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Bulk queue model with two phase's service, unreliable server and m phase of repair

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Abstract- In the present investigation, we have considered state dependent unreliable bulk queueing model with two phase of service and m phase of repair. The server provides first phase of service to each arriving unit while second phase is optional, provided as per demand of the unit. The server is subject to breakdown at any instant of services; whenever the server gets break down during the service (any phase), then it is immediately send for repair. The repair of server done in m different phases. We obtain the steady state queue size distribution in explicit and closed form in terms of the probability generating functions for the number of customers in the queue. Various performance indices viz. average number of customers in queue and system, long run probabilities of server being in different states, availability, mean time to failure etc. Some special cases of interest are discussed by setting appropriate parameters. To validate the analytical results, numerical results are obtained by taking an illustration.

Keywords: Queue, Bulk, Essential service, Optional service, Supplementary variable, Server break down, Queue length

I. INTRODUCTION

In most studies on the queueing system, it is assume that after getting essential service units leave the system. However, in many practical systems, we encounter situation where units demands for optional service apart from essential one. To illustrate we site the example of a call centre where server (machine) interact with their customer to resolve their queries. If the customer does not satisfy, he may further approach to next level machine (or customer executive) to get satisfactory answer. Queueing modelling with provision of optional service arises as one of the important areas of queueing theory in past few years.

It is general assumption in analysis of any queueing model that units arrive at service station with uniform arrival rate. However, in real life scenario we come across many queueing situation where the units arrive with varying rate. For example at the teller counter of the bank, the arrival rate of the customer at particular counter, depends upon the servers status i.e. whether the server is on vacation, breakdown or under repair and service rate of the server. Further, in some queueing situation units join the system in a group of random size.

While analyzing a queueing model it one of the general assumption that server is reliable i.e server work regularly without any random failure. However, in day-to-day life, one can easily observe that server failure is always associated with any working server. To illustrate we cite the example of manufacturing industry, where production of items stopped due random failure of any particular machine. Similar situation can be observe in telecommunication network, where communication is interrupted due random failure of any modem. Once the server failed it immediately send for repair. At the repair station, if the channel is free then the repair of the server starts immediately, otherwise the server has to wait for repair. The repair station facilitates the repair in single phase or multiphase as per requirement. The multiphase repair requirement can be observed in many real time systems including at motor vehicles servicing center where each failed vehicle has to go through a number of different phases of repair such as fixing of faulty part, washing, oiling, painting etc.

Choudhury and Tadj (2009) considered a model in which they have assumed that units arrive one by one with uniform arrival rate. But in real life we observe that some time the units may also join the service station in batch and arrival rates may be affected by server status, which may be busy in rendering services to another unit, may be waiting for repair or under the repair. Also in some real life congestion problem repair of server is done in different phases, for example at vehicle servicing center, each unit has to pass through a number of service. These fact motivates us to develop a model dealing with queueing situation, where in units arrive in bulk with state dependent arrival rate may demand optional service with unreliable server, delay time and m phase repair of the server.

The single server queue with uniforms arrival rate (single arrival or batch arrival) have been studied by numerous authors including Gaver [11], Choudhury [7], Avi-ltzhak [2], Bhorthakur[4] and many other. But in real life there may be different queueing scenario i.e. the unit may demands for additional optional service or arrival rate may not be uniform. Keilson and Kooharian [13] Choudhury [8], Baba [3], Medhi [21], Madan [19] have studies various form of second optional service under different assumptions. Rajadurai et.al [22] investigated retrial bulk queue model with modified vacation policy and option re service. They assume that units can rejoin the service without joining the orbit. Recently Chakravarthy[5] study a retrial queue model with optional cooperative service, in which arrival can demands for individual service or cooperative service.

Server failure is one of the important aspect that cannot be ignored in dealing with real life congestion situations. The queueing model with server breakdowns and repairs are more realistic representation of the real life queueing situations. Queueing models with unreliable server are effective tools for performance analysis of manufacturing systems, local area networks, and data communication systems. In queueing literature many prominent researcher contributed in this direction; to refer important contribution, we mention notable worksdone by Jayawardene and Kella [12], Aissani and Artalejo [1], Takine and Sengupta [23], Ke [13,14,15], Federgruen and So [9], Maraghi and Madan [20],Chaun [6] . Wang et al. [28] presented the cost analysis of a queueing system with some specific assumptions subject to server breakdowns. Madan [19] provided the time dependent as well steady state results for a queueing system, in which the service channel can break at any time, even in idle state. Huang et.al.(2014) considered multi-server unreliable queueing model with infinite capacity in which failed servers are sent for repair only if number of failed server reaches some threshold. Further Yang et.al.(2015) investigated an unreliable retrial queue model with assumption that server may take J optional vacation after the first essential vacation. More recently Kuo et.al.(2016) analyses a queueing model in which an unreliable server perform the repair or monitoring work of the failed components. The repairable system includes primary and standby components, in which failed primary component are immediately replaced by stand by components.

In this paper, we study a bulk queue with second optional service. The unit arrives at the service station in batch with state dependent arrival rate. The server is subject to breakdown at any instant during any phase of service. The breakdown server is it immediately sent for repair where it may have to wait some time for the repair called delay time. At the repair station, the repair of server is completed in m different compulsory phases (i.e. once the repair start it must have to pass through all the different phases of repair.)

II. MATHEMATICAL MODEL

In real life we face with many queue where service is provided in two phases. Furthermore flow of the units is influenced by the server status. Keeping in view above , Here we investigating a single server queueing system with two phase service, where server is subject to breakdown during any phase of service. The units arrive in batches of random size X according to Poisson process with different rates λ_1, λ_2 or λ_3 depending upon the status of the system, which may be busy in providing service , waiting for repair or under repair respectively. The following notation are used to formulate the model:

c_j : Probability that the batch of size j arrives

$X(z)$ = probability generating function for batch arrival

$E(X^{(j)})$: j^{th} moment of the random variable X

$B_i(x)$: The commutative distribution function of the service time for i^{th} phase service.

$D_i(y)$: The commutative distribution function of the waiting time of the server for repair during i^{th} phase of service.

$G_{i,l}(y)$:The commutative distribution function of the l^{th} phase repair during i^{th} phases of service.

$g_{il}^{(j)}$: j^{th} moment of $G_{i,l}(y)$

$B_i^*(s)$:Laplace transform of $B_i(x)$.

$\beta_i^{(j)}$: j^{th} moment of service time distribution of i^{th} (i=1,2) phase service .

$D_i^*(s)$: Laplace transform of $D_i(y)$

$G_{i,j}^*(s)$: Laplace transform $G_{i,l}(y)$

Let $N_Q(t)$ be the queue size (including one being served) at time t. Let $\varpi_i(t)$ denote elapsed service time of the customer for ith phase of service at time t and $\psi_i(t)$, $\varphi_{i,l}(t)$, $l = 1,2..m$ denote the elapsed delay time , elapsed repair time of the server respectively for ith phase of service during which breakdown occur in the system φ at time t, Here index i=1,2 denotes first phase service and second phase service respectively. Also we introduce a random variable

$$\zeta(t) = \begin{cases} 0, \ if\ the\ server\ is\ idle\ at\ time\ t \\ 1, \ if\ the\ server\ is\ busy\ with\ first\ phase\ service\ at\ time\ t \\ 2, \ if\ the\ server\ is\ busy\ with\ \sec ond\ phase\ service\ at\ time\ t \\ 3, \ if\ the\ server\ is\ waiting\ for\ repair\ during\ the\ first\ phase\ service\ at\ time\ t \\ 4, \ if\ the\ server\ is\ waiting\ for\ repair\ during\ the\ \sec ond\ phase\ service\ at\ time\ t \\ 5+i, \ if\ the\ server\ is\ under\ ith\ (i=1,2,..m)\ phase\ repair\ at\ time\ t \\ \quad when\ it\ fails\ during\ the\ first\ of\ phase\ service \\ 5+m+j, \ if\ the\ server\ is\ under\ jth\ (j=1,2,..m)\ phase\ repair\ at\ time\ t \\ \quad when\ it\ fails\ during\ the\ \sec ond\ of\ phase\ service \end{cases}$$

Then we have bivariate markav process $\{N_Q(t), X(t)\}$ where

$$X(t) = 0,\ \varpi_1(t),\ \varpi_2(t),\ \psi_1(t),\ \psi_2(t),\ \varphi_1^{(1)}(t),..\varphi_1^{(m)}(t),\ \varphi_2^{(1)}(t),...,\varphi_2^{(m)}(t)\ \ if\ \ \zeta(t) = 0,\ 1,\ 2,...5+2m\ \ respectively$$

With assumption that steady state exist we define the following limiting probabilities

$$P_0^{(0)} = \lim_{t\to\infty} \Pr.\{N_Q(t) = 0, X(t) = 0\}$$

$$P_n^{(i)}(x)dx = \lim_{t\to\infty} \Pr.\{N_Q(t) = n, X(t) = \varpi_i(t); x < \varpi_i(t) \le x + dx\}; n \ge 1, x > 0, i \in \{1,2\}$$

$$D_n^{(i)}(x) = \lim_{t\to\infty} \Pr.\{N_Q(t) = n, X(t) = \psi_i(t); y < \psi_i(t) \le y + dy / \varpi_i(t) = x\};$$
$$n \ge 0, x > 0, y > 0, i \in \{1,2\}$$

$$R_{l,n}^{(i)}(x,y)dy = \lim_{t\to\infty} \Pr.\{N_Q(t) = n, X(t) = \varphi_i^{(l)}(t); y < \varphi_i^{(l)}(t) \le y + dy / \varpi_i(t) = x\}; (x,y) > 0$$
$$\text{for } i \in \{1,2\} \text{ and } l = 1,2,3..m$$

Here we assumed that

$B_i(0) = 0, B_i(\infty) = 1, D_i(0) = 0, D_i(\infty) = 1, G_{i,l}(0) = 0,\ G_{i,l}(\infty) = 1$ for $i = 1,2,$,also $B_i(x)$ and $G_i(y)$, $D_i(y)$ are continuous at $x = 0$ and at $y = 0$ respectively. Then we have the hazard rate function

$$\mu_i(x)dx = \frac{dB_i(x)}{[1 - B_i(x)]},\ \ \eta_i(y)dy = \frac{dD_i(y)}{[1 - D_i(y)]},\ \ \ \xi_{i,l}(x)dx = \frac{dG_{i,l}(y)}{[1 - G_{i,l}(y)]} \text{ for } l = 1,2..m$$

of B_i, D_i and $R_{i,l}$ respectively for $i=1,2$

Further we define the following probability generating functions;

$$D^{(i)}(x,y,z)=\sum_{n=0}^{\infty} z^n D_n^{(i)}(x,y) \quad ; \quad D^{(i)}(x,0,z)=\sum_{n=0}^{\infty} z^n D_n^{(i)}(x,0)$$

$$R_l^{(i)}(x,y,z)=\sum_{n=0}^{\infty} z^n R_{l,n}^{(i)}(x,y) \quad ; \quad R_l^{(i)}(x,0,z)=\sum_{n=0}^{\infty} z^n R_{l,n}^{(i)}(x,0) \text{ Here } l=1,2...m$$

$$P^{(i)}(x,z)=\sum_{n=0}^{\infty} z^n P_n^{(i)}(x)$$

III. GOVERNING EQUATIONS

The steady state equations governing the model are;

$$\frac{d}{dx}P_n^{(i)}(x)+[\lambda_1+\alpha_i+\mu_i(x)]P_n^{(i)}(x)=\sum_{j=1}^{n}\lambda_1 c_j(1-\delta_{n,0})P_{n-j}^{(i)}(x)+\int_0^{\infty}\xi_{i,m}(y)R_{m,n}^{(i)}(x,y)dy \quad (1)$$

$$\frac{d}{dy}D_n^{(i)}(x,y)+[\lambda_2+\eta_i(y)D_n^{(i)}(x,y)=\sum_{j=1}^{n}\lambda_2 c_j(1-\delta_{n,0})D_{n-j}^{(i)}(x,y) \quad (2)$$

$$\frac{d}{dy}R_{l,n}^{(i)}(x,y)+[\lambda_3+\xi_{1,l}(y)]R_{l,n}^{(i)}(x,y)=\sum_{j-1}^{n}\lambda_3 c_j(1-\delta_{n,0})R_{l,n-j}^{(i)}(x,y) \quad (3)$$

$$\lambda_1 P_0^{(0)}=\int_0^{\infty}\mu_2(x)P_1^{(2)}(x)dx+q\int_0^{\infty}\mu_1(x)P_1^{(1)}(x)dx \quad (4)$$

where $\delta_{i,j}$ denotes kronecker's delta function

These set of equations (3.1)-(3.7) are to be solved under the boundary condition at $x=0$

$$P_n^{(1)}(0)=\int_0^{\infty}\mu_2(x)P_{n+1}^{(2)}(x)dx+q\int_0^{\infty}\mu_1(x)P_{n+1}^{(1)}(x)dx+\lambda_1 c_n P_0^{(0)} \quad (5)$$

$$P_n^{(2)}(0)=p\int_0^{\infty}\mu_1(x)P_n^{(1)}(x)dx \qquad n\geq 1 \quad (6)$$

At $y=0$ for $i=1,2$ and fixed value of x, we get

$$D_n^{(i)}(x,0)=\alpha_i P_n^{(i)}(x) \qquad x\geq 0, n\geq 1 \quad (7)$$

$$R_{1,n}^{(i)}(x,0)=\int_0^{\infty}\eta_i(y)D_n^{(i)}(x,y)dy \quad x>0, n\geq 1 \quad (8)$$

$$R_{l,n}^{(i)}(x,0)=\int_0^{\infty}\xi_{i,l-1}(y)R_{l-1,n}^{(i)}(x,y)dy \qquad 1<l\leq m \quad (9)$$

With normalizing condition is given by

$$P_0^{(0)}+\sum_{i=0}^{2}\sum_{n=1}^{\infty}\{\int_0^{\infty}P_n^{(i)}(x)dx+\int_0^{\infty}\int_0^{\infty}D_n^{(i)}(x,y)dxdy+\int_0^{\infty}\int_0^{\infty}\sum_{l=1}^{m}R_{l,n}^{(i)}(x,y)dxdy\}=1 \quad (10)$$

IV. MATHEMATICAL ANALYSIS

For brevity, we introduce some notation as follows;

$\phi_1(z) = \lambda_1(1-X(z))$, $\phi_2(z) = \lambda_2(1-X(z))$, $\phi_3(z) = \lambda_3(1-X(z))$

Multiplying equations (2)-(3), (7)-(9) by suitable powers of z^n and adding, after simplification we get

$$D^{(i)}(x,y,z) = D^{(i)}(x,0,z)\exp\{-\phi_2(z)y\}[1-D_i(y)] \quad for\ (x,y)>0 \tag{11}$$

$$R_l^{(i)}(x,y,z) = R_l^{(i)}(x,0,z)\exp\{-\phi_3(z)y\}[1-G_{i,l}(y)] \quad for\ 1\le l\le m \tag{12}$$

$$D^{(i)}(x,0,z) = \alpha_i P^{(i)}(x,z) \tag{13}$$

$$R_1^{(i)}(x,0,z) = D^{(i)}(x,0,z)\gamma_i^*(\phi_2(z)) \tag{14}$$

$$R_l^{(i)}(x,0,z) = R_{l-1}^{(i)}(x,0,z)G_{i,l-1}^*(\phi_3(z)) \qquad 1<l\le m \tag{15}$$

Multiplying equation (1) by suitable powers of z^n then taking summation over all possible of n and using equations (13)-(15) we get

$$P^{(i)}(x,z) = P^{(i)}(0,z)[1-B_i(x)]\exp\{-\tau_i(z)x\} \tag{16}$$

where $\tau_i(z) = \phi_1(z) + \alpha_i(1-\gamma_i^*(\phi_2(z))\prod_{j=1}^{m} G_{ij}^*(\phi_3(z)))$

Multiply (5) and (6) by z^n and then taking summation over all possible of n and after simplification we get

$$zP^{(1)}(0,z) = -z\phi_1(z)P_0^{(0)} + qP^{(1)}(0,z)B_1^*(\tau_1(z)) + P^{(2)}(0,z)B_2^*(\tau_2(z)) \tag{17}$$

$$P^{(2)}(0,z) = pP^{(1)}(0,z)B_1^*(\tau_1(z)) \tag{18}$$

From (17) and (18) equation, we get

$$P^{(1)}(0,z) = \frac{z\phi_1(z)P_0^{(0)}}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z)) - z]} \tag{19}$$

From equation (11) and (13) ,we have

$$D^{(i)}(x,y,z) = \alpha_i P^{(i)}(0,z)[1-B_i(x)]\exp\{-\tau_i(z)x\}[1-D_i(y)\exp\{-\phi_2(z)y\} \tag{20}$$

Similarly, from equations (12)-(15), we have

$$R_l^{(i)}(x,y,z) = \alpha_i P^{(i)}(0,z)[1-B_i(x)\exp\{-\tau_i(z)x\}\gamma_i^*(\phi_2(z))[\prod_{j=1}^{l-1} G_{i,j}^*(\phi_3(z))] \times[1-G_{i,l}(y)]\exp\{-\phi_3(z)y\} \quad l=1,2..m \tag{21}$$

Take $z\to 1$ in equation (18) and (19) we get

$$P^{(2)}(0,1) = pP^{(1)}(0,1) \tag{22}$$

$$P^{(1)}(0,1) = \frac{\lambda_1 P_0^{(0)}E(X)}{1-r_1} \tag{23}$$

Taking limit $z\to 1$ in equations (16),(20), (21) and using (22),(23) we get

$$p^{(1)}(x,1) = \frac{\lambda_1 P_0^{(0)}E(X)}{1-r_1}[1-B_1(x)] \tag{24}$$

$$P^{(2)}(x,1) = \frac{\lambda_1 pP_0^{(0)}E(X)}{1-r_1}[1-B_2(x)] \tag{25}$$

$$D^{(1)}(x,y,1) = \alpha_1\frac{\lambda_1 P_0^{(0)}E(X)}{1-r_1}[1-B_1(x)][1-D_1(y)] \tag{26}$$

$$D^{(2)}(x,y,1)=\alpha_2\frac{p\lambda_1 P_0^{(0)}E(X)}{1-r_1}[1-B_2(x)][1-D_2(y)] \quad (27)$$

$$R_l^{(1)}(x,y,1)=\alpha_1\frac{\lambda_1 P_0^{(0)}E(X)}{1-r_1}[1-B_1(x)][1-G_{1,l}(y)] \qquad 1\le l\le m \quad (28)$$

$$R_l^{(2)}(x,y,1)=\alpha_2\frac{p\lambda_1 P_0^{(0)}E(X)}{1-r_1}[1-B_2(x)][1-G_{2,l}(y)] \qquad 1\le l\le m \quad (29)$$

Using above values from (24) to (29) in normalizing condition (10) we have

$$P_0^{(0)}=\frac{1-r_1}{1+r_2}=1-\rho \quad (30)$$

$$r_1=\beta_1^{(1)}E(X)[\lambda_1+\alpha_1(\lambda_2\gamma_1^{(1)}+\lambda_3\sum_{j=1}^{m}g_{1j}^{(1)})]+p\beta_2^{(1)}E(X)[\lambda_1+\alpha_2(\lambda_2\gamma_2^{(1)}+\lambda_3\sum_{j=1}^{m}g_{2j}^{(1)})]$$

$$r_2=\beta_1^{(1)}E(X)\alpha_1[((\lambda_1-\lambda_2)\gamma_1^{(1)}+(\lambda_1-\lambda_3)\sum_{j=1}^{m}g_{1j}^{(1)})]+p\beta_2^{(1)}E(X)\alpha_2[((\lambda_1-\lambda_2)\gamma_2^{(1)}+(\lambda_1-\lambda_3)\sum_{j=1}^{m}g_{2j}^{(1)})]$$ and $\rho<1$

is the stability condition of the system.

From equations (16), (19)-(21) and (30), we get following result.

Theorem 1:*Under the stability condition* $\rho<1$ *the joint distribution of the server and the queue size has the following partial PGFs given by*

$$P^{(1)}(x,z)=\frac{z\phi_1(z)(1-\rho)[1-B_1(x)\exp\{\tau_1(z)x\}}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]} \quad (31)$$

$$P^{(2)}(x,z)=\frac{pz\phi_1(z)(1-\rho)B_1^*(\tau_1(z))[1-B_2(x)\exp\{-\tau_2(z)x\}}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]} \quad (32)$$

$$D^{(1)}(x,y,z)=\frac{\alpha_1 z\phi_1(z)(1-\rho)[1-B_1(x)][1-D_1(y)\exp\{-\tau_1(z)x\}\exp\{-\phi_2(z)y\}}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]} \quad (33)$$

$$D^{(2)}(x,y,z)=\frac{\alpha_2 z\phi_1(z)(1-\rho)B_1^*(\tau_1(z))[1-B_2(x)][1-D_2(y)]\times\exp\{-\tau_2(z)x\}\exp\{-\phi_2(z)y\}}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]} \quad (34)$$

$$R_l^{(1)}(x,y,z)=\frac{\alpha_1 z\phi_1(z)(1-\rho)[1-B_1(x)]\exp\{-(\tau_1(z)x\}\gamma_1^*(\phi_2(z))\times\prod_{j=1}^{l-1}G_{1,j}^*(\phi_3(z)))[1-G_{1,l}(y)\exp\{-\phi_3(z)y\}}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]} \quad (35)$$

$$R_l^{(2)}(x,y,z)=\frac{\alpha_2 pz\phi_1(z)(1-\rho)B_1^*(\tau_1(z))[1-B_2(x)]\exp\{-(\tau_2(z)x\}\gamma_2^*(\phi_2(z))\times\prod_{j=1}^{l-1}G_{2,j}^*(\phi_3(z)))[1-G_{2,l}(y)\exp\{-\phi_3(z)y\}}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]} \quad (36)$$

Here we assume that $1\le l\le m$ and $\prod_{j=1}^{0}G_{i,j}^*(\phi_3(z))=1$, for $i=1,2.$

***Theorem2**: Under the stability condition* $\rho<1$ *the marginal PGF of the queue size distribution of server state are given by*

$$P^{(1)}(z)=\frac{z\phi_1(z)(1-\rho)[1-B_1^*(\tau_1(z))]}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]\{\tau_1(z)\}} \quad (37)$$

$$P^{(2)}(z)=\frac{pz\phi_1(z)(1-\rho)[1-B_2^*(\tau_2(z))]B_1^*(\tau_1(z))}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]\{\tau_2(z)\}} \quad (38)$$

$$D^{(1)}(z)=\frac{\alpha_1 z\phi_1(z)(1-\rho)[1-B_1^*(\tau_1(z))][1-\gamma_1^*(\phi_2(z))]}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]\{\tau_1(z)\}\{\phi_2(z)\}} \quad (39)$$

$$D^{(2)}(z)=\frac{p\alpha_2 z\phi_1(z)(1-\rho)[1-B_2^*(\tau_2(z))]B_1^*(\tau_1(z))][1-\gamma_2^*(\phi_2(z))]}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]\{\tau_2(z)\}\{\phi_2(z)\}} \quad (40)$$

$$R_l^{(1)}(z)=\frac{\alpha_1 z\phi_1(z)(1-\rho)[1-B_1^*(\tau_1(z))]\gamma_1^*(\phi_2(z))\{\prod_{j=1}^{l-1}G_{1,j}^*(\phi_3(z))\}[1-G_{1,l}^*(\phi_3(z))]}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]\{\tau_1(z)\}\{\phi_3(z))\}} \quad (41)$$

$$R_l^{(2)}(z)=\frac{\alpha_2 pz\phi_1(z)(1-\rho)[1-B_2^*(\tau_2(z))]B_1^*(\tau_1(z))\gamma_2^*(\phi_2(z))\{\prod_{j=1}^{l-1}G_{2,j}^*(\phi_3(z))\}[1-G_{2,l}^*(\phi_3(z))]}{[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]\{\tau_2(z)\}\{\phi_3(z)\}}$$

(42)

Here we assume that $1\le l\le m$ and $\prod_{j=1}^{0}G_{i,j}^*(\phi_3(z))=1,\ for\, i=1,2.$

Proof: Appendix A.

***Theorem 3:** The probability generating function of the stationary queue size at departure epoch under the stability condition* $\rho<1$ *is given by*

$$\omega(z)=\frac{(1-r_1)\phi_1(z)[q+pB_2^*(\tau_2(z))]B_1^*(\tau_1(z))}{\lambda_1 E(X)[(q+pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))-z]} \quad (43)$$

Proof: Appendix B.

V. PERFORMANCE MEASURES

In this section we derive the expressions for long run probabilities of the server states and various performance measures as follows;

(a) Long run Probabilities of the server state:

(i) The probability that the server is busy with first phase of service.

$$P(B_1)=\lim_{z\to1}P^{(1)}(z)=\frac{\lambda_1\beta_1^{(1)}E(X)}{1+r_2} \quad (44)$$

(ii) The probability that the server is busy with second phase service.

$$P(B_2)=\lim_{z\to1}P^{(2)}(z)=\frac{p\lambda_1\beta_2^{(1)}E(X)}{1+r_2} \quad (45)$$

(iii) The probability that the server is waiting for repair during the first phase service.

$$P(D_1)=\lim_{z\to1}D^{(1)}(z)=\frac{\alpha_1\lambda_1\beta_1^{(1)}\gamma_1^{(1)}E(X)}{1+r_2} \quad (46)$$

(iv) The probability that the server is waiting for repair when failed during the second phase service.

$$P(D_2) = \lim_{z\to 1} D^{(2)}(z) = \frac{\alpha_2 p\lambda_1\beta_2^{(1)}\gamma_2^{(1)}E(X)}{1+r_2} \tag{47}$$

(v) The probability that server is under l phase repair when failed during the first phase service.

$$P(R_{1l}) = \lim_{z\to 1} R_l^{(1)}(z) = \frac{\alpha_1\lambda_1\beta_1^{(1)}g_{1l}^{(1)}E(X)}{1+r_2} \tag{48}$$

(vi) The probability that server is under l phase repair when failed during the second phase service.

$$P(R_{2l}) = \lim_{z\to 1} R_l^{(2)}(z) = \frac{\alpha_2 p\lambda_1\beta_2^{(1)}g_{2l}^{(1)}E(X)}{1+r_2} \tag{49}$$

The probability that server is idle is given by

$$P(I) = 1 - P(B_1) - P(B_2) - P(W_1) - P(W_2) - \sum_{l=1}^{m} P(R_{1l}) - \sum_{l1}^{m} P(R_{2l}) = 1-\rho \tag{50}$$

(b) Mean queue length

On differentiating equation (43) with respect to z and setting z=1 we get the mean queue length

$$E(N_Q(t) = r_1 + \frac{E(X^{(2)})}{2E(X)} + \frac{2p\beta_1^{(1)}\beta_2^{(1)}(E(X))^2((\lambda_1+\alpha_1(\lambda_2\gamma_1^{(1)}+\lambda_3\sum_{j=1}^{m}g_{1j}^{(1)}))((\lambda_1+\alpha_2(\lambda_2\gamma_2^{(1)}+\lambda_3\sum_{j=1}^{m}g_{2j}^{(1)}))}{2(1-r_1)}$$

$$+\frac{p\beta_2^{(1)}\left[\begin{array}{l}\lambda_1E(X^{(2)})+\alpha_2\{2\lambda_2\lambda_3(E(X))^2\gamma_2^{(1)}\sum_{j=1}^{m}(g_{2j}^{(1)})+2\lambda_3^2(E(X))^2\sum_{\substack{i\neq j\\1\le i<j\le m}}(g_{2i}^{(1)}g_{2j}^{(1)})\\+[\lambda_2E(X^{(2)})\gamma_2^{(1)}+\lambda_2^2(E(X))^2\gamma_2^{(2)}]+\sum_{j=1}^{m}\lambda_3E(X^{(2)})g_{2j}^{(1)}+\lambda_3^2(E(X))^2g_{2j}^{(1)}\end{array}\right] + p\beta_2^{(2)}\left[(\lambda_1+\alpha_2(\lambda_2\gamma_2^{(1)}+\lambda_3\sum_{j=1}^{m}g_{2j}^{(1)}))E(X)\right]^2}{2(1-r_1)}$$

$$+\frac{\beta_1^{(1)}\left[\begin{array}{l}\lambda_1E(X^{(2)})+\alpha_1\{2\lambda_2\lambda_3(E(X))^2\gamma_1^{(1)}\sum_{j=1}^{m}(g_{1j}^{(1)})+2\lambda_3^2(E(X))^2\sum_{\substack{i\neq j\\1\le i<j\le m}}(g_{1i}^{(1)}g_{1j}^{(1)})\\+[\lambda_2E(X^{(2)})\gamma_1^{(1)}+\lambda_2^2(E(X))^2\gamma_1^{(2)}]+\sum_{j=1}^{m}\lambda_3E(X^{(2)})g_{1j}^{(1)}+\lambda_3^2(E(X))^2g_{1j}^{(1)}\end{array}\right] + \beta_1^{(2)}\left[(\lambda_1+\alpha_1(\lambda_2\gamma_1^{(1)}+\lambda_3\sum_{j=1}^{m}g_{1j}^{(1)}))E(X)\right]^2}{2(1-r_1)} \tag{51}$$

$$E(W) = \frac{E(N_Q(t))}{\lambda E(X)} \tag{52}$$

Proof: See appendix D.

(c). Reliability indices

Let $A(t)$ be the system availability at time t. Then steady state availability A_v ,which is the probability that the server is either working for a unit or in an idle state, as

$$A_v = P_0^{(0)} + \lim_{z\to 1}\{P^{(1)}(z) + P^{(2)}(z)\} \tag{53}$$

$$A_v = 1 - \lambda E(X)\beta_1^{(1)}\alpha_1(\gamma_1^{(1)} + \sum_{j=1}^{m} g_{1j}^{(1)}) - \lambda E(X)\beta_2^{(1)}\alpha_2 p(\gamma_2^{(1)} + \sum_{j=1}^{m} g_{1j}^{(1)}) \tag{54}$$

Now, steady state failure frequency Ff is obtained using

$$Ff = \alpha_1 \int_0^\infty p_1(x,1)dx + \alpha_2 \int_0^\infty p_2(x,1)dx = \alpha_1\lambda\beta_1^1 E(X) + \alpha_2\lambda\beta_2^1 E(X) \tag{55}$$

VI. SPECIAL CASES

In this section, we evaluate some special case by setting appropriate parameter to validate our result with existing models.

Case (i): For $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ *and* $m = 1$ and $X(z) = z,\ E(X) = 1,$; equation (43) provides

$$\omega(z) = \frac{(1-\rho_1)(1-z)(q + pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))}{[(q + pB_2^*(\tau_2(z)))B_1^*(\tau_1(z)) - z]} \tag{56}$$

with $\tau_i(z) = \lambda(1-z) + \alpha_i\left[1 - \overline{D}_i(\lambda(1-z))\overline{G}_{i,1}(\lambda(1-z))\right]$

and $\rho_1 = \lambda E(B_1)\left[1 + \alpha_1(\gamma_1^{(1)} + g_{11}^{(1)})\right] + p\lambda E(B_2)\left[1 + \alpha_2(\gamma_2^{(1)} + g_{21}^{(1)})\right]$

The present model reduces to the model studied by Chaudhury and Tadj (2009).

Case (ii): If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ *and* $m = 1$ and $X(z) = z,\ E(X) = 1, \alpha_1 = \alpha_2 = 0$ in equation (43) we get

$$\omega(z) = \frac{(1-\rho_1)(1-z)(q + pB_2^*(\tau_2(z)))B_1^*(\tau_1(z))}{[(q + pB_2^*(\tau_2(z)))B_1^*(\tau_1(z)) - z]} \tag{57}$$

with $\tau_i(z) = \lambda(1-z)$ $\rho_1 = \lambda E(X)(E(B_1) + pE(B_2))$

This model reduced to model obtained by Medhi (2002) for single optional service.

VII. NUMERICAL ILLUSTRATION

In this section we present some tables and graph to show the effect of various parameters on the queue length, response time. To facilitate numerical results, first all we assume that batch size follows geometrical distribution with parameter a then the first and second moment of the batch size distribution is given by

$$E(X) = \frac{b}{a}, E(X^2) = \frac{b(1+b)}{a^2}; \quad \text{where } b = 1 - a.$$

The distribution of essential and optional service time are taken as k-Erlangian, so that their first and second moment used for numerical computation are obtained using

$$\beta_i^{(1)} = \frac{1}{\mu_i}, \beta_i^{(2)} = \frac{k+1}{k\mu_i^2} \quad \textit{for } i = 1,2,$$ where μ_i denote the service rate.

Further, delay time distribution is taken 2-Erlangian distribution with parameters $\gamma_i^{(1)} = \dfrac{1}{\gamma_i}, \gamma_i^{(2)} = \dfrac{3}{2\gamma_i^2}$ $\gamma_i(i = 1,2),$, with moments. Again the repair time distribution is taken exponential distribution with parameter g_{ij} with moments

$$g_{ij}^{(1)} = \frac{1}{g_{ij}}, g_{ij}^{(2)} = \frac{2}{g_{ij}^2} \quad \text{where } i = 1,2 \text{ and } j = 1,2..,m$$

To develop computer program, the coding is done in MATLAB. Now we summarize the numerical results to show the effect of various parameter on the various performance.
For figures 1-4, we set the default parameters as follows:

$$E(X)=3, \mu_2=1.2\mu_1, p=0.6, m=2, \alpha=0.1, \alpha_1=\alpha, \alpha_2=0.8\alpha, k=1, \lambda_1=1.2\lambda,$$
$$\lambda_2=\lambda, \lambda_3=0.8\lambda$$

In Figure (1),we examine the effect arrival rate on the average queue length (L_q) . By increasing λ , initially average queue length increase gradually then increases sharply;, further, as the number of phases of service increase the queue length decrease . Figure(2) show the effect of failure rate on the average queue length for different distribution; it is found that there is significant increase in the average queue length (L_q) with the increase in failure rate (α) . Figure (3) reveals the effect of number of phases of repair on the queue length (L_q). It clear that with the increase in number of phases of repair, the average queue length goes on increasing, As the number of phases of repair increase the server require more time under repair and hence queue length goes on increasing ; the increasing trend is more prominent for higher value of λ .

Figure (4) shows the effect of optional probability (p) on the average queue length (L_q); the increasing trend of (L_q) is quite significant for higher value.

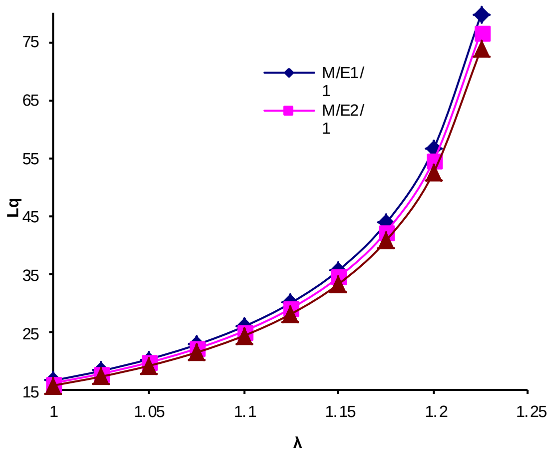

Fig. 1: L_q **vs.** λ for different service time

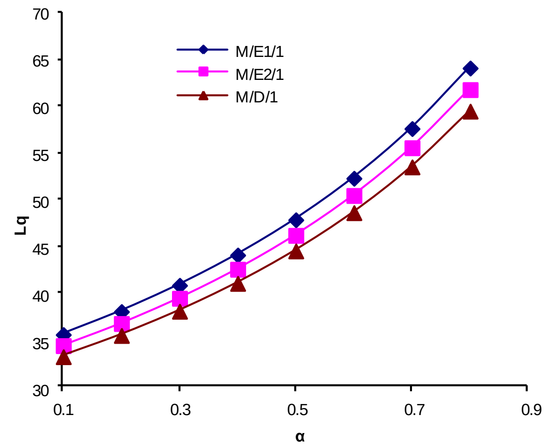

Fig. 2: L_q vs. α for different service time

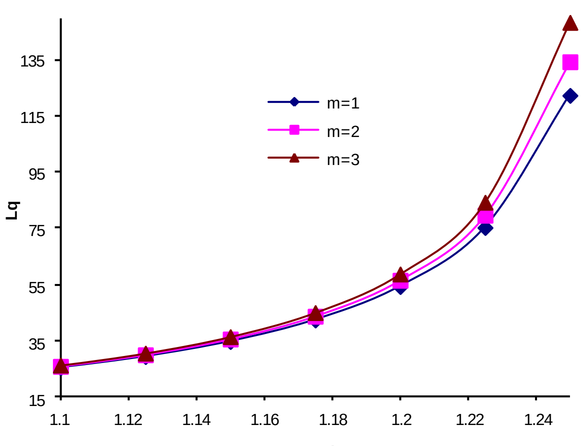

Fig. 3: L_q vs λ for different phases of repair

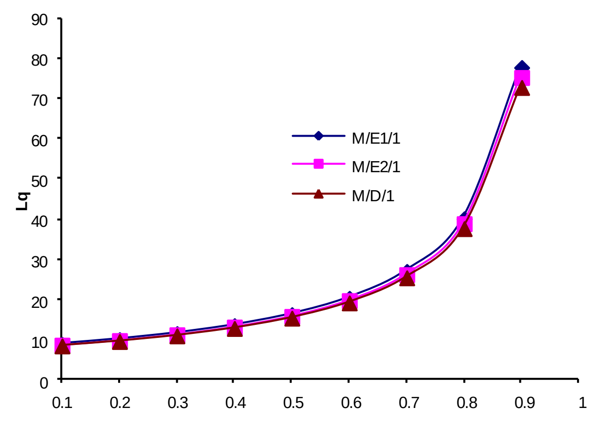

Fig. 4: L_q vs p for different service time

In tables 1 – 7, we summarize the performance indices by setting the fixed values of default parameters as follows:

Table 1: $E(X)=3, \alpha=0.1, \alpha_1=\alpha, \alpha_2=0.8\alpha, p=0.6, m=2, \mu_1=7, \mu_2=1.2\mu_1$.

Table 2: $E(X)=3, \alpha=0.1, \alpha_1=\alpha, \alpha_2=0.8\alpha, p=0.6, m=2, \mu_2=8.4, k=1, \lambda_1=1.2\lambda,$
$\lambda_2=\lambda, \lambda_3=0.8\lambda$.

Table 3: $E(X)=3, \alpha=0.1, \alpha_1=\alpha, \alpha_2=0.8\alpha, p=0.6, m=2, \mu_1=7, k=1, \lambda_1=1.2\lambda,$
$\lambda_2=\lambda, \lambda_3=0.8\lambda$.

Table 4: $E(X)=3, \alpha_1=\alpha, \alpha_2=0.8\alpha, p=0.6, m=2, \mu_1=7, \mu_2=8.4, k=1, \lambda_1=1.2\lambda,$
$\lambda_2=\lambda, \lambda_3=0.8\lambda$.

Table 5: $E(X)=3, p=0.6, m=2, \alpha=0.1, \alpha_1=\alpha, \alpha_2=0.8\alpha, \lambda_1=1.2\lambda, \lambda_2=\lambda, \lambda_3=0.8\lambda$

Table 6: $E(X)=3, m=2, k=1, \mu_1=7, \mu_2=8.4, \lambda=1, \lambda_1=1.2\lambda, \lambda_2=\lambda, \lambda_3=0.8\lambda$

Table 1 displays the effect of state dependent arrival rate on the queue length and response time. Tables 2 and 3 exhibits the effect of service rates on the long run probabilities of the server states for different arrival rates. From table 2, we observe that as μ_1 increases $P(I)$ increases but $P(B_1), P(D_1), P(R^1)$ decrease. It is noticed that $P(B_2), P(D_2), P(R^2)$ remain almost constant. In table 3 we observe that as μ_2 increases, $P(B_2), P(D_2), P(R^2)$ decrease but $P(B_1), P(D_1), P(R^1)$ remain almost constant.

Table 4 exhibits the variation in long run probabilities with failure rate α. From the table we see that as α increases $P(I)$, $P(B_1)$ and $P(B_2)$ decrease while $P(D_1)$, $P(D_2)$, $P(R^1)$ and $P(R^2)$ increase. Table 5 displays the effect of service rate (μ) and arrival rate (λ) on the queue length (L_q). we observe that the queue length increases with the increase in arrival rate; on the contrary it decreases with the increase in the service rate. Further, the queue length also decreases with the increase in the number of phases (k) of service time. Table 6 displays the effect of failure rate on steady state availability (A_v) and on failure frequency (Ff).

Table 1: Effect of arrival rates on the average queue length (L_q) and waiting time $E(W)$

			$M/E_1/1$		$M/E_2/1$		$M/D/1$	
λ_1	λ_2	λ_3	L_q	$E(W)$	L_q	$E(W)$	L_q	$E(W)$
1.20	0.80	1.00	16.65	4.63	16.22	4.51	15.79	4.39
1.25	0.80	1.00	19.59	5.24	19.05	5.09	18.50	4.94
1.30	0.80	1.00	23.74	6.10	23.03	5.92	22.32	5.74
1.35	0.80	1.00	30.00	7.43	29.05	7.19	28.09	6.95
1.30	1.10	1.10	23.89	6.14	23.18	5.95	22.47	5.77
1.30	1.15	1.10	23.91	6.14	23.19	5.95	22.48	5.77
1.30	1.20	1.10	23.92	6.14	23.20	5.96	22.49	5.77
1.30	1.25	1.10	23.93	6.14	23.21	5.96	22.50	5.78
1.35	1.10	1.10	30.25	7.48	29.28	7.24	28.31	7.00
1.35	1.10	1.15	30.32	7.50	29.35	7.26	28.38	7.02
1.35	1.10	1.20	30.39	7.51	29.42	7.27	28.45	7.03
1.35	1.10	1.25	30.46	7.53	29.49	7.29	28.51	7.05

Table 2: Effect of arrival rate (λ) service rate (μ_1) on the long run probabilities of server states

λ	μ_1	$P(I)$	$P(B_1)$	$P(B_2)$	$P(D_1)$	$P(D_2)$	$P(R^1)$	$P(R^2)$
0.5	7.0	0.6107	0.2568	0.1284	0.0006	0.0002	0.0024	0.0008
	7.1	0.6144	0.2532	0.1284	0.0006	0.0002	0.0024	0.0008
	7.2	0.6180	0.2497	0.1284	0.0006	0.0002	0.0023	0.0008
	7.3	0.6215	0.2463	0.1284	0.0006	0.0002	0.0022	0.0008
	7.4	0.6249	0.2430	0.1284	0.0005	0.0002	0.0022	0.0008
1.0	7.0	0.2223	0.5131	0.2566	0.0012	0.0004	0.0049	0.0016
	7.1	0.2297	0.5059	0.2565	0.0012	0.0004	0.0048	0.0016
	7.2	0.2368	0.4989	0.2565	0.0012	0.0004	0.0046	0.0016
	7.3	0.2438	0.4921	0.2566	0.0011	0.0004	0.00450	0.0016
	7.4	0.2506	0.4854	0.2566	0.0011	0.0004	0.0044	0.0016

Table 3: Effect of arrival rate (λ) and service rate (μ_2) on the long run probabilities of server states

λ	μ_2	$P(I)$	$P(B_1)$	$P(B_2)$	$P(D_1)$	$P(D_2)$	$P(R^1)$	$P(R^2)$
0.5	8.4	0.6107	0.2569	0.1284	0.0006	0.0002	0.0024	0.0008
	8.5	0.6122	0.2568	0.1269	0.0006	0.0002	0.0024	0.0008
	8.6	0.6137	0.2568	0.1254	0.0006	0.0002	0.0024	0.0008
	8.7	0.6152	0.2568	0.1240	0.0006	0.0002	0.0024	0.0008
	8.8	0.6166	0.2568	0.1226	0.0006	0.0002	0.0024	0.0007
1.0	8.4	0.2223	0.5130	0.2565	0.0012	0.0004	0.0049	0.0016
	8.5	0.2254	0.5130	0.2535	0.0012	0.0004	0.0049	0.0016
	8.6	0.2284	0.5130	0.2506	0.0012	0.0004	0.0049	0.0016
	8.7	0.2313	0.5131	0.2477	0.0012	0.0004	0.0049	0.0015
	8.8	0.2341	0.5131	0.2449	0.0012	0.0004	0.0049	0.0015

Table 4: Effect of arrival rate (λ) and failure rate (α) on the long run probabilities of the server states

λ	α	$P(I)$	$P(B_1)$	$P(B_2)$	$P(D_1)$	$P(D_2)$	$P(R^1)$	$P(R^2)$
0.5	0.1	0.6107	0.2569	0.1284	0.0006	0.0002	0.0024	0.0008
	0.2	0.6071	0.2566	0.1283	0.0012	0.0004	0.0049	0.0016
	0.3	0.6035	0.2562	0.1281	0.0018	0.0006	0.0073	0.0024
	0.4	0.5999	0.2559	0.1280	0.0024	0.0008	0.0097	0.0032
	0.5	0.5963	0.2556	0.1278	0.0030	0.0010	0.0122	0.0041
1.0	0.1	0.2223	0.5130	0.2566	0.0012	0.0004	0.0049	0.0016
	0.2	0.2161	0.5118	0.2559	0.0024	0.0008	0.0097	0.0032
	0.3	0.2099	0.5106	0.2553	0.0036	0.0012	0.0146	0.0049
	0.4	0.2037	0.5093	0.2546	0.0049	0.0016	0.0194	0.0065
	0.5	0.1976	0.5081	0.2540	0.0060	0.0020	0.0242	0.0081

Table 5: Effect of arrival rate (λ), service rate (μ) and k on the L_q

λ	$\mu=7.00$			$\mu=7.25$			$\mu=7.50$		
	$k=1$	$k=2$	$k\to\infty$	$k=1$	$k=2$	$k\to\infty$	$k=1$	$k=2$	$k\to\infty$
1.00	16.62	16.19	15.76	14.73	14.37	14.02	13.30	13.00	12.70
1.05	20.31	19.74	19.16	17.48	17.02	16.55	15.45	15.06	14.68
1.10	25.99	25.19	24.39	21.42	20.80	20.18	18.37	17.87	17.37
1.15	35.86	34.67	33.47	27.52	26.66	25.80	22.56	21.90	21.24
1.20	57.28	55.22	53.15	38.25	36.95	35.66	29.11	28.18	27.26
1.25	139.18	133.78	128.37	62.10	59.84	57.58	40.77	39.37	37.98

Table 6: Effect of failure rate and optional probability (p) on steady state availability (A_v) of the server and failure frequency (Ff)

(α_1, α_2)	$p = 0.1$		$p = 0.5$		$p = 0.9$	
	A_v	Ff	A_v	Ff	A_v	Ff
(0.1,0.4)	0.9950	0.0460	0.9916	0.0885	0.9888	0.1231
(0.2,0.4)	0.9911	0.0786	0.9881	0.1176	0.9857	0.1494
(0.3,0.4)	0.9873	0.1109	0.9847	0.1465	0.9826	0.1756
(0.4,0.4)	0.9835	0.1429	0.9813	0.1752	0.9795	0.2015
(0.5,0.4)	0.9797	0.1747	0.9779	0.2037	0.9764	0.2273
(0.4,0.1)	0.9843	0.1333	0.9848	0.1319	0.9853	0.1307
(0.4,0.2)	0.9840	0.1365	0.9836	0.1463	0.9834	0.1544
(0.4,0.3)	0.9837	0.1397	0.9825	0.1608	0.9814	0.1780
(0.4,0.4)	0.9835	0.1429	0.9813	0.1752	0.9795	0.2015
(0.4,0.5)	0.9832	0.1461	0.9801	0.1895	0.9776	0.2249

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Appendix A

Proof of Theorem 3: Integrating (31) and (32) with respect to x and using the result

$$\int_0^\infty e^{-sx}(1-A(x))dx = \frac{1-A^*(s)}{s} \tag{58}$$

we get required equation (37) and (38).

Here $A^*(s)$ denote laplace transform of $A(x)$.

Similarly integrating equations (33)-(36) with respect to y and using (58), then repeating the same process on resulting equation for integral with variable x , we get equations (39)-(42).

Appendix B

Proof of Theorem 4: To obtain the queue size distribution at departure epoch we will use the PASTA result. A departing customer will see' j ' customer in the queue just after a departure if and only if there' j ' customer in the FPS and SPS just before a departure. Now denoting $\{\omega_j; j \in Z^+\}$ as the probability that there are j units in the queue at a departure epoch, then for $j \in Z^+$ we may write

$$\omega_j = k_0\left(q\int_0^\infty \mu_1(x)P_j^{(1)}(x)dx + \int_0^\infty \mu_2(x)P_j^{(2)}(x)dx\right) \tag{59}$$

Where k_0 is the normalizing constant

Multiplying (59) by z^j and using $\omega(z) = \sum_{j=1}^{\infty}\omega_j z^j$ and after simplification we get

$$\omega(z) = \frac{k_0 A_0 \phi_1(z)[q + pB_2^*(\tau_2(z))]B_1^*(\tau_1(z))}{[(q + pB_2^*(\tau_2(z)))B_1^*(\tau_1(z)) - z]} \tag{60}$$

Utilizing the normalizing condition, $\omega(1) = 1$ we get

$$k_0 = \frac{1-r_1}{\lambda_1 P_0^{(0)} E(X)} \tag{61}$$

Putting the value of (60) in (61) we get equation (43).

Author Profile

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