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Common Fixed Point Theorems in Complete Hausdörff Uniform Spaces

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Abstract- Some common fixed points of point-valued mappings on a sequentially complete Hausdorff uniform space satisfying contractive type conditions have been obtained.

Keywords: Fixed Point, Sequentially complete, Hausdorff Uniform Space.

I. INTRODUCTION

A large number of literatures are available which deal with fixed and common fixed points of point-valued mappings in metric spaces, Banach spaces, Hilbert spaces, etc. Very few literatures are available which deal with common fixed points of point-valued mappings in Uniform Hausdorff spaces. In this paper attempt has been made to obtain some common fixed point theorems in sequentially complete Hausdorff Uniform spaces in such a way that they generalize some theorems which have already been proved in metric spaces.

Let (X, U) is a uniform space.

A family $\{d_\lambda : \lambda \in \Gamma, \Gamma$ is an index set$\}$ of pseudometrices is called an associated family for the uniformity u if the family $\beta = \{V(I,r) : i \in \Gamma, r>0\}$, where $V(i,r) = \{(x,y) : x,y \in X, d(x,y)<r\}$ is a sub-base for the uniformity u.

A family $\{d_\lambda\}, \lambda \in \Gamma, \Gamma$ is an index set of pseudometrices on X is called an augumented associated family for u if $\{d_{:\lambda \in \Gamma}\}$ is an associated family for u and has the additional property :

Given $a,b \in \Gamma$, there is a $v \in \Gamma$ such that

$$d_v(x,y) \geq \max\{d_a(x,y), d_b(x,y)\}.$$

An associated family and an augumented family will be denoted by P and P* respectively. For details one can see Kelley [2], Thorn [5], etc.

II. SOME DEFINITIONS

Let S and T be self-mapping of a sequentially complete Hausdorff uniform space (X, U) defined by $\{d_\lambda : \lambda \in F\} = P^*$. S and T are said to be weakly commutative on X if

$$d_\lambda(STx, TSx) \leq d_\lambda(Tx, Sx)$$

for all $x \in X$ and $\lambda \in \Gamma$.

Further S and T are said to be compatible if $\lim d_\lambda(STx_n, TSx_n) = 0$ for all $\lambda \in \Gamma$ whenever $\{x_n\}$ is a sequence in X and that $\lim d_\lambda(Sx_n, t) = \lim d_\lambda(Tx_n, t)$ for all $\lambda \in F$ and for some $t \in \lambda$.

The above two definitions are analogous to the definitioins as introduced by Sessa [4] and Jungck [1] in metric spaces while proving some common fixed point theorems in metric spaces. It is to be noted that weakly commutative mappings are compatible but the converse is not true [3].

Let $w : [0,\infty) \to [0,\infty)$ be such that w is continuous and $0<w(r) < r$ for $r>0$.

We now prove the following theorems :

Theorem 1. Let (X, U) be a sequentially complete Hausdorff uniform space defined by $\{d_\lambda : \lambda \in \Gamma\} = P^*$. Let f, g, h and J be four mappings on X satisfying

$$d_\lambda(fx, gy) \leq \max\{d_\lambda(Jx, hy), d_\lambda(fx, gx), d_\lambda(gy, hy),$$

$\frac{d_\lambda(Jx, gy) + d_\lambda(hy, fx)}{2}\}$ – w [max {d_λ (Jx,hy), d_λ (fx,gx), d_λ (gy,hy),

$\frac{d_\lambda(Jx, gy) + d_\lambda(hy, fx)}{2}\}$ for all $x,y \in X$, $\lambda \in \Gamma$.

Let h and J be continuous h and g be compatible and f and J be compatible. If $f(X) \subset h(X)$ and $g(X) \subset J(X)$ then f,g,h and J have a unique common fixed point in X.

Proof. Suppose $f(X) \subset h(X)$ and $g(X) \subset J(X)$.

Let $x_0 \in X$. Then there exists a sequence $\{x_n\}$ such that

$y_{2n} = fx_{2n} = hx_{2n+1}$ and

$y_{2n+1} = gx_{2n+1} = Jx_{2n+2}$ for all n=0,1,2,…

Now for all $x,y \in X$ and $\lambda \in \Gamma$. We have $d_\lambda(y_{2n}, y_{2n+1}) = d_\lambda(fx_{2n}, gx_{2n+1})$

$\leq \max\{d_\lambda(Jx_{2n}, hx_{2n+1}), d_\lambda(fx_{2n}, Jx_{2n}), d_\lambda(gx_{2n}, hx_{2n+1}), d_\lambda(Jx_{2n}, gx_{2n+1})$

$+ \frac{d_\lambda(hx_{2n}, fx_{2n})}{2}\}$

$- w[\max\{d_\lambda(Jx_{2n}, hx_{2n+1}), d_\lambda(fx_{2n}, Jx_{2n}), d_\lambda(gx_{2n}, hx_{2n+1}), \frac{d_\lambda(Jx_{2n}, gx_{2n+1}) + (d_\lambda(hx_{2n}, fx_{2n})}{2}\}]$

$\leq \max\{d_\lambda(gx_{2n-1}, fx_{2n}), d_\lambda(fx_{2n}, gx_{2n-1}), d_\lambda(gx_{2n+1}, fx_{2n}),$

$\frac{d_\lambda(gx_{2n-1}, gx_{2n+1}) + (d_\lambda(fx_{2n}, fx_{2n})}{2}\}$

$-w[\max\{d_\lambda(gx_{2n-1}, fx_{2n}), d_\lambda(fx_{2n}, gx_{2n-1}), d_\lambda(gx_{2n+1}, fx_{2n}), \frac{d_\lambda(gx_{2n-1}, gx_{2n+1}) + (d_\lambda(fx_{2n}, fx_{2n})}{2}\}]$

If $d_\lambda(y_{2n-1}, y_{2n}) < d_\lambda(y_{2n}, y_{2n+1})$ i.e., if

$d_\lambda(gx_{2n-1}, fx_{2n}) < d_\lambda(gx_{2n+1}, fx_{2n})$, then

$d_\lambda(gx_{2n-1}, gx_{2n+1}) \leq d_\lambda(gx_{2n-1}, fx_{2n}) + d_\lambda(fx_{2n}, gx_{2n+1})$

$< d_\lambda(fx_{2n}, gx_{2n+1}) + d_\lambda(fx_{2n}, gx_{2n+1})$

and $d_\lambda(y_{2n}, y_{2n+1}) \leq \{d_\lambda(y_{2n}, y_{2n+1})\} - w[d_\lambda(y_{2n-1}, y_{2n+1})] < d_\lambda(y_{2n}, y_{2n+1})$,

which is a contradiction and so

$d_\lambda(y_{2n}, y_{2n+1}) \leq d_\lambda(y_{2n-1}, y_{2n})$.

We next show that $\lim_{n\to\infty} d_\lambda(y_{n-1}, y_n) = 0$ for each $\lambda \in \Gamma$.

Since $d_\lambda(y_{n-1}, y_n)$ is a decreasing sequence of non-negative terms, then $\lim_{n\to\infty} d_\lambda(y_{n-1}, y_n) = d \in R$, say, we want to prove that d=0.

Suppose d>0 and since d is continuous and

since $\sum_{i=0}^{n} w(d_\lambda(y_2, y_{2+1})) \leq d_\lambda(y_0, y_1) - d_\lambda(y_n, y_{n+1}) < d_\lambda(y_0, y_1)$,

the series $\sum_{i=0}^{\infty} w(d_\lambda(y_i, y_{i+1}))$ is convergent.

Hence $\lim_{n\to\infty} w(d_n) = 0$.

Since $\{d_n\}$ is a decreasing sequence of non-negative terms, we have

$\lim_{n\to\infty} d_n = d$, say $\in R^+$

Since w is continuous it follows that

$\lim_{n\to\infty} w(d_n) = w(d)$

and therefore w(d) = 0.

But since w(r) > 0 for r>0 and so

d=0, i.e., $\lim_{n\to\infty} d_\lambda (y_{n-1}, y_n) = 0$ and $\lambda \in \Gamma$.

We now show that $\{y_n\}$ is a Cauchy sequence. Then for every positive number □ and for every positive integer K there exists two positive integers 2m(k) and 2n(k) such that 2m(k) > 2n(k) > k and $d_\lambda (y_{2m(k)}, y_{2m(k)}) >$ □.

Further let 2m(k) denote the smallest even integer for which 2m(k) >2n(k)>k,

$d_\lambda (y_{2m(k)}, y_{2n(k)}) >$ □ and $d_\lambda (y_{2m(k)-2}, y_{2n(k)}) \le$ □

Now, □ $< d_\lambda (y_{2m(k)}, y_{2n(k)}) \le d_\lambda (y_{2m(k)-2}, y_{2n(k)}) + d_\lambda (y_{2m(k)-2}, y_{2m(k)-1}) + d_\lambda (y_{2m(k)-1}, y_{2n(k)})$.

Letting $k\to\infty$, we get

$\lim_{k\to\infty} d_\lambda (y_{2m(k)}, y_{2n(k)}) =$ □ for each $\lambda \in \Gamma$.

By triangle inequality, we have

$|d_\lambda (y_{2m(k)}, y_{2n(k)+1}) - d_\lambda (y_{2m(k)} , y_{2n(k)})| \le d_\lambda (y_{2m(k)}, y_{2n(k)+1})$,

$|d_\lambda (y_{2m(k)+1}, y_{2n(k)+1}) - d_\lambda (y_{2n(k)} , y_{2n(k)+1})| \le d_\lambda (y_{2m(k)}, y_{2m(k)+1})$,

$|d_\lambda (y_{2m(k)+1}, y_{2n(k)+2}) - d_\lambda (y_{2m(k)+1} , y_{2n(k)+1})| \le d_\lambda (y_{2m(k)+1}, y_{2m(k)+2})$,

$|d_\lambda (y_{2m(k)+1}, y_{2n(k)+2}) - d_\lambda (y_{2m(k)+1} , y_{2n(k)+2})| \le d_\lambda (y_{2m(k)+1}, y_{2m(k)+1})$,

□ □ $= \lim_{k\to\infty} d_\lambda (y_{2m(k)}, y_{2n(k)+1})$

$= \lim_{k\to\infty} d_\lambda (y_{2m(k)+1}, y_{2n(k)+1})$

$= \lim_{k\to\infty} d_\lambda (y_{2m(k)+1}, y_{2n(k)+2})$

$= \lim_{k\to\infty} d_\lambda (y_{2m(k)}, y_{2n(k)+2})$ for each $\lambda \in \Gamma$.

By the given assumption,

$d_\lambda (y_{2m(k)+1}, y_{2n(k)+2})$

$= d_\lambda (gx_{2m(k)+1}, fx_{2n(k)+2})$

$\le \max \{d_\lambda (gx_{2n(k)+2}, hx_{2m(k)+2})$,

$d_\lambda(fx_{2n(k)+2}, Jx_{2n(k)+2}), d_\lambda (gx_{2m(k)+1}, hx_{2m(k)+1})$,

$$\frac{d_\lambda(Jx_{2n(k)+2}, gx_{2m(k)+1}) + d_\lambda(hx_{2m(k)}, fx_{2n(k)+2})}{2}$$

$-w [\max \{ d_\lambda(fx_{2n(k)+2}, Jx_{2n(k)+2}), d_\lambda (gx_{2m(k)+1}, hx_{2m(k)+1})$,

$$\frac{d_\lambda(Jx_{2n(k)+2}, gx_{2m(k)+1}) + d_\lambda(hx_{2m(k)}, fx_{2n(k)+2})}{2} \}]$$

$$= \max\{d_\lambda(y_{2m(k)},y_{2n(k)+1}), d_\lambda(y_{2m(k)+1},y_{2n(k)+2}), d_\lambda(y_{2m(k)}, y_{2m(k)+1}), \frac{d_\lambda(y_{2n(k)+1}, y_{2m(k)+1}) + d_\lambda(y_{2m(k)}, y_{2n(k)+2})}{2}$$

$$-w[\max\{d_\lambda(y_{2m(k)},y_{2n(k)+1}),d_\lambda(y_{2m(k)+1},y_{2n(k)+2}),d_\lambda(y_{2m(k)},y_{2m(k)+1}), \frac{d_\lambda(y_{2n(k)+1}, y_{2m(k)+1}) + d_\lambda(y_{2m(k)}, y_{2n(k)+2})}{2}\}]$$

for each $\lambda \in \Gamma$.

Letting $k\to\infty$ we get,

□□□□□–□□□□□□□□□□

which is a contradictions. Thus $\{y_n\}$ is a Cauchy sequence.

Since X is sequentially complete, three is a point $\xi \in X$ such that $\xi = \lim_{n\to\infty} y_n$. Consequently $\{fx_{2n}\} = \{hx_{2n+1}\}$ and $\{gx_{2n+1}\} = \{Jx_{2n+2}\}$ converge to ξ. The mapping J is continuous. Then we have for all $\lambda \in \Gamma$,

$d_\lambda (fJx_{2n}, gx_{2n+1}) \le \max \{d_\lambda (JJx_{2n}, hx_{2n+1}), d_\lambda (fJx_{2n}, JJx_{2n}), d_\lambda (gx_{2n}, hx_{2n+1})$,

$$\frac{d_\lambda(JJx_{2n}, gx_{2n+1}) + d_\lambda(hx_{2n+1}, hJx_{2n})}{2} \}$$

$-w[\max d_\lambda (fJx_{2n}, gx_{2n+1}) \le \max \{d_\lambda (JJx_{2n}, hx_{2n+1}), d_\lambda (fJx_{2n}, JJx_{2n})$,

d_λ (gx_{2n}, hx_{2n+1}), $\dfrac{d_\lambda(JJx_{2n}, gx_{2n+1}) + d_\lambda(hx_{2n+1}, hJx_{2n})}{2}$ }].

Since the mappings f and J are compatible, then we have

$d_\lambda (J\xi,\xi) \le \max \{d_\lambda (J\xi,\xi), d_\lambda (J\xi,J\xi), d_\lambda (\xi,\xi), d_\lambda (J\xi,\xi)\}$

$-w[\max d_\lambda (J\xi,\xi) \le \max \{d_\lambda (J\xi,\xi), d_\lambda (J\xi,J\xi), d_\lambda (\xi,\xi), d_\lambda (J\xi,\xi)\}]$

$= d_\lambda (J\xi,\xi) - w(d_\lambda (J\xi,\xi))$.

Now we consider $\xi \neq J\xi$. Since (X,U) is a Hausdorff space and $\xi \neq J\xi$, there is an index $\square \in \Gamma$ such that $d_\square (\xi,J\xi) \neq 0$.

Therefore, we have

$d_\square (J\xi,\xi) \le d_\square (J\xi,\xi) - w (d_\square (J\xi,\xi)) < d_\square (J\xi,\xi)$

which is a contradiction. Hence $J\xi=\xi$.

Further we have for $\lambda \in \Gamma$,

$d_\lambda (f\xi, gx_{2n+1}) \le \max \{d_\lambda(J\xi,hx_{2n+1}), d_\lambda(hx_{2n+1},f\xi), d_\lambda(gx_{2n+1},hx_{2n+1}), \dfrac{d_\lambda(J\xi, gx_{2n+1}) + d_\lambda(hx_{2n+1}, f\xi)}{2}\}$

$-w[\max\{ d_\lambda (f\xi, gx_{2n+1}) \le \max \{d_\lambda(J\xi,hx_{2n+1}), d_\lambda(hx_{2n+1},f\xi), d_\lambda(gx_{2n+1},hx_{2n+1}), \dfrac{d_\lambda(J\xi, gx_{2n+1}) + d_\lambda(hx_{2n+1}, f\xi)}{2}\}]$

Taking limit as $n\to\infty$, we have

$d_\lambda (f\xi,\xi) \le \max [d_\lambda (\xi,\xi), d_\lambda(f\xi,\xi), d_\lambda (\xi,\xi), \dfrac{1}{2} d_\lambda (f\xi,\xi)\}$

$-w [\max d_\lambda (f\xi,\xi) \le \max [d_\lambda (\xi,\xi), d_\lambda(f\xi,\xi), d_\lambda (\xi,\xi), \dfrac{1}{2} d_\lambda (f\xi,\xi)\}]$

i.e., $d_\lambda (f\xi,\xi) \le d_\lambda (f\xi,\xi) - w (d_\lambda (f\xi,\xi))$

for each $\lambda \in \Gamma$, which is a contradiction. Hence $f\xi=\xi$. Thus $f\xi=J\xi=\xi$.

Since h is continuous, we can show that $g\xi=h\xi=\xi$ and ξ is a common fixed point of f,g,h and J.

To prove the unicity of ξ, if possible let ξ' be another common fixed point of f,g,h and J and let $\xi\neq\xi'$. Then there is an index $v \in \Gamma$ such that

$d_v (\xi,\xi') = d_v (f\xi,g\xi')$

$\le \max \{d_v (J\xi,h\xi'), d_v (f\xi,J\xi), d_v (g\xi', h\xi'), \dfrac{d_v(J\xi, g\xi') + d_v(h\xi', f\xi')}{2}\}$

$-w[\max\{d_v (J\xi,h\xi'), d_v (f\xi,J\xi), d_v (g\xi', h\xi'), \dfrac{d_v(J\xi, g\xi') + d_v(h\xi', f\xi')}{2}\}]$

$= \max \{d_v (\xi,\xi'), d_v (\xi,\xi), d_v (\xi',\xi), d_v (\xi,\xi')\}$

$- w[\max\{d_v (\xi,\xi'), d_v (\xi,\xi), d_v (\xi',\xi), d_v (\xi,\xi')\}]$

$= d_v (\xi,\xi') - w[(d_v (\xi,\xi')) < d_v(\xi,\xi')$

which is a contradiction. Hence $\xi=\xi'$ and ξ is the unique common fixed point of f,g,h and J.

Corollary 1. Let (X,U) be a sequentially complete Hausdorff Uniform space defined by $\{d_\lambda : \lambda\in\Gamma\} = P^*$. Let f be a mapping on X satisfying the condition

$d_\lambda (fx,gy) \le \max \{d_\lambda(x,y), d_\lambda (hx,x), d_\lambda (hy,y), \dfrac{d_\lambda(x, hy) + d_\lambda(y, fx)}{2}\}$

$-w[\max d_\lambda (fx,gy) \le \max \{d_\lambda(x,y), d_\lambda (hx,x), d_\lambda (hy,y), \dfrac{d_\lambda(x, hy) + d_\lambda(y, fx)}{2}\}]$

for all $x,y\in X$, $\lambda\in\Gamma$. Then f has a unique fixed point in X.

Proof. Put f=g and J=h=I, identity mapping in Theorem 1, Corollary 1 follows.

Corollary 2. Let (X,U) be a sequentially complete Hausdorff uniform space defined by $\{d_\lambda : \lambda\in\Gamma\} = P^*$. Let f,g be mappings on X satisfying the condition

d_λ (fx,hy) $\leq$ max {d_λ (hx,hy), d_λ (fx,hx), d_λ (fy,hy), $\frac{d_\lambda(hx,hy)+d_\lambda(hy,fx)}{2}$}

$-$w[max {d_λ (hx,hy), d_λ (fx,hx), d_λ (fy,hy), $\frac{d_\lambda(hx,hy)+d_\lambda(hy,fx)}{2}$}]

for all x,y$\in$X, $\lambda\in\Gamma$.
Then f and h have a unique common fixed point in X, provided h is continuous, f and h are compatible and f(X) $\subset$ h(X).
Proof. Put f=g and J=h in Theorem 1, Corollary 2 follows.

Corollary 3. Let (X,U) be a sequentially complete Hausdorff uniform space defined by {d_λ : $\lambda\in\Gamma$ } = P*. Let f, g, h be three mappings on X satisfying the condition

d_λ (fx,fy) $\leq$ max {d_λ (hx,hy), d_λ (fx,hx), d_λ (fy,hy), $\frac{d_\lambda(hx,gy)+d_\lambda(hy,fx)}{2}$}

$-$w[max d_λ (fx,fy) $\leq$ max {d_λ (hx,hy), d_λ (fx,hx), d_λ (fy,hy), $\frac{d_\lambda(hx,gy)+d_\lambda(hy,fx)}{2}$}]

for all x,y$\in$X, $\lambda\in\Gamma$. Let h be continuous, h and f be compatible. If f(X)$\subset$h(X), then f, g, h have a unique common fixed point in X.
Proof. Put h=J in Theorem 1, Corollary 3 follows.

Corollary 4. Let (X, U) be a sequentially complete Hausdorff uniform space defined by {d_λ : $\lambda\in\Gamma$ } = P*. Let f, g, h be three mappings on X satisfying the condition

d_λ (fx,hy) $\leq$ max {d_λ (Jx,hy), d_λ (fx,Jx), d_λ (fy,hy), $\frac{d_\lambda(Jx,hy)+d_\lambda(hy,fx)}{2}$}

$-$w[max d_λ (fx,hy) $\leq$ max {d_λ (Jx,hy), d_λ (fx,Jx), d_λ (fy,hy), $\frac{d_\lambda(Jx,hy)+d_\lambda(hy,fx)}{2}$}]

for all x, y $\in$ X, $\lambda\in\Gamma$. Let h and J be continuous, h and f, J and f be compatible. If f(X) $\subset$ h(X) $\cap$ J(X) then f, J, h have a unique common fixed point in X.
Proof. Put f=g in Theorem 1, Corollary 4 follows.

REFERENCES

[1]. Jungck, G. Compatible mappings and common fixed points. *Internat. J. Math. & Math. Sci.*, 9, 1986, 771-779.
[2]. Kelley, J.L. General Topology. *Van Nostrand Reinholt,* Princeton, New Jersey, 1955.
[3]. Sastry, K.P.R., Babu, G.V.R. & Rao, D. Narayana Fixed point theorems in complete metric spaces. *Bull. Cal. Math. Soc.* 91, 1999, 493-502.
[4]. Sessa, S. On a weak commutativity condition of mappings in fixed point considerations. *Publ. Inst. Math.* 32, 1982, 149-153.
[5]. Thron, W.J. Topological structure, Rinehart and Winston, New York, 1966.