International Journal of Scientific Research in Mathematical and Statistical Sciences **Research Paper**
Vol.6, Issue.1, pp.147-154, February (2019) **E-ISSN:** 2348-4519
DOI: https://doi.org/10.26438/ijsrmss/v6i1.147154

Solution of European Call Option by the First Integral Method

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Available online at: www.isroset.org

Received: 10/Feb/2019, Accepted: 23/Feb/2019, Online: 28/Feb/2019

***Abstract*—**For finding the value of an option we use the widely used Black-Scholes formula in option pricing theory. In this paper, we convert the Black-Scholes PDE into ODE with boundary condition by using the First Integral Method and get accurate solution of Black-Scholes equation. Primarily, we address an error in a previously published work of other authors. We also show with numerical examples the effect of improved solution.

Keywords*—Black-Scholes equation, First integral method, European call option.*

I. INTRODUCTION

In an attempt to review and solve the Black-Scholes partial differential equation using the First integral method, an error was found in an earlier research work by Mehrdoust and Mirzazadeh[3]. The new analytical solution is hence derived which will correct the earlier work. To check the effect of the error some examples are worked out.

In financial mathematics with the help of Black-Scholes equation, one can obtain the option value. Black-Scholes equation is derived from Ito’s lemma, which is one of the most useful result for pricing European call option and put option. European options can only be exercised at the expiration date. Option price can be predicted by Black-Scholes model which is nearer to actual traded option price.

Since the derivative market have become important and growcontinuously, Black-Scholes equations are still useful in finance. Black-Scholes equation govern the price of the option over time and isuseful in the financial investment field. Also, in the field of financial engineering, Black-Scholes equation is a very easy way to obtain the price of call option [1,2].

Basically Black-Scholes equation depends on four quantities: time, risk free interest rate, underlying asset price and volatility. Time, asset price and risk free interest rate can be measured directly. The volatility of the asset depends on the change of its market value.

Several alternate methods can be used to solve the Black-Scholes equation. In this paper, we solve the Black-Scholes partial differential equation by applying first integral method for European calloption.We get exact solution of some non-linear partial differential equation using the first integral method which is one of the most direct and effective method [3].

II. LITERATURE REVIEW

Rouah (2013) derived Black Scholes partial differential equation using four kind of derivatives. He derived Black-Scholes partial differential equation by applying Hedging argument, Replicating portfolio, The Capital Asset Pricing Model, Limit of the Binomial Model and obtained Black-Scholes partial differential equation in terms of the Log Stock Price by applying Lognormal Distribution.

Park and Kim(2011) obtained analytic method for the value of options under stochastic volatility by using homotopy analysis method. Theprice is given by an infinite series whose value can be determined once the initial term is given well [5]. The initial term is given by the constant volatility Black-Scholes formula.

Aghajani and Ebadattalab(2015) solved the Black-Scholes equation by using the Reduced Differential Transform Method (RDTM). They obtained an efficient recurrent relation to solve Black-Scholes equation by using RDTM method. Due to less computation in RDTM, it was more useful and efficient than Differential Transform

Method(DTM). In financial mathematics some partial differential equationsare solved by this method.

Feng (2002) shows that First Integral method is efficient and widely applicable. This method can be applied to many nonlinear equations. He obtained the exact solution of Schrödinger equation, the generalized Klein-Gordon equation and higher order KDV by applying first integral method.

Tascan et al. (2009) applied First Integral method to solve complex nonlinear partial differential equation. They conclude that first integral method can be applied for solving nonlinear partial differential equation in other areas.

Taghizadeh et al. (2010) established exact traveling wave solution of non-linear partial differential equation by usingFirst Integral methodand they obtained exact solution for the Landau-Ginburg-Higgs equation and generalized form of a non-linear Schodinger equation. From this we can say that, by applying first integral method we get valid and effective solution for non linear PDE.

Mehrdoust and Mirzazadeh (2014) applied First Integral method and obtained analytical solution of Black & Scholes partial differential equation, which reduce non-linear Black-Scholes partial differential equation to ordinary differential equation.

III. THE BLACK-SCHOLES PDE

Asset price S_t is said to follow a GBM if it satisfies the following stochastic differential equation.

$$dS_t = \mu S_t + \sigma S_t dW_t \qquad (1)$$

where W is a Brownian motion, μ is the percentage drift, σ is the percentage volatility (μ and σ are constant) S_0 be the price of stock at initial time, S_t be the price of a stock at time t and E be the exercise price.

Let $V = V(S,t)$ denote the price of an option on a stock, T is a expiration time.

The value of call option at expiration is

$$V(S,t) = \max\{S - E, 0\}$$

The value of put option at expiration is

$$V(S,t) = \max\{E - S, 0\}$$

with the help of Δ-hedging technique we can derive Black-Scholes model.

Let us consider a portfolio

$$\pi = V - \Delta S \qquad (2)$$

where Δ = units of the underlying stock

Portfolio π starts at time t, and Δ remains unvaried in $(t, t+dt)$ interval, such that the π becomes risk-free means the return of the portfolio at expiration time.

$$rdt = \frac{\pi_{t+dt} - \pi_t}{\pi_t} \qquad (3)$$

or

As time changes, we immediately readjust the portfolio to account for the change in $V$and S_t. We ensure that $d\pi = r\pi dt$ at every moment time.

$$d\pi = r\pi dt$$

by the equation (2) we get

$$dV_t - \Delta dS_t = r\pi_t dt$$

$$dV_t - \Delta dS_t = r(V_t - \Delta S_t)dt \qquad (4)$$

$S_t$satisfies the stochastic differential equation (1)

From Ito's lemma, we obtain

$$dV_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S}\right)dt + \sigma S \frac{\partial V}{\partial S} dW_t \qquad (5)$$

From equation (4)

$$dV_t = r(V_t - \Delta S_t)dt + \Delta dS_t \qquad (6)$$

by equation (1)

$$dS_t = \mu S_t dt + \sigma S_t dB$$

$$dV_t = r(V - \Delta S)dt + \Delta(\mu S dt + \sigma S dW_t)$$

$$= r(V - \Delta S)dt + \Delta\mu S dt + \Delta\sigma S dW_t \qquad (7)$$

by equation (5) and (7) we have

$$r(V - \Delta S)dt + \Delta\mu S dt + \Delta\sigma S dW_t =$$

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{\partial V}{\partial S}\mu S\right)dt + \frac{\partial V}{\partial S}\sigma S dW_t$$

$$r(V-\Delta S)dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{\partial V}{\partial S}\mu S\right)dt$$

$$+\frac{\partial V}{\partial S}\sigma S dW_t - \Delta\mu S dt - \Delta\sigma S dW_t$$

$$r(V-\Delta S)dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{\partial V}{\partial S}\mu S - \Delta\mu S\right)dt$$

$$+\left(\frac{\partial V}{\partial S}\sigma S - \Delta\sigma S\right)dW_t \qquad (8)$$

Since we assume that the change over any time step $(t, t+dt)$ is non-random, the coefficient of the random term dW_t on the left hand side must be zero.

For this purpose, we choose $\Delta = \frac{\partial V}{\partial S}$

$$\left(rV - r\frac{\partial V}{\partial S}S\right)dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad (9)$$

This is called Black-Scholes equation which defines flow of the option price.

So, in order to determine the option pricing value at any time in $[0, T]$ We need to solve the following partial differential equation in the domain $\Omega = \{(S,t): 0 \le S < \infty, 0 \le t \le T\}$,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad (10)$$

with boundary condition, $V(S,t) = \max\{S-E, 0\}$ for call option and $V(S,t) = \max\{E-S, 0\}$ for put option.

By setting $x = lnS$ and $\tau = T - t$

For $S = e^x$,

by chain rule,

$$\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x}\cdot\frac{1}{S}, \frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 V}{\partial x^2}\cdot\frac{1}{S^2} - \frac{1}{S^2}\cdot\frac{\partial V}{\partial x}$$

For $\tau = T - t, t = T - \tau, \frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}$

putting all the values in Black& Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

$$\left(-\frac{\partial V}{\partial \tau}\right) + \frac{1}{2}\sigma^2 S^2\frac{1}{S^2}\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}\right) + r\left(\frac{\partial V}{\partial x}\right) - rV = 0$$

$$\left(-\frac{\partial V}{\partial \tau}\right) + \frac{1}{2}\sigma^2\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}\right) + r\left(\frac{\partial V}{\partial x}\right) - rV = 0$$

$$\left(-\frac{\partial V}{\partial \tau}\right) + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial x} - rV = 0$$

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} - \left(r - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial x} + rV = 0 \qquad (11)$$

Given equation (11) is a Cauchy problem of a parabolic equation with constant coefficients subject to the boundary condition, $V(S,t) = \max\{S-E, 0\}$ for call option and $V(S,t) = \max\{E-S, 0\}$ for put option.

IV. FIRST INTEGRAL METHOD

Let us take the non linear partial differential equation in the form

$$\Psi(u, u_x, u_t, u_{xx}, u_{xt} \ldots) = 0 \qquad (12)$$

where $u = u(x,t)$ is the solution of non-linear partial differential equation.

Here,we consider the following transformation

$$u(x,t) = f(\xi), \qquad (13)$$

where $\xi = x - ct$.

By chain rule,

$$\frac{\partial u}{\partial t} = -C\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}, \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}, \frac{\partial^2 u}{\partial x\partial t} = -C\frac{\partial^2 u}{\partial \xi^2}$$

which is same as,

$$\frac{\partial}{\partial t}(\cdot) = -C\frac{\partial}{\partial \xi}(\cdot), \frac{\partial}{\partial x}(\cdot) = \frac{\partial}{\partial \xi}(\cdot), \frac{\partial^2}{\partial x^2}(\cdot) = \frac{\partial^2}{\partial \xi^2}(\cdot),$$

$$\frac{\partial^2}{\partial x\partial t}(\cdot) = -C\frac{\partial^2}{\partial \xi^2}(\cdot), \qquad (14)$$

and so on with the help of equation (14) we can change the non-linear partial differential equation (12) to the following non-linear differential equation

$$G\left(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, \ldots\right) = 0. \qquad (15)$$

Let us take up a new independent variable

$$X(\xi) = f(\xi), Y(\xi) = \frac{\partial f(\xi)}{\partial \xi} \qquad (16)$$

which conducts a system of non-linear ordinary differential equations,

$$X(\xi) = f(\xi)$$

$$X'(\xi) = f'(\xi) = Y(\xi)$$

$$Y'(\xi) = f''(\xi) = \varphi\big(X(\xi), Y(\xi)\big) \qquad (17)$$

According to the qualitative theory of ordinary differential equation if we can find the integrals to equation (17) under the same condition we can find the general solution to (16) directly but given plane is in autonomous system so it is difficult for us to realize this even for one first integral. There is no similar theory that can tell us how to find its first integrals.

Division theorem:

Theorem: Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$ and also $P(w,z)$ is irreducible in $C[w,z]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ suchthat $Q(w,z) = P(w,z)G(w,z)$.

We obtain first integral by using the division theorem to equation (16) which reduces equation (15) to a first order integrable ordinary differential equation. By solving this equation we obtain equation (12).

V. SOLUTION OF BLACK-SCHOLE EQUATION WITH FIM

Take,

$$x = lnS$$

$$\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x}\cdot\frac{1}{S}, \frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 V}{\partial x^2}\cdot\frac{1}{S^2} - \frac{1}{S^2}\cdot\frac{\partial V}{\partial x}$$

$$\tau = \frac{\sigma^2}{2}(T-t), \frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau}\cdot\left(-\frac{\sigma^2}{2}\right)$$

Now, we take Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

$$-\frac{\sigma^2}{2}\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2\frac{1}{S^2}\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}\right) + rS\left(\frac{1}{S}\frac{\partial V}{\partial x}\right) - rV = 0$$

divide through by $-\frac{\sigma^2}{2}$,

$$-\frac{\partial V}{\partial \tau} + \left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}\right) + \frac{2r}{\sigma^2}\left(\frac{\partial V}{\partial x}\right) - \frac{2r}{\sigma^2}V = 0$$

$$-\frac{\partial V}{\partial \tau} + \frac{\partial^2 V}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial V}{\partial x} - \frac{2r}{\sigma^2}V = 0$$

$$-\frac{\partial V}{\partial \tau} + \frac{\partial^2 V}{\partial x^2} + (k-1)\frac{\partial V}{\partial x} - kV = 0$$

$$\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2} + (k-1)\frac{\partial V}{\partial x} - kV$$

Now, we solve this Black-Scholes PDE

$$\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2} + (k-1)\frac{\partial V}{\partial x} - kV \qquad (18)$$

with boundary condition, $V(T,S) = \max(e^x - E)$ for call option and $k = \frac{2r}{\sigma^2}$ is real constant.

Let us take following transformation,

$$V(x,,t) = f(\xi), \xi = x - ct$$

with the help of chain rule we get,

$$\frac{\partial V}{\partial \tau} = -c\frac{\partial V}{\partial \xi}, \frac{\partial V}{\partial x} = \frac{\partial V}{\partial \xi}, \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial \xi^2}$$

with the help of the chain rule we get,

$$-c\frac{\partial V}{\partial \xi} = \frac{\partial^2 V}{\partial \xi^2} + (k-1)\frac{\partial V}{\partial \xi} - kV$$

$$-c\frac{\partial f(\xi)}{\partial \xi} = \frac{\partial^2 f(\xi)}{\partial \xi^2} + (k-1)\frac{\partial f(\xi)}{\partial \xi} - kf(\xi)$$

$$-cf' = f'' + (k-1)f' - f \qquad (19)$$

by equation (16) and (17) we get,

$$X'(\xi) = Y(\xi) \qquad (20)$$

$$Y'(\xi) = f''(\xi)$$

$$Y'(\xi) = (1-c-k)f' + kf$$

$$Y'(\xi) = (1-c-k)Y(\xi) + kX(\xi) \qquad (21)$$

As stated in FIM, we suppose that $X(\xi)$ and $Y(\xi)$are nontrivial solutions of (20) and (21), and

$Q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$Q\big(X(\xi), Y(\xi)\big) = \sum_{i=0}^{m} a_i\big(X(\xi)\big)Y^i(\xi) = 0 \qquad (22)$$

where $a_i(X)$, $i = 0,1 \dots m$ are polynomials of X and $a_m(X) \neq 0$. Equation (22) is the first integral to (20) and (21). According to the division theorem we can say that there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dQ}{d\xi} = \frac{dQ}{dX}\frac{dX}{d\xi} + \frac{dQ}{dY}\frac{dY}{d\xi}$$

$$\frac{dQ}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i \qquad (23)$$

Suppose that $m = 1$ by comparing with the coefficients of Y^i, $i = 2,1,0$ on both sides of (23) we have

$$\frac{dQ}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i$$

$$Q\big(X(\xi), Y(\xi)\big) = \sum_{i=0}^{m} a_i\big(X(\xi)\big)Y^i(\xi)$$

We take $m = 1$

$$Q\big(X(\xi), Y(\xi)\big) = \sum_{i=0}^{1} a_i\big(X(\xi)\big)Y^i(\xi)$$

$$= a_0\big(X(\xi)\big)Y^0(\xi) + a_1\big(X(\xi)\big)Y^1(\xi)$$

$$\frac{dQ}{d\xi} = \frac{d}{d\xi}[a_0\big(X(\xi)\big)Y^0(\xi) + a_1\big(X(\xi)\big)Y^1(\xi)$$

$$= \frac{d}{d\xi}[a_0\big(X(\xi)\big)1 + a_1\big(X(\xi)\big)Y^1(\xi)]$$

$$= a_0'\big(X(\xi)\big)X'(\xi) + a_1'\big(X(\xi)\big)X'(\xi)Y^1(\xi)$$

$$+ a_1\big(X(\xi)\big)Y'(\xi)$$

$$\frac{dQ}{d\xi} = a_0'\big(X(\xi)\big)Y(\xi) + a_1'\big(X(\xi)\big)Y(\xi)Y^1(\xi)$$

$$+ a_1\big(X(\xi)\big)Y'(\xi)$$

For,

$$(g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i$$

$$= (g(X) + h(X)Y)\sum_{i=0}^{1} a_i(X)Y^i$$

$$= (g(X) + h(X)Y)[a_0(X) + a_1(X)Y^1]$$

$$= g(X)a_0(X) + g(X)a_1(X)Y^1 + h(X)Ya_0(X)$$

$$+ h(X)Ya_1(X)Y^1$$

putting all the value in equation (23),

$$a_0'\big(X(\xi)\big)Y(\xi) + a_1'\big(X(\xi)\big)Y(\xi)Y^1(\xi) + a_1\big(X(\xi)\big)Y'(\xi) = g(X)a_0(X) + g(X)a_1(X)Y^1 + h(X)Ya_0(X) + h(X)Ya_1(X)Y^1$$

$$a_0'\big(X(\xi)\big)Y(\xi) + a_1'\big(X(\xi)\big)Y(\xi)Y^1(\xi) + a_1\big(X(\xi)\big)[(1 - c - k)Y(\xi) + kX(\xi)] = g(X)a_0(X) + g(X)a_1(X)Y^1 + h(X)Ya_0(X) + h(X)Ya_1(X)Y^1$$

$$a_0'\big(X(\xi)\big)Y(\xi) + a_1'\big(X(\xi)\big)Y(\xi)Y^1(\xi) + a_1\big(X(\xi)\big)(1 - c - k)Y(\xi) + a_1\big(X(\xi)\big)kX(\xi) = g(X)a_0(X) + g(X)a_1(X)Y^1 + h(X)Ya_0(X) + h(X)Ya_1(X)Y^1$$

Now we comparing with coefficients Y^i, $i = 2,1,0$

$$a_1'(X)YY^1 = h(X)Ya_1(X)Y^1$$

$$a_1'(X) = h(X)a_1(X) \qquad (24)$$

$$a_0'(X)Y + a_1(X)(1 - c - k)Y = g(X)a_1(X)Y^1 + h(X)Ya_0(X)$$

$$a_0'(X) + a_1(X)(1 - c - k) = g(X)a_1(X) + h(X)a_0(X) \qquad (25)$$

$$a_1(X)kX = g(X)a_0(X) \qquad (26)$$

Since$a_i(X)$, $i = 0,1$ are polynomials, from equation (24) we deduce that $a_1(X)$ is constant and $h(X)$ is zero. We take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we deduce that $deg\big(g(X)\big) = 0$.

Suppose that $g(X) = A_1$ then we find $a_0(X)$.

By the equation (25)

$$a_0'(X) + a_1(X)(1 - c - k) = g(X)a_1(X) + h(X)a_0(X)$$

$$a_0'(X) + (1)(1 - c - k) = A_1(1) + (0)a_0(X)$$

$$a_0'(X) + (1 - c - k) = A_1$$

$$a_0'(X) = A_1 + c + k - 1$$

$$\int a_0'(X) = \int (A_1 + c + k - 1)$$

$$a_0(X) = (A_1 + c + k - 1)X + A_0 \quad (27)$$

where A_0 is integration constant.

Substituting $a_0(X)$ and $g(X)$ into (26).

$$a_1(X)kX = g(X)a_0(X)$$

$$a_1(X)(kX) = A_1A_0 + \left(A_1{}^2 + cA_1 + kA_1 - A_1\right)X$$

$$A_1A_0 + \left(A_1{}^2 + cA_1 + kA_1 - A_1 - ka_1(X)\right)X = 0$$

Setting all the coefficients of power X to be zero. Then we obtain a system of non-linear algebraic equations and by solving it we obtain a system of algebraic equations and by solving it we obtain.

$$A_1{}^2 + cA_1 + kA_1 - A_1 - ka_1(X) = 0, A_1A_0 = 0$$

$$A_1{}^2 + cA_1 + kA_1 - A_1 - k = 0, A_0 = 0$$

$$a = 1b = c + k - 1c = -k$$

$$\Delta = b^2 - 4ac$$

$$= (c + k - 1)^2 - 4 \cdot 1 \cdot (-k)$$

roots,

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-(c+k-1) \pm \sqrt{(c+k-1)^2 + 4(k)}}{2}$$

$$\frac{-(c+k-1)}{2} \pm \frac{\sqrt{(c+k-1)^2 + 4(k)}}{2}$$

$$\frac{-(c+k-1)}{2} \pm \frac{\sqrt{c^2+k^2+1+2ck-2k-2c+4(k)}}{2}$$

$$A_1 = \frac{-(c+k-1)}{2} \pm \frac{\sqrt{c^2+k^2+1+2ck-2k-2c+4(k)}}{2}$$

$$A_1 = -\frac{c}{2} - \frac{k}{2} + \frac{1}{2} \pm \frac{\sqrt{c^2+k^2+1+2ck-2k-2c+4(k)}}{2} \quad (28)$$

where k and c are arbitrary constants (also verified using Mathematica).

Using the conditions (28) and (22), We obtain

$$Q\left(X(\xi), Y(\xi)\right) = \sum_{i=0}^{m} a_i\left(X(\xi)\right)Y^i(\xi) = 0$$

$$\sum_{i=0}^{m} a_i\left(X(\xi)\right)Y^i(\xi) = 0$$

$$a_0\left(X(\xi)\right)Y^0(\xi) + a_1\left(X(\xi)\right)Y^1(\xi) = 0$$

$$a_0\left(X(\xi)\right) + 1 \cdot Y^1(\xi) = 0$$

$$a_0\left(X(\xi)\right) + Y^1(\xi) = 0$$

$$Y(\xi) = -a_0\left(X(\xi)\right)$$

$$Y(\xi) = -[(A_0 + (A_1 + c + k - 1)X(\xi)]$$

$$Y(\xi) = -A_0 - (A_1 + c + k - 1)X(\xi)$$

$$Y(\xi) = (-A_1 - c - k + 1)X(\xi)$$

$$= \left(-\frac{c}{2} - \frac{k}{2} + \frac{1}{2} \pm \frac{\sqrt{c^2+k^2+1+2ck-2k-2c+4(k)}}{2}\right)X(\xi)$$

$$Y(\xi) = \left(-\frac{c}{2} - \frac{k}{2} + \frac{1}{2} \pm \frac{\sqrt{c^2+k^2+1+2ck+2k-2c}}{2}\right)X(\xi) \quad (29)$$

By combining (15) and (28), We obtain

$$X'(\xi) = \left(-\frac{c}{2} - \frac{k}{2} + \frac{1}{2} \pm \frac{\sqrt{c^2+k^2+1+2ck+2k-2c}}{2}\right)X(\xi)$$

So, the exact solution of equation (20) is

$$\frac{\partial X(\xi)}{\partial \xi} = \left(-\frac{c}{2} - \frac{k}{2} + \frac{1}{2} \pm \frac{\sqrt{c^2+k^2+1+2ck+2k-2c}}{2}\right)X(\xi)$$

$$\frac{\partial X(\xi)}{X(\xi)} = \left(-\frac{c}{2} - \frac{k}{2} + \frac{1}{2} \pm \frac{\sqrt{c^2+k^2+1+2ck+2k-2c}}{2}\right)\partial\xi$$

$$\int \frac{\partial X(\xi)}{X(\xi)} = \left(-\frac{c}{2} - \frac{k}{2} + \frac{1}{2} \pm \frac{\sqrt{c^2+k^2+1+2ck+2k-2c}}{2}\right)\int \partial\xi$$

$$log X(\xi) = \left(-\frac{c}{2} - \frac{k}{2} + \frac{1}{2} \pm \frac{\sqrt{c^2+k^2+1+2ck+2k-2c}}{2}\right)x$$

$$-ct + \xi_0$$

$$X(\xi) = exp\left(-\frac{c}{2} - \frac{k}{2} + \frac{1}{2} \pm \frac{\sqrt{c^2+k^2+1+2ck+2k-2c}}{2}\right)x$$

$$-ct + \xi_0$$

where ξ_0 is arbitrary integrationconstant.

Then $X(\xi) = f(\xi)$ which is $V(x,t)$ so the exact solution to the Black-Scholes equation can be written as follows,

$$V(x,t) = \exp\left\{\left(-\frac{c}{2}-\frac{k}{2}+\frac{1}{2}\pm\frac{\sqrt{c^2+k^2+1+2ck+2k-2c}}{2}\right)x - ct+\xi_0\right\} \quad \text{(A)}$$

where c is an arbitrary constant, $k=\frac{\sigma^2}{2}$ and ξ_0 is an arbitrary integration constant.

The similar term obtained by Mehrdoust and Mirzazadeh [3], in which we have found some possible error is given as,

$$V(x,t) = \exp\left\{-\frac{c}{2}-\frac{k}{2}+\frac{1}{2}\pm \sqrt{c^2+k^2+1+2ck+2k-2c}(x-ct+\xi_0)\right\} \quad \text{(B)}$$

where c is arbitrary constant, $k=\frac{\sigma^2}{2}$ and ξ_0 is arbitrary integration constant.

While calculating all steps in detail to find option premium we found the error in final equation.

To check whether the equation is negligible or not, we substituted different values of 'c' in equation.

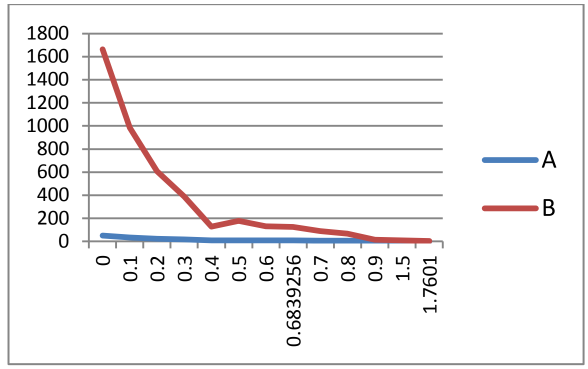
We now show the comparison and differences in the value of the option for different value of c by equation (A) and (B).

VI. EXAMPLE

If we take r = 0.05, σ^2 = 0.375, x = 74.625, t = 1.646

Table 1 Comparison of numerical value of (A) and (B)

Value of "c"	From equation (A)	From equation (B)
0	73.9952	2940.43
0.1	49.5996	1663.97
0.2	34.2684	984.106
0.3	24.3724	606.729
0.4	17.819	388.852
0.5	8.25501	128.644
0.6	10.2827	177.215
0.6839256	8.3722	131.315
0.7	8.08949	125.194
0.8	6.49979	90.7547
0.9	5.3249	67.2824
1.5	2.21191	14.8635
1.7601	1.71503	8.27935
2.0	1.4189	4.78545



Graph 1 Comparison of (A) and (B)

As it is shown in graph, the red line in graph indicates values with error and blue line in graph indicates improved values.

We can see that the red line shows extreme fluctuation. Comparing both the graphs there is a huge change in results.

VII. CONCLUSION

In [3], Mehrdoust and Mirzazadeh gave an analytical solution of Black-Scholes equation with the help of First integral method. We observed an error in the given solution. So, after correcting an error in the existing solution, we work out some numerical examples. We observe from the graph that the current solution is better than solution as proposed in [3].

REFERENCES

[1] A. Shinde,K.Takale, "*Study of Black-Scholes Model and its Applications*", Procedia Engineering, Vol.**38**, pp.**270-279**, **2012.**

[2] Y.Saedi, G.Tularam,*"AReview of the Recent Advances Made in the Black-Scholes Models and Respective Solutions Methods"*, Journal of Mathematics and Statistics, Vol.**14**, Issue.**1**, pp.**29-39**, **2018.**

[3] F. Mehrdoust, M. Mirzazadeh, *"On Analytical Solution of the Black-Scholes Equation by the First Integral Method"*, UPB Sci. Bull., Series A, Vol.**76**, Issue.**4**, pp.**85-90**, **2014**.

[4] F.Rouah,*"Four Derivations of the Black-Scholes PDE"*, FRouah.com, Volopta.com,**2011**.

[5] S.Park, J.Kim,*"Homotopy analysis method for option pricing under stochastic volatility"*, Applied Mathematics Letters, Vol.**24**, Issue.**10**, pp.**1740-1744**, **2011**.

[6] Aghajani, M.Ebadattalab,*"The reduced differential transform method for the Black-Scholes pricing model of European option valuation"*, Int. J. Adv. Appl. Math. and Mech., Vol.**3**, Issue.**1**, pp.**135 – 138**, ISSN: **2347-2529**, **2015**.

[7] Z. Feng, *"The first-integral method to study the Burgers–Korteweg–de Vries equation"*, Journal of Physics A: Mathematical and General, Vol.**35**, Issue.**2**, pp.**343-349**, **2002**.

[8] F. Tascan, A. Bekir, M. Koparan, *"Travelling wave solutions of nonlinear evolution equations by using the first integral*

method", Communications in Nonlinear Science and Numerical Simulation, Vol.**14**, Issue.**5**, pp.**1810-1815**, **2009**.

[9] N. Taghizadeh, M. Mirzazadeh, A. SamieiPaghaleh., *"The first-integral method applied to the Eckhaus equation"*, Applied Mathematics Letters, Vol.**25**, Issue.**5**, pp.**798-802**,**2012**.

[10] M. Bohner, Y. Zheng, *"On analytical solutions of the Black–Scholes equation"*, Applied Mathematics Letters, Vol.**22**, Issue.**3**, pp.**309-313**, **2009**.

[11] M. El-Sabbagh, S. El-Ganaini. *"The first integral method and its applications to nonlinear equations"*, Applied Mathematical Sciences, Vol.**6**, Issue.**77-80**, pp.**3893-3906**,**2012**.

[12] V. Glkac,*"The homotopy perturbation method for the BlackScholes equation"*, Journal of Statistical Computation and Simulation,Vol.**80**, pp.**1349-1354**, **2010**.

[13] J. C. Hull, *"Options, Futures, and Other Derivatives"*, New Jersey: Prentice-Hall, **USA**, 1999.

[14] K. Raslan, *"The first integral method for solving some important nonlinear partial differential equations"*, Nonlinear Dynamics, Vol.**53**, Issue.**4**, pp.**281-286**, 2008.

[15] V. Goodman, J.Sampfli, *"The Mathematics of Finance: Modeling and Hedging"*, The American Society, First Indian Edition, 2013.

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