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A Mathematical Model on Chain of Competitions in Different Trophic Levels

Nabajyoti Bhattacharjee

Department of Mathematics, Assam University, Silchar

Corresponding Author: bhattacharjeenabajyoti@gmail.com, Tel.: +91-9706276296

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Abstract—In this paper we shall consider a sequence of trophic levels and obtain a recurrence for the total biomass production in each trophic level consisting of exactly two species competing for resource in its immediate lower trophic level and prove that if the consumption factor of both the species in each trophic level is same and the consumption factor at each trophic level forms a strictly decreasing sequence and all are greater than one, then biomass production at each trophic level forms a strictly decreasing sequence of total biomass production starting from lowest trophic level.

Keywords— *Mathematical modelling, Competition, Trophic level, Superiority factor, chain of competition.*

I. INTRODUCTION

In an ecosystem individual interact with each other and these interactions depend upon climatic conditions, amount of resources present in the given region, population density of both the species, ability to consume resources and reproduce. Competition is one of such interaction in which both the competing species have a negative impact on their population size. Many research work have been done in this area to study competition between two related and unrelated species some of the work we can find in [1], [2], [3], [4]. We often observe that, in a natural ecosystem, the biomass production of one trophic level decides the production of biomass in higher trophic level follow the 10 percent law (for details see [5]) and due to this lack of energy obtained from food creates a natural negative effect on reproduction which ultimately results in the declination of population size in the higher trophic levels. Further competition both inter- and intra-specific between the individuals plays an important role in the rate of competition. In this paper, we shall consider a finite sequence of trophic levels each consists of two different species except the lowest trophic level which consists resources available for the immediate higher trophic level and individuals of each trophic level competing with each other to consume resources available from its immediately lower trophic level. Thus, we get an open chain consisting of finite knots representing trophic level and in each trophic level species competing for the resources available in immediate lower trophic level except the last one, in other words we get a chain of competition. Finally we shall prove that population size of trophic levels forms a strictly decreasing sequence starting from first trophic level. Due to mathematical constraints and lack of data we shall assume the following so as to establish a mathematical model to describe the above situation. Throughout the paper total biomass production and total resources available in a given trophic level are considered to be same

II. ASSUMPTIONS

- Competition between individuals in a given trophic level implies both inter-specific and intra-specific competition.
- Climatic effect on competition is neglected.
- A resource partitioning and other adjustment between the individuals in a given trophic level is not allowed.
- There are exactly two species in each of the trophic level competing for resources from the immediate lower trophic level in a food chain.

III. MATHEMATICAL MODEL FOR CHAIN COMPETITION

Let us consider a terrestrial ecosystem covering a geographical region R and let $S_0, S_1, \dots, S_k$ be $k+1$ trophic levels in the food chain in a order, $S_0 < S_1 < \cdots < S_k$. It is to be noted that S_0 is the lowest trophic level and S_k is the highest trophic level. Let A_{1j} and A_{2j} be two species in the trophic level S_j and competing for food in S_{j-1} for all $j = 1,2,\dots,k$ we define $|S_0| = |R|$ the total resource available in the region R for the population in the trophic level S_1. Let x_j and y_j be the population density of A_{1j} and A_{2j} at any time t. Let μ_{1j} and μ_{2j} be the consumption factor of both the species in the trophic level S_j that is the amount of resource consumed by an individual of a given species in unit time.

Superiority Factor: Let T be the period of competition. Then superiority factor between two species A_{1j} and A_{2j} in a trophic level S_j is defined to be the ratio

$$\eta_j = \frac{\mu_{1j} \int_0^T x_j(t)dt}{\mu_{2j} \int_0^T y_j(t)dt}$$

Relation: Let $|S_{j-1}|$ be the total resource available for the individuals of trophic level S_{j-1} then from the relations in we have the following,

a) $\int_0^T x_j(t)dt \leq \frac{|S_{j-1}|}{\mu_{1j}T(\eta_j+1)}$
b) $\int_0^T y_j(t)dt \leq \frac{\eta_j|S_{j-1}|}{\mu_{2j}T(\eta_j+1)}$
c) $Total\ population\ of\ S_j \leq \frac{|S_{j-1}|(\mu_{1j}+\eta_j\mu_{2j})}{\mu_{1j}\mu_{2j}T(\eta_j+1)}$
d) $If\ \mu_{1j} = \mu_{2j} = \mu_j\ then$

$$Total\ population\ of\ S_j \leq \frac{|S_{j-1}|}{\mu_j T}$$

Let $|S_0|, |S_1|, |S_2|, \dots, |S_{k-1}|$ be the total resource available for the consumption by the individuals of the immediate higher trophic levels $S_1, S_2, S_3, \dots, S_k$ respectively. Let η_j be the superiority factor between two species A_{1j} and A_{2j} in the trophic level S_1 cannot exceed the limit $\frac{|S_0|(\mu_{11}+\eta_1\mu_{21})}{\mu_{11}\mu_{21}T(\eta_1+1)}$, where T is the period of competition that is the period for which we analyse the effect of competition.

$$|S_1| \leq \frac{|S_0|(\mu_{11}+\eta_1\mu_{21})}{\mu_{11}\mu_{21}T(\eta_1+1)}$$

But the total amount of consumable resources available for the individuals belonging to trophic level S_2 is $|S_1|$ then by the same argument we have

$$|S_2| \leq \frac{|S_1|(\mu_{12}+\eta_2\mu_{22})}{\mu_{12}\mu_{22}T(\eta_2+1)} \leq \frac{|S_0|(\mu_{11}+\eta_1\mu_{21})(\mu_{12}+\eta_2\mu_{22})}{\mu_{11}\mu_{12}\mu_{21}\mu_{22}T^2(\eta_1+1)(\eta_2+1)}$$

In general at any tropic level S_n the amount of consumable resource available for its immediate higher trophic level S_n is given by;

$$(1) \dots \quad |S_n| \leq \frac{|S_{n-1}|(\mu_{1n}+\eta_n\mu_{2n})}{\mu_{1n}\mu_{2n}T(\eta_n+1)}$$

$$(2) \dots \quad |S_n| \leq \frac{|S_0|(\mu_{11}+\eta_1\mu_{21}) \dots \dots (\mu_{1n}+\eta_n\mu_{2n})}{\mu_{11} \dots \mu_{1n}\mu_{21} \dots \mu_{2n}T^n(\eta_1+1) \dots (\eta_n+1)}$$

For all n, $1 \leq n \leq k$

Thus, the total resource at each trophic level depends on the consumption and superiority factor and any mild fluctuation in the level of competition at any trophic level will induces a natural disturbance in the resources in the higher trophic levels. Further for a given rate of consumption and superiority factor, we can observe that the biomass production of S_n is less than the total biomass production of the nature $|S_0|$ and it tends to zero as period of competition T tends to infinity which results in the diminishing of species from the higher trophic level. Thus, in nature there must be a few trophic level can be possible in any food chain or food web for a long run of competition and total population density reduces as move high in the trophic level. If the consumption factor of both the species in a given trophic level is same then the total biomass production in a given trophic level is independent of superiority factor. As we move to higher trophic level biomass production starts decreasing given that consumption factor in the lowest trophic level is the highest. We shall express the situation by a result stated below;

IV. MAIN RESULT

RESULT: ***If the competition factor*** $\boldsymbol{\mu_j = \mu_{1j} = \mu_{2j}}$ ***for all*** $\boldsymbol{1 \leq j \leq k}$ ***and*** $\boldsymbol{\mu_1 > \mu_2 > \dots > \mu_k > 1}$ ***then*** $\boldsymbol{|S_0| > |S_1| > \dots > |S_k|}$ ***for sufficiently large period of competition.***

To establish the main result we shall prove some sub-results

CLAIM I: ***If*** $\boldsymbol{\mu_{1j} = \mu_{2j} = \mu_j}$ ***for all*** $\boldsymbol{1 \leq j \leq k}$ ***be the consumption factor of both the species in each trophic level then,***

$$\boldsymbol{TP(A_{1j}) \leq \frac{|S_0|}{\mu_1\mu_2 \dots \mu_j T^j(\eta_j+1)}} \text{ and } \boldsymbol{TP(A_{2j}) \leq \frac{\eta_j|S_0|}{\mu_1\mu_2 \dots \mu_j T^j(\eta_j+1)}},$$

where $\boldsymbol{TP(A_{ij})}$ ***is the total population of*** $\boldsymbol{A_{ij}}$ ***in*** $\boldsymbol{S_j}$ ***for*** $\boldsymbol{i = 1, 2}$

Proof: Let $\mu_{1j} = \mu_{2j} = \mu_j$ for all $1 \leq j \leq k$ be the consumption factor of both the species in each trophic level. Then, recurrence formula (2) reduces to;

$$(3) \dots \quad |S_n| \leq \frac{|S_0|}{\mu_1\mu_2 \dots \mu_n T^n}\ for\ all\ n, 1 \leq n \leq k$$

Let η_j be the superiority factor between the species A_{1j} and A_{2j} for all j, $1 \leq j \leq k$ then from the relation (a) and (b) we have;

$$(4) \dots \quad \begin{cases} TP(A_{1j}) \leq \dfrac{\eta_j|S_{j-1}|}{\mu_j T(\eta_j+1)} \\ TP(A_{2j}) \leq \dfrac{|S_{j-1}|}{\mu_j T(\eta_j+1)} \end{cases}$$

Putting $n = j - 1$ in recurrence relation (3) we get;

$$(5) \ldots \qquad |S_{j-1}| \le \frac{|S_0|}{\mu_1\mu_2 \ldots \mu_{j-1}T^{j-1}}$$

Using relation (5) in inequality (4) we get;

$$\begin{cases} TP(A_{1j}) \le \dfrac{\eta_j|S_0|}{\mu_1\mu_2 \ldots \mu_j T^j(\eta_j+1)} \\ TP(A_{2j}) \le \dfrac{|S_0|}{\mu_1\mu_2 \ldots \mu_j T^j(\eta_j+1)} \end{cases}$$

Hence Proved

CLAIM II: ***If the consumption factor $\mu_j = \mu_{1j} = \mu_{2j}$ and η_j be the superiority factor for all $1 \le j \le k$ such that μ_{j+1} for some j then $|S_p| = 0$ for every $j \le p \le k$ as $T \longrightarrow \infty$.***

Proof: Let $\mu_j = \mu_{1j} = \mu_{2j}$ and superiority factor η_j for all $1 \le j \le k$ be given. For a fix j , $1 \le j \le k$, $|S_j| = TP(A_{1j}) + TP(A_{2j})$. Also $\mu_j \ge 1$ and $\eta_j \ge 1$ for all $j, 1 \le j \le k$.

$$\frac{1}{\mu_1\mu_2 \ldots \mu_j T^j} \ge \frac{1}{\mu_1\mu_2 \ldots \mu_{j+1}T^{j+1}}$$

$$(6) \ldots \qquad \frac{|S_0|}{\mu_1\mu_2 \ldots \mu_j T^j} \ge \frac{|S_0|}{\mu_1\mu_2 \ldots \mu_{j+1}T^{j+1}}$$

By CLAIM I, we see that

$$|S_j| = TP(A_{1j}) + TP(A_{2j}) \le \frac{|S_0|}{\mu_1\mu_2 \ldots \mu_j T^j}$$

From inequality (5) we have;

$$\frac{|S_0|}{\mu_1\mu_2 \ldots \mu_j T^j} \ge |S_{j+1}|$$

Then,

$$\frac{|S_0|}{\mu_1\mu_2 \ldots \mu_j T^j} \ge \max\{|S_j|, |S_{j+1}|\}$$

Now taking limit as $T \longrightarrow \infty$ we see that $\max\{|S_j|, |S_{j+1}|\} = 0$, it follows that, $|S_j| = |S_{j+1}| = 0$ and from inequality (1) we have $|S_p| = 0$ for every $j \le p \le k$.

CLAIM III: ***If the competition factor $\mu_j = \mu_{1j} = \mu_{2j}$ and η_j be the superiority factor for all $1 \le j \le k$ such that $\mu_{j+1} \ge \mu_{j+2} \ge \cdots \ge \mu_k > 1$ for some j then $|S_j| > |S_p|$ for all $j+1 \le p \le k$ for sufficiently large T.***

Proof: Let the competition factor $\mu_j = \mu_{1j} = \mu_{2j}$ and η_j be the superiority factor for all $1 \le j \le k$ such that $\mu_p > 1$ for all $j+1 \le p \le k$ and for some j then by CLAIM II, for every positive number $\Delta > 0$ there exist T such that $|S_p| < \Delta$ for all $j+1 \le p \le k$. Also by inequality (1) and using $\mu_p > 1$ for all $j+1 \le p \le k$ we see that,

$$(7) \ldots \qquad |S_p| \le \frac{|S_{p-1}|}{\mu_p T} < \frac{\Delta}{\mu_p T} < \Delta$$

Let $\Delta = \frac{1}{2}|S_j|$ then for given $\Delta > 0$ there exist T such that $|S_{j+1}| < \Delta$ and using inequality (7) we see that $|S_p| < \Delta$ for all $j+1 \le p \le k$ it follows that $|S_j| > |S_p|$ for all $j+1 \le p \le k$.

Proof of the RESULT directly follows from CLAIM III.

V. CONCLUSION

From the above result, we can observe that if the consumption factor of both the species for each trophic level S_j is same μ_j say then instead of having higher consumption factor in the lower trophic level, to maintain the balance in the food chain, the biomass production in a given trophic level is less than its immediately lower trophic level whenever competition of resources occurs for a sufficiently large period. It is to be noted that, population density of highest trophic level is the least among all the preceding trophic levels. Further, for a given trophic level consumption factor of both the species are considered to be same as a result the total biomass production in a given trophic level is independent of the superiority factor which is quite obvious, as in this case the superiority factor depends only on population size of each species i.e., consumption factor does not affect the population size of a species in a given trophic level.

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