$$\mathcal{J} = x^{-A-\alpha-\alpha'}\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n\Gamma(n+1)}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}\frac{(ax^{-\upsilon})^n}{n!}$$

$$\times\left[\frac{\Gamma(A+\alpha+\alpha'+\upsilon n)\ \Gamma(A+\alpha+\beta'+\upsilon n)\ \Gamma(A-\beta+\gamma+\upsilon n)}{\Gamma(A+\alpha+\alpha'+\beta'+\upsilon n)\ \Gamma(A+\gamma+\upsilon n)\ \Gamma(A+\alpha-\beta+\gamma+\upsilon n)}\right] \quad (3.4)$$

where $A=\rho+\upsilon\xi$

Interpreting the right hand side of (3.4) in view of (2.10) we arrive at the required result.

Theorem 3.3. Let $\Re(\xi)>0, \Re(\upsilon)>0$, then the fractional differential $D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the generalized Miller-Ross function under the conditions $\Re(\gamma)>0, \Re(\lambda)>0, \Re(\rho+\upsilon\xi)>max\{0,\Re(\gamma-\alpha-\alpha'-\beta'),\Re(\beta-\alpha)\}$ exists and is given by

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{\rho-1}\ {}_0^aN_{p,q}^{\lambda,\xi}\ (t^{\upsilon})\right)(x)$$

$$= x^{A+\alpha+\alpha'-\gamma-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times\ {}_{p+4}\psi_{q+4}\left[\begin{matrix}(a_1,1),\dots(a_p,1),(1,1),(A,\upsilon)\\(b_1,1),\dots(b_q,1),(\xi+1,\lambda),(A-\beta,\upsilon)\end{matrix}\right.$$

$$\left.\begin{matrix}(A-\gamma+\alpha+\alpha'+\beta',\upsilon),(A-\beta+\alpha,\upsilon)\\(A-\gamma+\alpha+\alpha',\upsilon),(A-\gamma+\beta'+\alpha,\upsilon)\end{matrix};ax^{\upsilon}\right] \quad (3.5)$$

where $A=\rho+\upsilon\xi$

Proof. Taking the LHS of (3.5) as $\mathcal{I}$ and using (2.15) we get

$$\mathcal{I}=\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\ (t^{\rho-1}\right.$$

$$\left.\times\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}(t^{\upsilon})^{n+\xi})\right)(x)$$

Using (2.5) we get

$$\mathcal{I}=\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}$$

$$\times\left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}\ (t^{\rho+\upsilon n+\upsilon\xi-1})\right)(x)$$

Applying (2.8) with $\rho=(\rho+\upsilon n+\upsilon\xi)$, we get

$$\mathcal{I}=x^{A+\alpha+\alpha'-\gamma-1}\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n\Gamma(n+1)}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}\frac{(ax^{\upsilon})^n}{n!}$$

$$\left[\frac{\Gamma(A+\upsilon n)\ \Gamma(A-\gamma+\alpha+\alpha'+\beta'+\upsilon n)\quad \Gamma(A-\beta+\alpha+\upsilon n)}{\Gamma(A-\beta+\upsilon n)\ \Gamma(A-\gamma+\alpha+\alpha'+\upsilon n)\Gamma(A-\gamma+\beta'+\alpha+\upsilon n)}\right] \quad (3.6)$$

where $A=\rho+\upsilon\xi$

Interpreting the right hand side of (3.6) in view of (2.10) we arrive at the required result.

Theorem 3.4. Let $\Re(\xi)>0, \Re(\upsilon)>0$, then the fractional differential $D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the generalized Miller-Ross function under the conditions $\Re(\gamma)>0, \Re(\lambda)>0$, $\Re(1-\gamma-\rho-\upsilon\xi)<1+min\{\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\}$and is given by

$$\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{\gamma-\rho}\ {}_0^aN_{p,q}^{\lambda,\xi}\ (t^{-\upsilon})\right)(x)$$

$$= x^{-A+\alpha+\alpha'}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times\ {}_{p+4}\psi_{q+4}\left[\begin{matrix}(a_1,1),\dots,(a_p,1),(1,1),(A-\alpha-\alpha',\upsilon)\\(b_1,1),\dots(b_q,1),(\xi+1,\lambda),(A-\alpha-\alpha'-\beta,\upsilon)\end{matrix}\right.$$

$$\left.\begin{matrix}(A-\beta-\alpha',\upsilon),(A+\beta'-\gamma,\upsilon)\\(A-\gamma,\upsilon),(A-\alpha'+\beta'-\gamma,\upsilon)\end{matrix};ax^{-\upsilon}\right] \quad (3.7)$$

where $A=\rho+\upsilon\xi$

Proof. Taking the LHS of (3.7) as $\mathcal{I}$ and using (2.15) we get

$$\mathcal{I}=\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}\ (t^{\gamma-\rho}\right.$$

$$\left.\times\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}(t^{-\upsilon})^{n+\xi})\right)(x)$$

Using (2.6) we get

$$\mathcal{I}=\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}$$

$$\left(I_{0-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}\ (t^{\gamma-\rho-\upsilon n-\upsilon\xi})\right)(x)$$

Applying (2.9) with $\rho=(1+\gamma-\rho-\upsilon n-\upsilon\xi)$, we get

$$\mathcal{I}=x^{-A+\alpha+\alpha'}\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n\Gamma(n+1)}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}\frac{(ax^{-\upsilon})^n}{n!}$$

$$\times\left[\frac{\Gamma(A-\alpha-\alpha'+\upsilon n)\ \Gamma(A-\alpha'-\beta+\upsilon n)\ \Gamma(A+\beta'-\gamma+\upsilon n)}{\Gamma(A-\alpha-\alpha'-\beta+\upsilon n)\ \Gamma(A-\gamma+\upsilon n)\ \Gamma(A-\alpha'+\beta'-\gamma+\upsilon n)}\right] \quad (3.8)$$

where $A=\rho+\upsilon\xi$

Interpreting the right hand side of (3.8) in view of (2.10) we arrive at the required result.

Image formulas related to Beta transform

The beta transform of the function $f(z)$ is defined by [24]

$$B\{f(z):\ l,m\} = \int_0^1 z^{l-1}(1-z)^{m-1}f(z)dz. \tag{3.9}$$

Theorem 3.5. Let $\Re(\xi)>0, \Re(\upsilon)>0, \Re(\gamma)>0$, $\Re(\lambda)>0$, be such that $\Re(\rho+\upsilon\xi) > max\{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\}$ is satisfied, then

$$B\left\{\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}{}_0^{\alpha}N_{p,q}^{\lambda,\xi}(zt^{\upsilon})\right)\right)(x): l,m\right\} = x^{A-\alpha-\alpha'+\gamma-1}\Gamma(m)\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times\ {}_{p+5}\psi_{q+5}\left[\begin{matrix}(a_1,1),\dots,(a_p,1),(A,\upsilon),(A+\beta'-\alpha',\upsilon)\\(b_1,1),\dots(b_q,1),,(A+\beta',\upsilon),(A+\gamma-\alpha-\alpha',\upsilon)\end{matrix}\right.$$

$$\left.\begin{matrix}(A+\gamma-\alpha-\alpha'-\beta,\upsilon),(l+\xi,1),(1,1)\\(A+\gamma-\beta-\alpha',\upsilon),(\xi+1,\lambda)(l+m+\xi,1)\end{matrix};ax^{\upsilon}\right] \tag{3.10}$$

where $A=\rho+\upsilon\xi$

Proof. Taking the LHS of (3.10) as $\mathcal{J}$ and using (3.9) we get

$$\mathcal{J} = \int_0^1 z^{l-1}(1-z)^{m-1}\left\{\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}{}_0^{\alpha}N_{p,q}^{\lambda,\xi}(zt^{\upsilon})\right)\right)(x)\right\}dz \tag{3.11}$$

using (2.15) in (3.11) we get

$$\mathcal{J} = \int_0^1 z^{l-1}(1-z)^{m-1}\left\{\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\times\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n(zt^{\upsilon})^{n+\xi}}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}\right)(x)\right\}dz$$

$$\mathcal{J} = \int_0^1 z^{l-1}(1-z)^{m-1}\times\left\{\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n z^{n+\xi}}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}\times\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho+\upsilon n+\upsilon\xi-1}\right)\right)(x)\right\}dz \tag{3.12}$$

Applying (2.8) with $\rho=(\rho+\upsilon n+\upsilon\xi)$, we get

$$\mathcal{J} = \int_0^1 z^{l-1}(1-z)^{m-1}\times\left\{x^{A-\alpha-\alpha'+\gamma-1}\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n z^{n+\xi}(ax^{\upsilon})^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}\right.$$

$$\left.\times\left[\frac{\Gamma(A+\upsilon n)\ \Gamma(A+\gamma-\alpha-\alpha'-\beta+\upsilon n)\ \Gamma(A+\beta'-\alpha'+\upsilon n)}{\Gamma(A+\beta'+\upsilon n)\ \Gamma(A+\gamma-\alpha-\alpha'+\upsilon n)\ \Gamma(A+\gamma-\beta-\alpha'+\upsilon n)}\right]\right\}dz$$

where $A=\rho+\upsilon\xi$

Interchanging the order of integration and summation we get

$$\mathcal{J} = x^{A-\alpha-\alpha'+\gamma-1}\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n(ax^{\upsilon})^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)}\times\left[\frac{\Gamma(A+\upsilon n)\ \Gamma(A+\gamma-\alpha-\alpha'-\beta+\upsilon n)\ \Gamma(A+\beta'-\alpha'+\upsilon n)}{\Gamma(A+\beta'+\upsilon n)\ \Gamma(A+\gamma-\alpha-\alpha'+\upsilon n)\ \Gamma(A+\gamma-\beta-\alpha'+\upsilon n)}\right]\times\int_0^1 z^{l+n+\xi-1}(1-z)^{m-1}dz$$

$$\mathcal{J} = x^{A-\alpha-\alpha'+\gamma-1}\times\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n\Gamma(n+1)(ax^{\upsilon})^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\lambda n+\xi+1)n!}\times\left[\frac{\Gamma(A+\upsilon n)\ \Gamma(A+\gamma-\alpha-\alpha'-\beta+\upsilon n)\ \Gamma(A+\beta'-\alpha'+\upsilon n)}{\Gamma(A+\beta'+\upsilon n)\ \Gamma(A+\gamma-\alpha-\alpha'+\upsilon n)\ \Gamma(A+\gamma-\beta-\alpha'+\upsilon n)}\right]\frac{\Gamma(l+n+\xi)\Gamma(m)}{\Gamma(l+m+n+\xi)} \tag{3.13}$$

Interpreting the right hand side of (3.13) in view of (2.10) we arrive at the required result.

Theorem 3.6. Let $\Re(\xi)>0, \Re(\upsilon)>0, \Re(\gamma)>0$, $\Re(\lambda)>0$, be such that $\Re(1-\gamma-\rho-\upsilon\xi) < 1 + min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}$ is satisfied then

$$B\left\{\left(I_{0-}^{\alpha,\alpha\prime,\beta,\beta',\gamma}\left(t^{-\gamma-\rho}\ {}_0^{\alpha}N_{p,q}^{\lambda,\xi}\ (zt^{-\upsilon})\right)\right)(x): l,m\right\} = x^{-A-\alpha-\alpha'}\Gamma(m)\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times\ {}_{p+5}\psi_{q+5}\left[\begin{matrix}(a_1,1),\dots(a_p,1),(1,1),(A+\alpha+\beta',\upsilon)\\(b_1,1),\dots(b_q,1),(A+\alpha+\alpha'+\beta',\upsilon),(\xi+1,\lambda)\end{matrix}\right.$$

$$\left.\begin{matrix}(A+\alpha+\alpha',\upsilon),(A-\beta+\gamma,\upsilon),(l+\xi,1);ax^{-\upsilon}\\(A+\gamma,\upsilon),(A+\alpha-\beta+\gamma,\upsilon),(l+m+\xi,1)\end{matrix}\right] \tag{3.14}$$

where $A=\rho+\upsilon\xi$

Proof. The proof of the theorem can be established by following the same steps as that of the above theorem. So we omit the details here.

Theorem 3.7. Let $\Re(\xi)>0, \Re(\upsilon)>0, \Re(\gamma)>0$, $\Re(\lambda)>0$, be such that $\Re(\rho+\upsilon\xi) > max\{0, \Re(\gamma-\alpha-\alpha'-\beta'), \Re(\beta-\alpha)\}$ is satisfied then

$$B\left\{\left(D_{0+}^{\alpha,\alpha\prime,\beta,\beta',\gamma}\left(t^{\rho-1}\ {}_0^{\alpha}N_{p,q}^{\lambda,\xi}\ (zt^{\upsilon})\right)\right)(x): l,m\right\} = x^{A+\alpha+\alpha'-\gamma-1}\Gamma(m)\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times\ {}_{p+5}\psi_{q+5}\left[\begin{matrix}(a_1,1),\dots(a_p,1),(1,1),(A-\gamma+\alpha+\alpha'+\beta',\upsilon)\\(b_1,1),\dots(b_q,1),(A-\beta,\upsilon),(A-\gamma+\alpha+\alpha',\upsilon)\end{matrix}\right.$$

$$\left.\begin{matrix}(A-\beta+\alpha,\upsilon),(l+\xi,1),(A,\upsilon),\\(A-\gamma+\beta'+\alpha,\upsilon),(\xi+1,\lambda)(l+m+\xi,1)\end{matrix};ax^{\upsilon}\right] \tag{3.15}$$

where $A = \rho + \upsilon\xi$

Proof. Taking the LHS of (3.15) as $\mathcal{I}$ and using (3.9) we get

$$\mathcal{I} = \int_0^1 z^{l-1}(1-z)^{m-1} \left\{\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}\, {}_0^{a}N_{p,q}^{\lambda,\xi}(zt^{\upsilon})\right)\right)(x)\right\} dz \qquad (3.16)$$

Using (2.15) in (3.16) we get

$$\mathcal{I} = \int_0^1 z^{l-1}(1-z)^{m-1} \left\{\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\, t^{\rho-1} \sum_{n=0}^{\infty} \frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n (zt^{\upsilon})^{n+\xi}}{(b_1)_n,(b_2)_n,\dots,(b_q)_n \Gamma(\lambda n+\xi+1)}\right)(x)\right\} dz$$

Using (2.5) we get

$$\mathcal{I} = \int_0^1 z^{l-1}(1-z)^{m-1} \left\{\sum_{n=0}^{\infty} \frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n z^{n+\xi}}{(b_1)_n,(b_2)_n,\dots,(b_q)_n \Gamma(\lambda n+\xi+1)} \times \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}(t^{\rho+\upsilon n+\upsilon\xi-1})\right)(x)\right\} dz \qquad (3.17)$$

Applying (2.8) with $\rho = (\rho + \upsilon n + \upsilon\xi)$, we get

$$\mathcal{I} = \int_0^1 z^{l-1}(1-z)^{m-1} \{x^{A+\alpha+\alpha'-\gamma-1} \times \sum_{n=0}^{\infty} \frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n \Gamma(n+1) z^{n+\xi}(ax^{\upsilon})^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n \Gamma(\lambda n+\xi+1) n!}$$

$$\times \left[\begin{matrix}\Gamma(A+\upsilon n)\ \Gamma(A-\gamma+\alpha+\alpha'+\beta'+\upsilon n)\Gamma(A-\beta+\alpha+\upsilon n)\\ \Gamma(A-\beta+\upsilon n)\ \Gamma(A-\gamma+\alpha+\alpha'+\upsilon n)\Gamma(A-\gamma+\beta'+\alpha+\upsilon n)\end{matrix}\right]\Bigg\} dz$$

where $A = \rho + \upsilon\xi$

Interchanging the order of integration and summation we get

$$\mathcal{I} = x^{A+\alpha+\alpha'-\gamma-1} \sum_{n=0}^{\infty} \frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n \Gamma(n+1)\ (ax^{\upsilon})^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n \Gamma(\lambda n+\xi+1) n!}$$

$$\left[\begin{matrix}\Gamma(A+\upsilon n)\ \Gamma(A-\gamma+\alpha+\alpha'+\beta'+\upsilon n),\Gamma(A-\beta+\alpha+\upsilon n)\\ \Gamma(A-\beta+\upsilon n)\,\Gamma(A-\gamma+\alpha+\alpha'+\upsilon n),\Gamma(A-\gamma+\beta'+\alpha+\upsilon n)\end{matrix}\right]$$

$$\times \int_0^1 z^{l+n+\xi-1}(1-z)^{m-1} dz$$

$$\mathcal{I} = x^{A+\alpha+\alpha'-\gamma-1} \sum_{n=0}^{\infty} \frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n \Gamma(n+1)\ (ax^{\upsilon})^n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n \Gamma(\lambda n+\xi+1) n!}$$

$$\left[\begin{matrix}\Gamma(A+\upsilon n)\ \Gamma(A-\gamma+\alpha+\alpha'+\beta'+\upsilon n),\Gamma(A-\beta+\alpha+\upsilon n)\\ \Gamma(A-\beta+\upsilon n)\,\Gamma(A-\gamma+\alpha+\alpha'+\upsilon n),\Gamma(A-\gamma+\beta'+\alpha+\upsilon n)\end{matrix}\right]$$

$$\times \frac{\Gamma(l+n+\xi)\Gamma(m)}{\Gamma(l+m+n+\xi)} \qquad (3.18)$$

Interpreting the right hand side of (3.18) in view of (2.10) we arrive at the required result.

Theorem 3.8. Let $\Re(\xi) > 0, \Re(\upsilon) > 0, \Re(\gamma) > 0$, $\Re(\lambda) > 0$, be such that $\Re(1-\gamma-\rho-\upsilon\xi) < 1 + \min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}$. Then

$$B\left\{\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\gamma-\rho}\ {}_0^{a}N_{p,q}^{\lambda,\xi}\ (zt^{-\upsilon})\right)\right)(x): l, m\right\}$$

$$= x^{-A+\alpha+\alpha'}\Gamma(m) \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$${}_{p+5}\psi_{q+5}\left[\begin{matrix}(a_1,1),\dots,(a_p,1),(A-\alpha-\alpha',\upsilon),(A-\beta-\alpha',\upsilon)\\ (b_1,1),\dots(b_q,1),(A-\alpha-\alpha'-\beta,\upsilon),(A-\gamma,\upsilon)\end{matrix}\right.$$

$$\left.\begin{matrix}(A+\beta'-\gamma,\upsilon),(l+\xi,1),(1,1)\\ (A-\alpha'+\beta'-\gamma,\upsilon),(\xi+1,\lambda)(l+m+\xi,1)\end{matrix}; ax^{-\upsilon}\right] \qquad (3.19)$$

where $A = \rho + \upsilon\xi$

Proof. The proof of the theorem can be established by following the same steps as that of the above theorem. So we omit the details here.

Image formulas related to Whittaker transform

Theorem 3.9. Let $\Re(\xi) > 0, \Re(\upsilon) > 0, \Re(\gamma) > 0$, $\Re(\lambda) > 0, \Re(\delta \pm \eta) > -\frac{1}{2}$ be such that $\Re(\rho+\upsilon\xi) > max\{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\}$ is satisfied, then the following Whittaker transform formula holds:

$$\int_0^{\infty} z^{\delta-1}e^{-\frac{z}{2}}\left[W_{\sigma,\eta}\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{\rho-1}\ {}_0^{a}N_{p,q}^{\lambda,\xi}\ (zt^{\upsilon})\right)(x)\right] dz$$

$$= x^{A-\alpha-\alpha'+\gamma-1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$${}_{p+6}\psi_{q+5}\left[\begin{matrix}(a_1,1),\dots,(a_p,1),(A,\upsilon),(A+\beta'-\alpha',\upsilon),(1,1)\\ (b_1,1),\dots(b_q,1),,(A+\beta',\upsilon),(A+\gamma-\alpha-\alpha',\upsilon)\end{matrix}\right.$$

$$\left.\begin{matrix}(A+\gamma-\alpha-\alpha'-\beta,\upsilon),(B+\eta,1),(B-\eta,1)\\ (A+\gamma-\beta-\alpha',\upsilon),(\xi+1,\lambda),(B-\sigma,1)\end{matrix}; ax^{\upsilon}\right] \qquad (3.20)$$

where $A = \rho + \upsilon\xi$ and $B = \delta + \xi + \frac{1}{2}$

Proof. Taking the LHS of (3.20) as $\mathcal{I}$, and using (2.15) we get

$$\mathcal{I} = \int_0^{\infty} z^{\delta-1}e^{-\frac{z}{2}}\left[W_{\sigma,\eta}\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{\rho-1} \sum_{n=0}^{\infty} \frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n a^n z^{n+\xi}}{(b_1)_n,(b_2)_n,\dots,(b_q)_n \Gamma(\lambda n+\xi+1)}(t^{\upsilon})^{n+\xi}\right)(x)\right] dz \qquad (3.21)$$

Applying (3.2) in (3.21) we get

$$\mathcal{I} = \int_0^\infty z^{\delta+n+\xi-1} e^{-\frac{z}{2}} W_{\sigma,\eta}\{x^{A-\alpha-\alpha'+\gamma-1}$$
$$\times \sum_{n=0}^\infty \frac{(a_1)_n,(a_2)_n,\ldots,(a_p)_n \Gamma(n+1)}{(b_1)_n,(b_2)_n,\ldots,(b_q)_n \Gamma(\lambda n+\xi+1)} \frac{(ax^\upsilon)^n}{n!}$$
$$\times \left[\frac{\Gamma(A+\upsilon n)\ \Gamma(A+\gamma-\alpha-\alpha'-\beta+\upsilon n)\ \Gamma(A+\beta'-\alpha'+\upsilon n)}{\Gamma(A+\beta'+\upsilon n)\ \Gamma(A+\gamma-\alpha-\alpha'+\upsilon n)\ \Gamma(A+\gamma-\beta-\alpha'+\upsilon n)}\right]\Big\} dz$$

where $A=\rho+\upsilon\xi$

Interchanging the order of integration and summation and applying (2.17) we get

$$\mathcal{I} = x^{A-\alpha-\alpha'+\gamma-1} \sum_{n=0}^\infty \frac{(a_1)_n,(a_2)_n,\ldots,(a_p)_n \Gamma(n+1)}{(b_1)_n,(b_2)_n,\ldots,(b_q)_n \Gamma(\lambda n+\xi+1)} \frac{(ax^\upsilon)^n}{n!}$$
$$\times \left[\frac{\Gamma(A+\upsilon n)\ \Gamma(A+\gamma-\alpha-\alpha'-\beta+\upsilon n)\ \Gamma(A+\beta'-\alpha'+\upsilon n)\ \Gamma(B+\eta+n)\Gamma(B-\eta+n)}{\Gamma(A+\beta'+\upsilon n)\ \Gamma(A+\gamma-\alpha-\alpha'+\upsilon n)\ \Gamma(A+\gamma-\beta-\alpha'+\upsilon n)\Gamma(B-\sigma+n)}\right] \tag{3.22}$$

where $A=\rho+\upsilon\xi$ and $B=\delta+\xi+\frac{1}{2}$

Interpreting the right hand side of (3.22) in view of (2.10) we arrive at the required result.

Theorem 3.10. Let $\Re(\xi)>0, \Re(\upsilon)>0, \Re(\gamma)>0$, $\Re(\lambda)>0, \Re(\delta\pm\eta)>-\frac{1}{2}$ be such that $\Re(1-\gamma-\rho-\upsilon\xi) < 1 + min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}$ is satisfied, then the following Whittaker transform formula holds:

$$\int_0^\infty z^{\delta-1} e^{-\frac{z}{2}} \left[W_{\sigma,\eta}\left(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{-\gamma-\rho}\ {}_0^aN_{p,q}^{\lambda,\xi}\ (zt^{-\upsilon})\right)(x)\right] dz$$
$$= x^{-A-\alpha-\alpha'} \sum_{n=0}^\infty \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{i=1}^p \Gamma(a_i)}$$
$$\times\ {}_{p+6}\psi_{q+5}\left[\begin{matrix}(a_1,1),\ldots(a_p,1),(A+\alpha+\beta',\upsilon),(A+\alpha+\alpha',\upsilon)\\(b_1,1),\ldots(b_q,1),(A+\alpha+\alpha'+\beta',\upsilon),(A+\gamma,\upsilon)\end{matrix}\right.$$
$$\left.\begin{matrix}(A-\beta+\gamma,\upsilon),(1,1),(B+\eta,1),(B-\eta,1)\\(A+\alpha-\beta+\gamma,\upsilon),(\xi+1,\lambda),(B-\sigma,1)\end{matrix}; ax^{-\upsilon}\right] \tag{3.23}$$

where $A=\rho+\upsilon\xi$ and $B=\delta+\xi+\frac{1}{2}$

Proof. The proof of the theorem can be established by following the same steps as that of the above theorem. So we omit the details here.

Theorem 3.11. Let $\Re(\xi)>0, \Re(\upsilon)>0, \Re(\gamma)>0$, $\Re(\lambda)>0, \Re(\delta\pm\eta)>-\frac{1}{2}$ be such that $\Re(\rho+\upsilon\xi) > max\{0, \Re(\gamma-\alpha-\alpha'-\beta'), \Re(\beta-\alpha)\}$ is satisfied then the following Whittaker transform formula holds:

$$\int_0^\infty z^{\delta-1} e^{-\frac{z}{2}} \left[W_{\sigma,\eta}\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{\rho-1}\ {}_0^aN_{p,q}^{\lambda,\xi}\ (zt^{\upsilon})\right)(x)\right] dz$$
$$= x^{A+\alpha+\alpha'-\gamma-1} \sum_{n=0}^\infty \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{i=1}^p \Gamma(a_i)}$$
$$\times\ {}_{p+6}\psi_{q+5}\left[\begin{matrix}(a_1,1),\ldots(a_p,1),(A,\upsilon),(A-\gamma+\alpha+\alpha'+\beta',\upsilon)\\(b_1,1),\ldots(b_q,1),(A-\beta,\upsilon),(A-\gamma+\alpha+\alpha',\upsilon)\end{matrix}\right.$$
$$\left.\begin{matrix}(A-\beta+\alpha,\upsilon),(B+\eta,1),(B-\eta,1),(1,1)\\(A-\gamma+\beta'+\alpha,\upsilon),(\xi+1,\lambda),(B-\sigma,1)\end{matrix}; ax^{\upsilon}\right] \tag{3.24}$$

where $A=\rho+\upsilon\xi$ and $B=\delta+\xi+\frac{1}{2}$

Proof. Taking the LHS of (3.24) as $\mathcal{I}$, and using (2.15) we get

$$\mathcal{I} = \int_0^\infty z^{\delta-1} e^{-\frac{z}{2}} \Big[W_{\sigma,\eta}\Big(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{\rho-1}$$
$$\sum_{n=0}^\infty \frac{(a_1)_n,(a_2)_n,\ldots,(a_p)_n a^n z^{n+\xi}}{(b_1)_n,(b_2)_n,\ldots,(b_q)_n \Gamma(\lambda n+\xi+1)} (t^\upsilon)^{n+\xi}\Big)(x)\Big] dz \tag{3.25}$$

Applying (3.6) in (3.25) we get

$$\mathcal{I} = \int_0^\infty z^{\delta+n+\xi-1} e^{-\frac{z}{2}} W_{\sigma,\eta}\{x^{A-\alpha-\alpha'+\gamma-1}$$
$$\times \sum_{n=0}^\infty \frac{(a_1)_n,(a_2)_n,\ldots,(a_p)_n \Gamma(n+1)}{(b_1)_n,(b_2)_n,\ldots,(b_q)_n \Gamma(\lambda n+\xi+1)} \frac{(ax^\upsilon)^n}{n!}$$
$$\left[\frac{\Gamma(A+\upsilon n)\ \Gamma(A-\gamma+\alpha+\alpha'+\beta'+\upsilon n), \Gamma(A-\beta+\alpha+\upsilon n)}{\Gamma(A-\beta+\upsilon n)\ \Gamma(A-\gamma+\alpha+\alpha'+\upsilon n), \Gamma(A-\gamma+\beta'+\alpha+\upsilon n)}\right]\Big\} dz$$

where $A=\rho+\upsilon\xi$

Interchanging the order of integration and summation and applying (2.17) we get

$$\mathcal{I} = x^{A+\alpha+\alpha'-\gamma-1} \sum_{n=0}^\infty \frac{(a_1)_n,(a_2)_n,\ldots,(a_p)_n \Gamma(n+1)}{(b_1)_n,(b_2)_n,\ldots,(b_q)_n \Gamma(\lambda n+\xi+1)} \frac{(ax^\upsilon)^n}{n!}$$
$$\times \left[\frac{\Gamma(A+\upsilon n)\ \Gamma(A-\gamma+\alpha+\alpha'+\beta'+\upsilon n)}{\Gamma(A-\beta+\upsilon n)\ \Gamma(A-\gamma+\alpha+\alpha'+\upsilon n)}\right.$$
$$\left.\frac{\Gamma(A-\beta+\alpha+\upsilon n)\Gamma(B+\eta+n)\Gamma(B-\eta+n)}{\Gamma(A-\gamma+\beta'+\alpha+\upsilon n)\Gamma(B-\sigma+n)}\right] \tag{3.26}$$

where $A=\rho+\upsilon\xi$ and $B=\delta+\xi+\frac{1}{2}$

Interpreting the right hand side of (3.26) in view of (2.10) we arrive at the required result.

Theorem 3.12. Let $\Re(\xi)>0, \Re(\upsilon)>0, \Re(\gamma)>0$, $\Re(\lambda)>0, \Re(\delta\pm\eta)>-\frac{1}{2}$ be such that $\Re(1-\gamma-\rho-\upsilon\xi) < 1 + \min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}$ is satisfied then the following Whittaker transform formula holds:

$$\int_0^\infty z^{\delta-1} e^{-\frac{z}{2}} \left[W_{\sigma,\eta}\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}\, t^{\gamma-\rho}\, {}_0^{\alpha}N_{p,q}^{\lambda,\xi}\,(zt^{-\upsilon})\right)(x)\right] dz$$

$$= x^{-A+\alpha+\alpha'} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)}$$

$$\times\, {}_{p+6}\psi_{q+5}\left[\begin{matrix}(a_1,1),\dots(a_p,1),(A-\alpha-\alpha',\upsilon),(A-\beta-\alpha',\upsilon)\\ (b_1,1),\dots(b_q,1),(A-\alpha-\alpha'-\beta,\upsilon),(A-\gamma,\upsilon)\end{matrix}\right.$$

$$\left.\begin{matrix}(A+\beta'-\gamma,\upsilon),(1,1),(B+\eta,1),(B-\eta,1)\\ (A-\alpha'+\beta'-\gamma,\upsilon),(\xi+1,\lambda),(B-\sigma,1)\end{matrix};ax^{-\upsilon}\right] \quad (3.27)$$

where $A=\rho+\upsilon\xi$ and $B=\delta+\xi+\frac{1}{2}$

Proof. The proof of the theorem can be established by following the same steps as that of the above theorem. So we omit the details here.

IV. CONCLUSION

In this paper we presented the different generalized theorems associated with the generalized fractional operators given by Marichev-Saigo Maeda. We implement the generalized fractional operators induced by Saego-Maeda using the Appell's $F_3(.)$ function and establish the image formulas associated with the generalized Miller-Ross function in terms of the generalized Fox- Wright function. The main fractional operators operated on the generalized Miller-Ross function given in section III, are quite general in nature and can be specialized to yield the large number of simpler and different results. The main results may find the large number of applications in various fields of science and engineering.

REFERENCES

[1] D. Baleanu, O. G. Mustafa, "On the Global Existence of Solutions to a Class of Fractional Differential Equations", Com. Math. Appl, Vol. 59 pp.1835-1841. 2010

[2] D. Baleanu, P. Agarwal, S. D. Purohit, "Certain fractional integral formulas involving the product of generalized Bessel functions", Sci. World J. pp.1-9, 2013

[3] A.A. Kilbas, "Fractional Calculus of the Generalized Wright Function", Fract. Calc. Appl. Anal, vol.8 No.2 pp.113-126, 2005

[4] Miller, K. And Ross, B., "An introduction to the Fractional Calculus and Fractional differential Equations", John Wiley Sons, New York,1993.

[5] Saigo M., "A Remark on Integral Operators Involving the Gauss Hyper-geometric Functions" Math. Rep. College General Ed. Kyushu Univ., Vol. 11, pp. 135-143, 1978

[6] G.M. Mittag-Leffler, "Sur la representation analytique de'une branche Uniforme une function monogene", Acta. Math., 29, pp.101-181. 1905

[7] T.R. Prabhakar, "A Singular Integral Equation with a Generalized Mittag-Leffler Function in the Kernel", Yokohama Math. J., vol 19, pp.7-15, 1971

[8] D. Kumar, S. D. Purohit, J. Choi, "Generalized fractional integrals involving product of multivariable H-function and a general class of polynomials", J. Nonlinear Sci. Appl, 9 821, 2016

[9] J. Ram, D. Kumar, "Generalized fractional integration involving Appell hyper-geometric of the product of two H-functions", Vijnana Parishad Anusandhan Patrika, Vol. 54,No.3 pp. 33-43, 2011

[10] J. Zhao, "Positive solutions for a class of q-fractional boundary value problems with p-Laplacian", J. Nonlinear Sci.Appl., 8 , 442-450.1, 2015

[11] P. Agarwal, J. Choi, R. B. Paris, Extended Riemann-Liouville fractional derivative operator and its applications. Nonlinear Sci. Appl. Vol. 8 ,pp. 451-466, 2015

[12] R. K. Saxena, J. Ram, D. Kumar, "Generalized fractional integration of the product of Bessel functions of the First kind", Proceedings of the 9th Annual Conference, Soc. Spec. Funct. Appl., vol.9 , pp.15-27, 2011

[13] S. D. Purohit, S. L. Kalla, D. L. Suthar, "Fractional integral operators and the multi index Mittag-Leffler functions", Sci. Ser. A Math. Sci. ,vol. 21,pp. 87-96, 2011

[14] S. D. Purohit, D. L. Suthar, S. L. Kalla, "Marichev-Saigo-Maeda fractional integration operators of the Bessel function", Matematiche (Catania), 67, pages. 21-32, 2012

[15] S. R. Mondal, K. S. Nisar, "Marichev-Saigo-Maeda fractional integration operators involving generalized Bessel function's", Math. Probl. Eng., vol. 11 pages.1-9, 2014

[16] Javid Majid, Imtiyaz, Aarif and Jain, "Certain image formulas associated with the fractional integrals and derivatives of the product of Srivstava polynomials and the K-function", IJRAR ,Vol. 5,issue 4,pages 32-47,2018

[17] Shweta Pandey, Sandeep Dixit, 'Solution of System of Fractional Differential equations using Variational Iteration Method", International Journal of Computer Sciences and engineering, Vol. 6, Issue 8,pp. 797-802,2018.

[18] Saigo M., "A Remark on Integral Operators Involving the Gauss Hyper-geometric Functions" Math. Rep. College General Ed. Kyushu Univ., Vol. 11, pp. 135-143, 1978

[19] Appell, P. and Kampe de Feriet, J., "Functions Hyper-geometriqueset Hyperspheriques Polynomes d'Hermite" . Gauthier-Viliars, Paris. 1926

[20] Wright E. M., "The asymptotic expansion of the generalized hyper-geometric functions" J. London Math. Soc., Vol. 10, pp. 286-293, 1935

[21] Rainville E. D., "Special Functions", the Macmillan Company, New York, 1963

[22] Farhan, Sharma and Jain, "Generalized Miller-Ross Function and its Integral Representation", International Journal of Mathematics Trends and Technology- Vol. 16 Number 1, pp. 16-18,2014

[23] Mathai AM, Saxena RK, Haubold HJ. "The H-Function Theory and Applications" New York, NY: Springer-Verlag 2010

[24] I.N.Sneddon, "The use of integral transforms", Tata McGraw-Hill, New Delhi, 1979

AUTHORS PROFILE

Mr. Javid Majid is a research scholar who has pursued his B.Sc and B.Ed degrees from the University of Kashmir, M.Sc. Mathematics from Jiwaji University, Gwalior (M.P). He has also pursued M.Phil Mathematics under the Supervision of Professor Renu Jain from Jiwaji University, Gwalior (M.P.), India.

Mr. Imtiyaz Majid is a research scholar who has pursued his B.Sc degree from university of Kashmir, M.Sc. from Barkatullah University, Bhopal (M.P).He has also pursued B.Ed & M.Ed from university of Kashmir.