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Separation Axioms in Topological Spaces Via Ideal

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***Abstract*—**In this article, $D^\star$ sets used to characterize some weak separation axioms in ideal topological spaces and to study some of their essential properties.

Keywords *— Ideal topological spaces, Open set.*

I. INTRODUCTION

The idea of topology via ideal was presented in 1990 by Jankovic and Hamlet [1]. Setup $D^\star$ sets in ideal topological space and built up certain its noteworthy properties. The Concept of the D_μ-set was introduced by Sarsak [3] in 2011. We have defined some Separation axioms in ideal topological space using the concept of D_μ-set.

II. RELATED WORK

First we review the definition of generalized topological space, g-open sets and g-closed sets.

Definition 2.1 [1] Let X be a non-empty set and let τ be a family of subsets of X. Then τ is said to be a **topology** on X, if following two conditions are fulfilled viz,:

(i). $\phi \in \tau$;
(ii). $\cup_{i\in\wedge} G_i \in \tau$ for $G_i \in \tau$.
(iii).$\cap_{i=1}^{n} G_i \in \tau$ for $G_i \in \tau$.

Definition 2.1 [2] Let X be a non empty set and I be a family of subsets of X. Then I is said to be an ideal on X, if satisfies the following two conditions viz,:

(a) $A \in I$ and $B \subset A$ then $B \in I$
(b) $A, B \in I$ then $A \cup B \in I$.

Then the topology τ with ideal I, (X, τ, I) is known as an **ideal topological space**. The members of ideal topology $\tau^\star$are called $\star$-open sets and their complements are called $\star$-closed sets.

Definition **2.2[2]:** Let (X, τ, I) be an ideal topological space and $A \subseteq X$. The set $A^\star = \{x \in X: A \cap U \in I$ for each neighbourhood U of $x\}$ is **Local function of A with respect to I and τ.** The local function $A^\star$ is also denoted by $A^\star(I)$.

Definition 2.2[2]: The ideal topology $\tau^\star$ is characterized as $\tau^\star = \{X - Cl^\star(A): Cl^\star(A) = A\ ,\ A \subset X\}$, where $Cl^\star(A) = A \cup A^\star$ is a topology through ideal concerning topology on X. We denote ideal topological space (X, τ, I) by $(X, \tau^\star)$, where $\tau^\star$ is an ideal topology generated through ideal I with respect to topology τ on X. The ideal topological space $\tau^\star$is generated by the basis set $B = \{U - A: U \in \tau\ and\ A \in I\}$. The ideal Topology $\tau^\star$ is finer than topology τ i.e., $\tau^\star \supseteq \tau$.

Example 2.1 Let $X = \{a, b, c\}$ and let $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ and ideal $I = \{\phi, \{b\}\}$. Then ideal topology $\tau^\star$=$\{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$ is a topology on X.

Definition 2.3[2]: Let$(X, \tau^\star)$ be an ideal topological space and let $A \subseteq X$.Then the intersection of all $\star$-closed sets in X containing A is $\star$-closure of A. The $\star$-closure of A is symbolized by $c^\star(A)$.

Remark [2]: Since arbitrary intersection of $\star$-closed sets is a $\star$-closed set, it follows that the smallest $\star$-closed set in $(X, \tau^\star)$ containing A is $c^\star(A)$.

Definition 2.3: Let $(X, \tau^\star)$ be a generalized topological space and $A \subseteq X$. Then the union of all $\star$-open sets in X contained in A is $\star$-interior of A. The $\star$-interior of A is symbolized by $i^\star(A)$.

Remark: Since arbitrary union of $\star$-open sets is a $\star$-open set, it follows that the largest $\star$-open set in $(X, \tau^\star)$ contained in A is $i^\star(A)$.

III. RESULTS AND DISCUSSION

A. $D^\star$-sets and their properties

Definition 3.1: Let $(X, \tau^\star)$ be an ideal topological space and $B \subseteq X$. Then A is a $D^\star$-set if there are two $U, V \in \tau^\star$ such that $U \neq X$ and $B = U - V$.

Remark 3.1: Note that every proper $\star$-open set is a $D^\star$-set

Definition 3.2: An ideal topological space $(X, \tau^\star)$ is a $D_0^\star$-space if for any pair of distinct points x and y of X there exists a $D^\star$-set of X in which x but not y or a $D^\star$-set of X in which y but not x.

Definition 3.3: An ideal topological space $(X, \tau^\star)$ is a $D_1^\star$-space if for any pair of distinct points x and y of X there exists a $D^\star$-set A, B of X in which x, y respectively such that $y \notin A$ and $x \notin B$.

Definition 3.4: An ideal topological space $(X, \tau^\star)$ is a $D_2^\star$-space if for any pair of distinct points x and y of X there exists disjoint $D^\star$-set A, B of X in which x, y respectively.

Definition 3.5: An ideal topological space $(X, \tau^\star)$ is a $T_0^\star$-space if for any pair of distinct points x and y of X there exists a $\star$-open set of X in which x but not y or a $\star$-open set of X in which y but not x.

Definition 3.3: An ideal topological space $(X, \tau^\star)$ is a $T_1^\star$-space if for any pair of distinct points x and y of X there exists a $\star$-open set A and B of X in which x and y respectively such that $y \notin A$ and $x \notin B$.

Definition 3.4: An ideal topological space $(X, \tau^\star)$ is a $T_2^\star$-space if for any pair of distinct points x and y of X there exists disjoint $\star$-open set A, B of X in which x, y respectively.

Proposition 3.1: Every $T_i^\star$ space is a $T_{i-1}^\star$ space.
Proof: Its immediately follows from definitions.

Proposition 3.2: Every $T_i^\star$ space is a $D_i^\star$ space.
Proof: Since every proper $\star$-open set is $D^\star$-set. It means that every $T_i^\star$ space is a $D_i^\star$ space.

Proposition 3.3: Every $D_i^\star$ space is a $D_{i-1}^\star$ space.
Proof: Its immediately follows from definitions.

Theorem 3.1: **A**n ideal topological space $(X, \tau^\star)$ is $D_0^\star$-space iff it is $T_0^\star$-space.
Proof: suppose that $(X, \tau^\star)$ is $D_0^\star$-space and $a, b \in X$ then there exists $D^\star$-set G containing one of a, b (say a) but no other ($ie. b \notin G$). Suppose that $G = V_1 - V_2$ where $V_1 \neq X$ and $V_1, V_2 \in \tau^\star$. Clearly $a \in V_1$. For $b \notin G$, we have two cases;
(i). $b \notin V_1$.
(ii). $b \in V_1$ & $b \in V_2$.
In case (a), V_1 contain a but not b. In case (ii), V_2 contains b but not a. Hence $(X, \tau^\star)$ is $T_0^\star$-space. Conversely follows from proposition 3.2.

Theorem 3.2: **A**n ideal topological space $(X, \tau^\star)$ is $D_1^\star$-space iff it is $D_2^\star$-space.

Proof: suppose that $(X, \tau^\star)$ is $D_1^\star$-space and $a, b \in X$ then there exists $D^\star$-set G_1 and G_2 in which a, b but not b, a respectively. Suppose $G_1 = V_1 - V_2$ and $G_2 = V_3 - V_4$. Since $a \notin G_2$ ie. Either $a \notin V_3$ or a in V_3, V_4 both.
When $a \notin V_3$ and $b \notin G_1$, we have following two cases;

(a) $b \notin V_1$. Since $a \in (V_1 - V_2)$ but $a \notin V_3$, then $a \in \left(V_1 - (V_2 \cup V_3)\right)$. From $b \in (V_3 - V_4)$ but $b \notin V_1$, then $b \in \left(V_3 - (V_4 \cup V_1)\right)$. Clearly, $\left(V_1 - (V_2 \cup V_3)\right) \cap \left(V_3 - (V_4 \cup V_1)\right) = \phi$.

(b) When b in V_1, V_2 both. We have $a \in (V_1 - V_2)$ and $b \in V_2$ and $(V_1 - V_2) \cap V_2 = \phi$.

When a in V_3, V_4 both. Then we have $b \in (V_3 - V_4)$, $x \in V_4$ and $(V_3 - V_4) \cap V_4 = \phi$. Hence $(X, \tau^\star)$ is $D_2^\star$-space. Conversely follows from proposition 3.3.
Corollary 3.1: **A**n ideal topological space $(X, \tau^\star)$ is $D_1^\star$-space then it is $T_o^\star$-space.
Proof: Since every $D_1^\star$-space is $D_0^\star$-space and by the theorem 3.1 we have $D_1^\star$-space is $T_o^\star$-space.

From the above observation, we have following diagram:

$$\begin{array}{ccc} \boldsymbol{T_2^\star} & \rightarrow & \boldsymbol{D_2^\star} \\ \downarrow & & \updownarrow \\ \boldsymbol{T_1^\star} & \rightarrow & \boldsymbol{D_1^\star} \\ \downarrow & & \downarrow \\ \boldsymbol{T_0^\star} & \leftrightarrow & \boldsymbol{D_0^\star} \end{array}$$

Example 3.1: Let $X = \{a, b, c\}$ be ideal topological space with respect to topology $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and ideal $I = \{\phi, \{b\}\}$. Then the ideal topology is $\tau^\star = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ on X. Clearly $(X, \tau^\star)$ is $T_0^\star$-space but not $T_1^\star$ and $T_2^\star$. Thus, $(X, \tau^\star)$ is a $D_0^\star$-space but not $D_1^\star$ and $D_2^\star$.

Example 3.2: Let $X = \{a, b, c\}$ be ideal topological space with respect to topology $\tau = \{\phi, X\}$ and ideal $I = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Then the ideal topology is $\tau^\star = \{\phi, \{c\}, \{b, c\}, \{a, c\}, X\}$ on X. Clearly $(X, \tau^\star)$ is $T_0^\star$-space but not $T_1^\star$ and $T_2^\star$. Thus, $(X, \tau^\star)$ is a $D_0^\star$, $D_1^\star$ and $D_2^\star -$ space.

Theorem 3.3: An ideal topological space $(X, \tau^\star)$ is $T_0^\star$ iff for each pair of distinct points x, y of X, $c^\star(\{x\}) \neq c^\star(\{y\})$.
Proof: Suppose $(X, \tau^\star)$ is $T_0^\star$-space and x, y be a distinct point of X. Then, there exists a $\star$-open set G containing x but not y, therefore x in compliment of G (say G^c) but $y \notin G^c$. Since $c^\star(\{y\})$ is smallest $\star$-closed set containing y, ie. $c^\star(\{y\}) \subset G^c$. Therefore $x \notin c^\star(\{y\})$. Thus, $c^\star(\{x\}) \neq c^\star(\{y\})$. Conversely suppose x, y be a distinct point of X and $c^\star(\{x\}) \neq c^\star(\{y\})$. Let point $z \in X$ such that $z \in c^\star(\{x\})$ and $z \notin c^\star(\{y\})$. Then we will show that $x \notin c^\star(\{y\})$. Suppose that $x \in c^\star(\{y\})$ then $c^\star(\{x\}) \subset c^\star(\{y\})$. This implies that $z \in c^\star(\{y\})$, which is a contradiction. Thus, $x \in \left(c^\star(\{y\})\right)^c$ which is $\star$-open set does not contain y. Hence $(X, \tau^\star)$ is $T_0^\star$.

Theorem 3.4: An ideal topological space $(X,\tau^\star)$ is $T_1^\star$iff the singletons of X are $\star$-closed.

Proof: Suppose $(X,\tau^\star)$ is $T_0^\star$-space and x be a any point of X.Let $y\in\{x\}^c$, then $x\neq y$, so there exists $\star$-open set U containing y but not x. This implies $y\in U\subset\{x\}^c$, ie. $\{x\}^c=\cup\{U\colon U$ containing$y\}$ which is $\star$-open. Conversely let $\{a\}$ is $\star$-closed set for every $a\in X$. Let x,y be a distinct point of X. Then $\{x\}^c$ is a $\star$-open set containing y but not x. Similarly $\{y\}^c$ is a $\star$-open set containing x but not y. Hence $(X,\tau^\star)$is $T_1^\star$.

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