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On Common Fixed Points of Mappings in Hilbert Spaces

Kalishankar Tiwary[1]*, Chandan Kamelia[2], Biplab Kumar Bag[3]

[1,2,3]Deparetment of Mathematiics, Raiganj University, Raiganj,733134 U/D, India

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Abstract -In this paper some common fixed points theorems have been proved in Hilbert spaces using the definition of normal structure etc.

Keywords- Fixed Point, Normal Structure, Hilbert Space.

I. INTRODUCTION

Recently a large number of literatures are available in the area of fixed point theory which deal with a common fixed point of two or more mappings. Actually the notion of commutativity has been weakened in various ways by many research workers such as Jungck [1,2], Sessa [6], Sharma and Sahu [7], etc. Using these definition as introduced by the above research workers time to time, several authors have proved a large number of common fixed point theorems. For references one can see the literatures in the references.

Let S be a closed subset of a Hilbert space H. Let $\{T_n\}$ be a sequence of mappings of S into itself. Koparde and Waghmode [4] have proved common fixed point theorems for a sequence $\{T_n\}$ of mappings satisfying the condition

$$\| T_i x - T_j y \|^2 \le a(\| x - T_i x \|^2 + \| y - T_j y \|^2) \text{ for } x, y \in S,\ x \ne y, \text{where } 0 \le a < \frac{1}{2}.$$

Pandhare and Waghmade [5] have proved common fixed point theorem for a sequence $\{T_n\}$ of mapping satisfying the condition

$$\| T_i x - T_j y \|^2 \le a(\| x - y \|^2 + b(\| x - T_i x \|^2 + \| y - T_j y \|^2)$$

for all x, y $\in$ S and x$\neq$y, where $0 \le a$, $0 \le b < 1$ and $a+2b<1$.

Veerapandi and Kumar [9] have generalized the above conditions in the following ways and have proved several fixed point theorems.

There exist real numbers a, b, c, satisfying $0 \le a,b,c<1$ and $a+2b+2c<1$ such that for each $x,y \in S$ and $x \neq y$.

(A)
$$\| T_i x - T_j y \|^2 \le a(\| x - y \|^2 + b(\| x - T_i x \|^2 + \| y - T_j y \|^2) + \frac{c}{2}(\| x - T_j y \|^2 + \| y - T_i x \|^2)$$

(B) There exists a real number h satisfying $0 \le h < 1$ such that for all x, y$\in$S and x$\neq$y

$$\| T_i x - T_j y \|^2 \le h \max\{\| x - y \|^2 . \frac{1}{2}(\| x - T_i x \|^2 + \| y - T_j y \|^2). \frac{1}{4}(\| x - T_j y \|^2) + (\| y - T_i x \|^2)\}$$

(C) There exists a real number h satisfying $0 \le h<1$ such that for all x,y$\in$S and x$\neq$y

$$\| T_i x - T_j y \|^2 \le h \max\{\| x - y \|^2 . \| x - T_i x \|^2 . \| y - T_j y \|^2). \frac{1}{4}(\| x - T_j y \|^2) + (\| y - T_i x \|^2)\}$$

(D) There exists a real number h satisfying $0 \underline{<} h<1$ such that for all $x \in S$

$$\| T_i x - T_j T_i x \|^2 \leq h \| x - T_i x \|.$$

In the above conditions the constants are taken in such a way that their sums is less than one and this helps then to show that the sequence of iteration is Cauchy and because the Hilbert space H is complete the converging point ultimately becomes a fixed point. In this paper the constants are extended so that the technique of the proof has become non-routine, in the proof of our theorems.

II. DEFINITION

Let A be a bounded subset of Banach space X. A point $a \in X$ is said to be a non-diametral point of A is Sup $\{\|x-a\|.x \in a\} < \delta(A)$. A bounded convex subset K of X is said to be have normal structure if for each convex subset H of K which contains more than one point there exists an $x \in H$ which is a non-diametral point of H.

Hilbert space has always a normal structure.

We have proved the following theorems.

Theorem 1. Let X be Hilbert space and K be a non-empty bounded closed convex subset of X. Let $T_1, T_2 : K \to K$ be such that

(A) $$\| T_1 x - T_2 y \|^2 \leq a_1(x,y) \| x - y \|^2 + a_2(x,y) \| x - T_1 x \|^2 + a_3(x,y) \| y - T_2 y \|^2$$
$$+ a_4(x,y) \| x - T_2 y \|^2 + a_5(x,y) \| y - T_1 x \|^2$$

where $a_i(x,y) \geq 0 \ \forall \ x,y \in K$ and

$$\sup_{x,y \in k} \{2a_1(x,y) + 11a_2(x,y) + 11a_3(x,y) \,|\, 8a_4(x,y) \,|\, 8a_5(x,y) = 2$$

(B) $T_1 F \subset F$ iff $T_2 F \subset F$ for every convex subset F of K.

(C) Either $\sup_{x \in F} \| x - T_1 x \| < \delta(F)$

$$\sup_{x \in F} \| x - T_2 x \| < \delta(F)$$

for every non-empty bounded closed convex subset F of K which is mapped into itself by T_1 or T_2. Then T_1 and T_2 have a common fixed point in X.

Proof. Let Y be the family of all non-empty bounded closed convex subsets of K ordered by set inclusion which are mapped into itself by T_2. Since X is a Hilbert space, it is a reflexive Banach space and hence by Smulian's result [8] every decreasing sequence of non-empty bounded closed convex subsets of X has non-empty intersection and by Zorn's lemma, it follows that X possesses a minimal element F, say. If F contains only element then that element becomes a fixed point of T_2. We shall show that F contains only one element. We suppose on the contrary that F contains more than one point, which we will show implies a contradiction.

Let $A = \sup_{y \in F} \| T_2 y - y \|$. By the condition (C) $A > \delta(F)$

We now define the following terms for $x \in F$.

Let $$\gamma_x(F) = \max\{\sup_{y \in F} \| x - y \| .A\}$$

$$\gamma(F) = \inf\{\gamma_x(F), x \in F\},$$

and $$F_c = \{x \in F; \gamma_x(F) = \gamma(F)\}.$$

We now show that F_c is non-empty closed and convex. For a positive integer n and for $x \in F$, let $F(x,n) = \{y \in F; \|x-y\| \underline{<} \gamma(F) + 1/n\}$ and $C_n = \bigcap_{x \in F} F(x,n)$.

We show first that C_n is non-empty. If possible let $C_n = \varphi$, then there exist x_1 and $x_2 \in F$ such that $F(x_1,n) \cap F(x_2,n) = \varphi$. By construction $F(x_1,n) = \{y \in F : \|x_1 - y\| \underline{<} \gamma(F) + 1/n\}$ and similarly $F(x_2,n)$.

$$\| x_1 - x_2 \| \geq 2\gamma(F) + \frac{2}{n} \quad (1)$$

Now for $x \in F$, $\sup_{y \in F} \|x-y\| \geq \frac{\delta(F)}{2}$ and $\gamma_x(F) \geq \frac{\delta(F)}{2}$ and this implies $\frac{\delta(F)}{2} \underline{<} \gamma(F)$.

Therefore $\delta(F) < 2\gamma(F) + 2/n$. So from (1) $\|x_1 - x_2\| > \delta(F)$ which is a contradiction because $x_1, x_2 \in F$.
Therefore C_n is non-empty.
It may further be verified that C_n is closed, convex and that $C_{n+1} \subset C_n$.

We wish to show that $F_c = \bigcap_{n=1}^{\infty} C_n$.

For this let $y \in F_c$. Then $v_y(F) = \gamma(F)$

So $\max\{\sup_{X \in F} \| y - x \|, A\} = \gamma(F)$ and so

$$\{\sup_{X \in F} \| y - x \| < \gamma(F) \qquad (2)$$

We verify that $y \in F(x,n)$ for all $x \in F$ and for all n. If possible let $y \notin F(x,n)$ for some x and for some n.

Then $\|x - y\| > \gamma(F) + (1/n)$ (3)

From (2) we see that $\|x - y\| \leq \gamma(F)$ which is a contradiction to (3).

So $y \in \bigcap_{n=1}^{\infty} C_n$ and so $F_c \subset \bigcap_{n=1}^{\infty} C_n$

next let $y \in \bigcap_{n=1}^{\infty} C_n$. Then $y \in F(x,n)$ for all x and for all n and this implies that $\sup_{x \in F} \|x - y\| \leq \gamma(F)$. Also $A \leq \gamma(F)$. These two give $\gamma_y(F) \leq \gamma(F)$. But $\gamma(F) \leq \gamma_y(F)$ always and then $\gamma_y(F) = \gamma(F)$ and this gives $\gamma \in F_c$. So $\bigcap_{n=1}^{\infty} C_n \subset F_C$. Thus $F_C = \bigcap_{n=1}^{\infty} C_n$.

This equality further gives that F_c is closed and convex and by Smulian's result [8] non-empty. Next we show that $\delta(F_c) < \delta(F)$.
Since K has a normal structure and $A < \delta(F)$ three exists a point $x \in F$ such that $\gamma_x(F) < \delta(F)$.

If $x_1, x_2 \in F_c$, then $\|x_1 - x_2\| \leq \gamma_{x_1}(F) = \gamma(F)$.

So $\delta(F_c) = \sup\{\|x_1 - x_2\| : x_1, x_2 \in F_c\} \leq \gamma(F) \leq \gamma_x(F) < \delta(F)$ (4)

If $x \in F_c$ and y is an arbitrary elements of F we obtain

$\|T_2y - T_1x\|^2 \leq a_1(x,y)\|x - y\|^2 + a_2(x,y)\|y - T_2y\|^2 + a_3(x,y)\|x - T_1x\|^2$
$+ a_4(x,y)\|y - T_1x\|^2 + a_5\|x - T_2y\|^2$
$\leq a_1(x,y)\|x - y\|^2 + a_2(x,y)\|y - T_2y\|^2 + a_3(x,y)\{2\|x - y\|^2 + 2\{2\|y - T_2y\|^2 + 2\|T_2y - T_1x\|^2\}]$
$+ a_4(x,y)\{2\|y - T_2y\|^2 + 2\|T_2y - T_1x\|^2\} + a_5(x,y)\{2\|x - y\|^2 + 2\|y - T_2y\|^2\}$
$\Rightarrow (1 - 2a_4(x,y) - 4a_3(x,y))\|T_2y - T_1x\|^2 \leq (a_1(x,y) + 2a_3(x,y) + 2a_5(x,y))\|x - y\|^2 +$
$+ (a_2(x,y) + 4a_3(x,y) + 2a_4(x,y) + 2a_5(x,y))\|y - T_2y\|^2$ (A)

Similarly,
$(1 - 4a_2(x,y) - 2a_5(x,y))\|T_2y - T_1x\|$-2- $\leq (a_1(x,y) + 2a_2(x,y) + 2a_4(x,y))\|x - y\|^2$
$+ (4a_2(x,y) + a_3(x,y) + 2a_4(x,y) + 2a_5(x,y))\|y - T_2y\|^2$ (B)

Adding (A), and (B), we get

$(2 - 4a_2(x,y) - 4a_3(x,y) - 2a_4(x,y) - 2a_5(x,y))\|T_2y - T_1x\|^2$
$\leq (2a_1(x,y) + 2a_2(x,y) + 2a_3(x,y) + 2a_4(x,y) + 2a_5(x,y))\|x - y\|^2$
$+ (5a_2(x,y) + 5a_3(x,y) + 4a_4(x,y) + 4a_5(x,y))\|y - T_2y\|^2$
$\leq (2a_1(x,y) + 7a_2(x,y) + 7a_3(x,y) + 6a_4(x,y) + 6a_5(x,y))\max(\sup_{y \in F}\|x - y\|^2, \sup_{y \in F}\|y - T_2y\|^2$

or, $\|T_2y - T_1x| \leq \frac{2a_1(x,y) + 7a_2(x,y) + 7a_3(x,y) + 6a_4(x,y) + 6a_5(x,y)}{2 - 4a_2(x,y) - 4a_3(x,y) - 2a_4(x,y) - 2a_5(x,y)} \max\{\sup_{y \in F}\|x - y\|^2$
$\sup_{y \in F}\| y - T_2 y \|^2\}$.

Using the condition $2a_1(x,y) + 11a_2(x,y) + 11a_3(x,y) + 8a_4(x,y) + 8a_5(x,y) = 2$.

We can show that $\frac{2a_1(x,y) + 7a_2(x,y) + 7a_3(x,y) + 6a_4(x,y) + 6a_5(x,y)}{2 - 4a_2(x,y) - 4a_3(x,y) - 2a_4(x,y) - 2a_5(x,y)} \leq 1$

Taking the positive square root, we get

$$\|T_2y-T_1x\| \le \max\{\sup_{y\in F}\| x-y\|^2, \sup_{y\in F}\| y-T_2y\|\}$$

$$= \max\{\sup_{y\in F}\| x-y\|, A\} = \gamma_x(F) = \gamma(F).$$

So the set $T_2(F)$ is contained in a closed sphere with centre at T_1x and radius v(F). We denote this sphere by $\overline{U}$.

Clearly T_2 (F $\bigcap \overline{U}$) $\subset$ F $\bigcap \overline{U}$ and because F is minimal, F $\bigcap \overline{U}$ and so

$$\sup_{y\in F}\|T_1x-y\| < \gamma(F) \tag{5}$$

Now $\gamma_{T_1x}(F) = \max\{\sup_{y\in F}\|T_1x-y\|.A\}$

$\le \max\{\gamma(F),A\}$, from (5)

$= \gamma(F)$, because $\gamma(F) \ge A$.

Hence $\gamma_{T_1x}(F) \le \gamma(F)$. But we always have $\gamma(F) \le \gamma T_1x(F)$

So $\gamma_{T_1x}(F) = \gamma(F)$.

This implies that $T_1(x) \in F_c$ and by (B) $T_2(x) \in F_c$.

Therefore F_c is a non-empty, closed, convex subset of F which is mapped into itself by T_1 and T_2 and because of (4) $\delta(F_c) < \delta(F)$.

Therefore, F_c is a proper subset of F. This contradicts the fact that F is minimal. Therefore, F cannot contain more than one element, but F is not empty. Hence F contains only one element which is clearly a fixed point of T_1 and T_2.

Note : If $T_1 = T_2$ and $a_2(x,y) = a_3(x,y) = a_4(x,y) = a_5(x,y) = 0$ for all $x,y\in X$ and $a_1(x,y) = a$, 'a' constant the theorem proved in Kirk [3] follows.

Theorem 2. Let X be a Hilbert space and K be a non-empty bounded closed convex subset of X. Let $\{T_n\}$ be a sequence of mappings which map K into itself and satisfy

(A) $\|T_ix-T_jy\|^2 \le a_1(x,y)\|x-y\|^2 + a_2(x,y)\|x-T_ix\|^2 + a_3(x,y)\|y-T_jy\|^2 + a_4(x,y)\|x-T_jy\|^2 + a_5(x,y)\|y-T_ix\|^2$

where, $a_i \ge 0$; i=1,2,3,4,5 and $\sup_{x,y\in K}\{2a_1(x,y)+11a_2(x,y)+11a_3(x,y)+8a_4(x,y) +8a_5(x,y)\}\le 2$.

(B) $T_1F \subset F$ iff $T_jF \subset F$ for every convex subset F of K,

(C) Either $\sup_{x\in F}\|x-T_ix\|<\delta(F)$

$\sup_{x\in F}\|x-T_jx\|<\delta(F)$

for every empty bounded closed, convex subset of K which are mapped into itself by either T_i or T_j.

Then $\{T_n\}$ has a common fixed point in X.

Proof. Picking any two mappings T_i and T_j from $\{T_n\}$ and following the proof of the Theorem 1, it follows that T_i and T_j have a common fixed point in X. Since T_i and T_j are any two mappings it follows that $\{T_n\}$ has a common fixed point in X.

This completes the proof.

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