International Journal of Scientific Research in Mathematical and Statistical Sciences **Research Paper**
Vol.6, Issue.1, pp.155-161, February (2019) **E-ISSN:** 2348-4519
DOI: https://doi.org/10.26438/ijsrmss/v6i1.155161

New Modification of Third-Order Iterative Method with Optimal Fourth-Order Convergence for Solving Nonlinear Equations

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Available online at: www.isroset.org

Received: 10/Feb/2019, Accepted: 22/Feb/2019, Online: 28/Feb/2019

Abstract— Behl's method is a third-order iterative method involving three functional evaluations for solving nonlinear equation. In this paper, we presented a new iterative method free second derivative with three real parameters by using second-order Taylor expansion. In order to avoid the second derivative, it is approximated by using equality of Chun-Kim's method and Newton-Steffensen's method. The result of study shows that the proposed method converges quartically and requires three evaluation of functions per iteration with efficiency index equal to 1.587401. Numerical simulation is presented to examine the performance of the proposed method by using several real test functions. The final results show that the proposed method is more efficient and perform better than some other kind of methods.

Keywords*— Behl's method, Chun-Kim's method, Third-order iterative method, Newton-Steffensen's method, Numerical simulation*

I. INTRODUCTION

Nonlinear equation is mathematical model in many problems of sciences, engineering and technology. For the most actual cases, an engineer frequently encounter the problem because of the complicated nonlinear equations cannot be solve using an analytical procedure. So, finding a solution of nonlinear equation is one of the important topics that is often discussed in numerical analysis [1].

The problem of nonlinear equation is how to solve the equation as form

$$f(x) = 0, \tag{1}$$

where $f : D \subset R \to R$ is the scalar function in the open interval D.
The alternative solution to find a root of (1) is calculated by arithmetics processing that is known as an iterative method.

Generally, the various classical iterative method are constructed by using Taylor series expansion.

Let α be a solution of nonlinear equation (1) and x_n is an approximation solution at n-th iteration, then Taylor series expansion for $f(x)$ about x_n as following

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{f''(x_n)}{2!}(x - x_n)^2 + \cdots \tag{2}$$

The classical iterative method that known widely as basic approximation to solve (1) is written as form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{3}$$

The method of (2) converges quadratically with efficiency index equal to $2^{1/2} \approx 1.414213$.

The Newton method in (2) is a second-order iterative method which is constructed by using first-order Taylor expansion. Other second-order iterative methods are Schroder's method [2] and Steffensen's method [3].

Then from the second-order Taylor expansion, we get an iterative method having cubically convergence. It is known as Chebyshev's, Halley's and Euler's (Irrational Halley) method that are written as

$$x_{n+1} = x_n - \left(1 + \frac{L_f}{2}\right)\frac{f(x_n)}{f'(x_n)},\ n = 0,\ 1,\ 2, \ldots, \tag{4}$$

$$x_{n+1} = x_n - \left(1 + \frac{2}{2 - L_f}\right)\frac{f(x_n)}{f'(x_n)},\ n = 0,\ 1,\ 2, \ldots, \tag{5}$$

and

$$x_{n+1} = x_n - \left(\frac{2}{1 + \sqrt{1 - 2L_f}}\right)\frac{f(x_n)}{f'(x_n)},\ n = 0,\ 1,\ 2, \ldots, \tag{6}$$

where

$$L_f = \frac{f(x_n) f''(x_n)}{f'(x_n)^2}.$$

Recently, some reseacher have been used several technical approximation to construct either the family of second-order or the family of third-order iterative methods, such as : geometrical intepretation [4, 5, 6, 7, 8, 9, 10, 11], circle of curvature [12], Homotopy pertubation method [13, 10], Adomian decomposition method [13, 14, 15, 16], modified Adomian decomposition method [17], and variant of Newton's method [18, 19, 20, 21].

Now, Behl [6] use an exponential function to construct a third-order iterative method in form

$$x_{n+1} = x_n - \left(\frac{4-L_f}{4-3L_f}\right)\frac{f(x_n)}{f'(x_n)}. \tag{7}$$

All of the third-order iterative method above involves three evaluation of functions. Based on Kung-Traub [22] conjectured that a multipoint iterative method without memory based on *n* functional evaluations will be to achieve an optimal order of convergence 2^{n-1}. So, the methods don't have an optimal order of convergence.

The aim of this paper is to develop third-order iterative method in (7) by using second-order Taylor expansion with three real parameters. The proposed method has second derivative, then we reduced its second derivative by using approximation of equality of two third-order iterative method [23, 24, 25].

This paper is organized as follows. In Section 2, we provide the short reviewing for some definitions and the describing of the proposed iterative method. Section 3 describes the analitycal method used to find the order of convergence. The examining performance of the proposed method by using numerical simulation is shown in Section 4. Finally, A conclusion is presented in Section 5.

II. PRELIMINARIS AND THE PROPOSED METHOD

In this section, we give some basic definitions which will be used in this paper. Furthermore, we construct a fourth-order iterative method with optimal order of convergence.

Definition 1. A sequence of iterates $\{x_n : n \geq 0\}$ is said to converge with order $p \geq 1$ to a point α if

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = c, \tag{3}$$

and the error equation is

$$e_{n+1} = ce_n^p + O(e_n^{p+1}), \tag{4}$$

where $e_n = x_n - \alpha$ and *c* is the asymptotic constant error.

For some $c > 0$. If $p = 1$, the equance is said to converge linearly to α. In that case, we require $c < 1$; the constant *c* is called the rate of linear convergence of x_n to α.

Definition 2. Let α be a root of the function *f* and suppose that x_{n-1}, x_n, x_{n+1} are closer to the root α, then the computational order of convergence (COC) ρ can be approximated using the formula

$$\rho \approx \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}. \tag{5}$$

Definition 3. Let *d* be the number of new pieces of information required by a method. A piece of information typically is any evaluation of a function or one of its derivatives. The efficiency of the method is measured by the concept of efficiency index [11] and is defined by

$$E = p^{1/d}, \tag{6}$$

where *p* is the order of the method.

To develop the two-point iterative method (7) with optimal order of convergence, we begin deriving the third-order iterative method [6].

Consider an exponentially fitted straight line that is written as

$$y(x) = e^{p(x-x_n)}\left(A(x-x_n)+B\right), \tag{7}$$

with its first derivative and second derivative

$$y'(x) = pe^{p(x-x_n)}\left(A(x-x_n)+B\right)+e^{p(x-x_n)}A, \tag{8}$$

$$y''(x) = p^2e^{p(x-x_n)}\left(A(x-x_n)+B\right)+2pe^{p(x-x_n)}A. \tag{9}$$

Substitute $x = x_n$ into (7) and (8) and suppose $y(x_n) = f(x_n)$, $y'(x_n) = f'(x_n)$, we get

$$A = f'(x_n) - pf(x_n),\ B = f(x_n). \tag{10}$$

Suppose that the straight line (7) through the *x*-axis at $x = x_{n+1}$, then $y(x_{n+1}) = 0$. So, we have

$$x_{n+1} = x_n - \frac{B}{A}, \tag{11}$$

or

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)-pf(x_n)}, n \geq 0. \tag{12}$$

Iterative method (12) is the well-known one parameter family of Newton's method.

Again, from (9) we find

$$p^2 f(x_n) - 2pf'(x_n) + f''(x_n) = 0, \tag{13}$$

when $|p| \ll 1$, then p^2 can be negleted. So, we get *p* in form

$$p=\frac{f''(x_n)}{2f'(x_n)}. \tag{14}$$

Based on (13), we can write (13) as

$$p\left(pf(x_n)-2f'(x_n)\right)+f''(x_n)=0. \tag{15}$$

From (15), we find implicit form of p that is written as

$$p=\frac{f''(x_n)}{2f'(x_n)-p^*f(x_n)}, \tag{16}$$

where p^* is defined by (14).
Substitute (14) into (16), we get a new form of p,

$$p=\frac{2f'(x_n)f''(x_n)}{4f'(x_n)-f(x_n)f''(x_n)} \tag{17}$$

Based on (12) and (17), we get

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\left(\frac{4f'(x_n)^2-f(x_n)f''(x_n)}{4f'(x_n)^2-3f(x_n)f''(x_n)}\right). \tag{18}$$

Equation (18) is third-order convergence including three functional evalutions.

To improve (18), we consider third-order iterative method (18) with two real parameter that is written as

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\left(\frac{4f'(x_n)^2-\lambda f(x_n)f''(x_n)}{4f'(x_n)^2-3\beta f(x_n)f''(x_n)}\right). \tag{19}$$

Furthermore, we construc a iterative method by using Taylor expansion. So, we consider second order Taylor series expansion $f(x)$ about x_n as form:

$$f(x)=f(x_n)+(x-x_n)f'(x_n)+\frac{(x-x_n)^2}{2!}f''(x_n), \tag{20}$$

If x_{n+1} is an approximation root at $(n+1)$ iteration, then (20) can be written as

$$f(x_{n+1})\approx f(x_n)+(x_{n+1}-x_n)f'(x_n)+\frac{(x_{n+1}-x_n)^2}{2!}f''(x_n) \tag{21}$$

Let x_{n+1} is closely to α, then $f(x_{n+1})\approx 0$, So, from (21) we get

$$x_{n+1}-x_n=-\frac{f(x_n)}{f'(x_n)}-\frac{(x^*_{n+1}-x_n)^2}{2}\frac{f''(x_n)}{f'(x_n)}. \tag{22}$$

where x^*_{n+1} is third-order iterative method thas is defined by (19).

Subtitute (19) into (22), we find a new iterative method with second derivative as following

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\left(\frac{f''(x_n)f(x_n)^2}{2f'(x_n)^3}\right)\left(\frac{-4f'(x_n)^2+\lambda f(x_n)f''(x_n)}{-4f'(x_n)^2+3\beta f(x_n)f''(x_n)}\right)^2 \tag{23}$$

Equation (23) still contains second derivative of $f(x_n)$. Therefore, we reduce it by using equality of Newton Steffensen's method [3] with one real parameter and Halley's method [4] that are written in form

$$x_{n+1}=x_n-\frac{f(x_n)^2}{f'(x_n)\left(f(x_n)-\theta f(y_n)\right)}, \tag{24}$$

and

$$x_{n+1}=x_n-\frac{2f(x_n)f'(x_n)}{2f'(x_n)^2-f(x_n)f''(x_n)}, \tag{25}$$

where y_n is defined by (2), respectively.

Using (24) and (25), we have a new approximation for $f''(x_n)$ in form

$$f''(x_n)\approx\frac{2\theta f'(x_n)^2f(y_n)}{f(x_n)^2}. \tag{26}$$

Based on (26), susbtitute it into (23) and simplify it, then we get a new iterative method with three real parameters

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \tag{27}$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\frac{\theta f(y_n)\left(-2f(x_n)+\theta\lambda f(y_n)\right)^2}{f'(x_n)\left(-2f(x_n)+3\beta\theta f(y_n)\right)^2}, \tag{28}$$

where $\theta, \lambda, \beta \in R$.

Equation (28) is a two point iterative methods involving three evaluation of functions namely $f(x_n)$, $f'(x_n)$ dan $f(y_n)$.

III. ORDER OF CONVERGENCE

Based on the Kung and Traub [22], the convergence order of a two point iterative method will be optimal if the method converges quartically.

Theorem following will describe the convergence order of the method in (28).

Theorem 1. Let $f: D\subset\Re\rightarrow\Re$ be a differentiable function in open interval D. Then asume that is a simple root of $f(x)=0$. Suppose x_0 is a given value that is sufficiently close to α, then iterative method (18) has fourth-order of convergence for $\theta = -1$, $\beta = 0$ and $\lambda = 2$ that satisfy the following error

$$e_{n+1} = (-c_2c_3 + 4c_2^3)e_n^4 + O(e_n^5) . \quad (29)$$

where $c_k = \dfrac{f^{(k)}(\alpha)}{k!\, f'(\alpha)}, k = 2,3,4,5,\ldots$.

Proof : Let α is an exact root of $f(x)$ then $f(\alpha) = 0$. Furthermore, assume $f'(\alpha) \neq 0$ and $x_n = e_n + \alpha$, then Taylor series expansion for $f(x_n)$ and $f'(x_n)$ about α, we have

$$f(x_n) = f'(\alpha)\left(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)\right), \quad (30)$$

and

$$f'(x_n) = f'(\alpha)\left(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_4e_n^4 + \mathrm{O}(e_n^5)\right). \quad (31)$$

Use (30) and (31), we find

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + (2c_2^3 - 2c_3)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 + O(e_n^5) \quad (32)$$

From (27) and $x_n = e_n + \alpha$, we get

$$y_n = \alpha + c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + \cdots + \mathrm{O}(e_n^5)) . \quad (33)$$

Furthermore, by using Taylor series expansion $f(x)$ about α at $x = y_n$, we get

$$f(y_n) = f'(\alpha)\left(c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 + 5c_2^3 - 7c_2c_3)e_n^4 + \mathrm{O}(e_n^5)\right). \quad (34)$$

By using (30), (31), and (34) we get

$$\theta f(y_n)\left(-2f(x_n) + \theta f(y_n)\right)^2 = f'(\alpha)^3\left(4\theta c_2e_n^4 + O(e_n^5)\right), \quad (34)$$

and

$$f'(x_n)\left(-2f(x_n) + 3\beta\theta f(y_n)\right)^2 = f'(\alpha)^3\left(4e_n^2 + 4(4-3\theta\beta)c_2e_n^3 \right.$$
$$\left.((20 - 12\beta\theta + 9\theta^2\beta^2)c_2 + 4(5 - 6\theta\beta)c_3\right)e_n^4 + O(e_n^5)\Big). \quad (35)$$

Equation (34) is divided by (35), and is found

$$\frac{\theta f(y_n)\left(-2f(x_n) + \theta f(y_n)\right)^2}{f'(x_n)\left(-2f(x_n) + 3\beta\theta f(y_n)\right)^2} = \theta c_2e_n^2 + \left((-4\theta + \theta^2(3\beta - \lambda))c_2^2 + 2\theta c_3\right)e_n^3 + \cdots + O(e_n^5) . (36)$$

Furthermore, by using (23) and (28), we get:

$$\frac{f(x_n)}{f'(x_n)} - \frac{\theta f(y_n)\left(-2f(x_n) + \theta f(y_n)\right)^2}{f'(x_n)\left(-2f(x_n) + 3\beta\theta f(y_n)\right)^2} = e_n - (\theta+1)c_2e_n^2$$
$$+\left(-2(\theta+1)c_3 - (2 + 4\theta + \theta^2(\lambda - 3\beta))c_2^2\right)e_n^3 + \cdots + O(e_n^5) \quad (37)$$

Substitute (37) into (28) and take $x_{n+1} = e_{n+1} + \alpha$ and $x_n = e_n + \alpha$, then we get convergence order of (18) in form

$$e_{n+1} = -(\theta+1)c_2e_n^2 + \left((-2 - 4\theta + \theta^2(3\beta - \lambda))c_2^2 + 2(\theta+1)c_3\right)e_n^3$$
$$+\left(\left(4 + 13\theta + 7(\lambda - 3\beta)\theta^2 + \frac{1}{4}\left(27\beta^2 - 12\beta + \lambda\right)\theta^3\right)c_2^3 + \left(-7 - 14\theta + 4(\lambda - 3\beta)\theta^2\right)c_2c_3 + 3(1+\theta)c_4\right)e_n^4$$
$$+ O(e_n^5) \quad (38)$$

Equation (38) give an information that the convergence order of (28) will increase if we take $\theta = -1$. So, by subtituting the value of θ into (38), we get

$$e_{n+1} = \left(2 + 3\beta - \lambda\right)c_2^2e_n^3 + \left(\left(7 + 12\beta - 4\lambda\right)c_2^2 + \frac{1}{4}\left(-\lambda^2 + 4(3\beta+7)\lambda - (36 + 48\beta + 27\beta^2)\right)c_2^3\right)e_n^4 + O(e_n^5) . \quad (39)$$

Base on (39), the convergence order of the proposed method is at least three. So, (33) still involve two real parameters, the we get increase again by using relationship $\lambda = 3\beta + 2$, $\beta \in \Re$. Substitute it into (39), then we get

$$e_{n+1} = \left((4 + 3\beta)c_2^3 - c_2c_3\right)e_n^4 + O(e_n^5) . \quad (40)$$

The proof is completed. □

Equation (40) is the convergence order of (27) – (28) for $\theta = -1$, $\lambda = 3\beta + 2$, and $\beta \in \Re$.

$$x_{n+1} = x_n - \left(1 - \frac{f(y_n)\left(2f(x_n) + (3\beta+2)f(y_n)\right)^2}{f(x_n)\left(2f(x_n) + 3\beta f(y_n)\right)^2}\right)\frac{f(x_n)}{f'(x_n)} \quad (41)$$

Furthermore by taking any value of β, it will apperar some fouth-iterative methods.

For $\beta = 0$,

$$x_{n+1} = x_n - \left(1 - \frac{f(y_n)\left(f(x_n) + f(y_n)\right)^2}{f(x_n)f(x_n)^2}\right)\frac{f(x_n)}{f'(x_n)} . \quad (42)$$

For $\beta = -\dfrac{2}{3}$,

$$x_{n+1} = x_n - \left(1 - \frac{f(y_n)f(x_n)}{\left(f'(x_n) - f(x_n)\right)^2}\right)\frac{f(x_n)}{f'(x_n)}. \qquad (43)$$

IV. NUMERICAL SIMULATION

In this section, numerical examples are presented to illustrate the efficiency of the proposed method in (27) – (28) by using several test functions. The zeros approximation of the test functions was displayed round up to 16th decimal places.

To show the performance of Eq. (27) - (28) for θ = −1, β = 0 and λ = 2 (MBM4), we compare it with Newton's method (NM)[11], classical Chebyshev-Halley's method with β = 1/2 (HM)[3], Newton-Steffensen's method (NSM)[20], third-order iterative method (BM3)[6].

Table 1 Comparison of efficiency index

Iterative Method	Orde of Convergence	Functional evaluation	Efficiency index
NM	2	2	1.41421356
HM	3	3	1.44224957
MNS	3	3	1.44224957
BM3	3	3	1.44224957
MBM4	4	3	1.58740105

All computations are performed by using Maple 13.0 with 850 digits floating point arithmetics for the following several test functions.

$$|x_{n+1} - x_n| \le \varepsilon \qquad (44)$$

dengan $\varepsilon = 10^{-95}$.

The used real functions are as following:

$f_1(x) = xe^{-x} - 0{,}1$, α = 0.111832559158,

$f_2(x) = e^x - 4x^2$, α = 4.306584728220,

$f_3(x) = \cos(x) - x$, α = 0.739085133215,

$f_4(x) = x^3 + 4x^2 - 10$, α = 1.365230013414,

$f_5(x) = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1$, α = −1.0000000000000000,

$f_6(x) = \sin^2(x) - x^2 + 1$, α = 1,4044916482153412.

The number of iteration and COC for compared iterative methods shown at the Table 1 following.

Table 2 Comparison of the number of iterations **(IT)** for several iterative methods

$f(x)$	x_0	NM	HM	NSM	BM3	MBM4
$f_1(x)$	−0.2 0.3	8 (2.0000) 8 (2.0000)	5 (3.0000) 5 (3.0000)	5 (3.0000) 5 (3.0000)	5(3.0000) 4(3.0000)	4(4.0000) 4(4.0000)
$f_2(x)$	4.0 4.5	8 (2.0000) 7 (2.0000)	5 (3.0000) 5 (3.0000)	5 (3.0000) 5 (3.0000)	4(3.0000) 4(3.0000)	4(4.0000) 4(4.0000)
$f_3(x)$	0.1 1.5	8 (2.0000) 7 (2.0000)	5 (3.0000) 5 (3.0000)	5 (3.0000) 5 (3.0000)	5 (3.0000) 5 (3.0000)	4(4.0000) 4(4.0000)
$f_4(x)$	1.0 2.0	8 (2.0000) 8 (2.0000)	5 (3.0000) 5 (3.0000)	5 (3.0000) 5 (3.0000)	5 (3.0000) 5 (3.0000)	4(4.0000) 4(4.0000)
$f_5(x)$	− 1.5 0.0	7 (2.0000) 7 (2.0000)	5 (3.0000) 6 (3.0000)	5 (3.0000) 5 (3.0000)	5 (3.0000) 5 (3.0000)	4(4.0000) 4(4.0000)
$f_6(x)$	1.2 2.0	8 (2.0000) 8 (2.0000)	5 (3.0000) 5 (3.0000)	5 (3.0000) 5 (3.0000)	5(3.0000) 5(3.9999)	4(4.0000) 4(3.9999)

Based on Table 1, the proposed method converges quartically for all test function. This gives an information that the convergene order of the method is four. Beside in, for all compared iteration method, the proposed method has the least number of iteration.

Furthermore, to find an accuration for compared iterative methods, we use absolute value of $f(x_n)$ and of relatif error to measure the accuration with *Total Number of Functional Evaluation* (TNFE) = 12 which given at Table 3 and 4.

Table 3. Value of $|f(x_{n+1})|$ for TNFE = 12

$f(x)$	x_0	NM	HM	NSM	BM3	MBM4
$f_1(x)$	−0.2 0.3	3.0850 (e−36) 1.0735 (e−42)	2.7757 (e−55) 3.5153 (e−66)	1.2725 (e−45) 9.0538 (e−54)	1.6466 (e−093) 3.7952 (e−115)	2.6777 (e−131) 2.0988 (e−149)
$f_2(x)$	4.0 4.5	5.0253 (e−33) 3.1919 (e−52)	2.1103 (e−53) 5.2464 (e−76)	5.5769 (e−42) 2.4261 (e−66)	1.3675 (e−100) 8.3696 (e−111)	2.6862 (e−114) 4.5289 (e−198)

$f_3(x)$	0.1 1.5	2.0345 (e–46) 3.7607 (e–64)	3.9683 (e–49) 1.1496 (e–51)	4.7468 (e–58) 3.5077 (e–80)	6.4494 (e–61) 6.1628 (e–53)	1.6745 (e–138) 2.9422 (e–197)
$f_4(x)$	1.0 2.0	3.9823 (e–43) 1.2361 (e–37)	2.2349 (e–60) 4.6600 (e–52)	9.1052 (e–55) 7.8139 (e–48)	2.9042 (e–86) 1.6196 (e–66)	4.3768 (e–150) 1.3781 (e–137)
$f_5(x)$	– 1.5 0.0	5.7389 (e–66) 1.9261 (e–65)	1.5261 (e–43) 6.3918 (e–26)	5.1899 (e–92) 9.3636 (e–73)	3.4133 (e–40) 2.2303 (e–23)	1.2949 (e–173) 2.9415 (e–154)
$f_6(x)$	1.2 2.0	2.0864 (e–47) 2.2623 (e–32)	1.5527 (e–64) 8.6200 (e–39)	7.4954 (e–60) 8.2994 (e–41)	3.6987 (e–88) 1.5465 (e–45)	6.9377 (e–163) 5.7027 (e–110)

Table 4. Value of $|x_{n+1} - x_n|$ for TNFE = 12

$f(x)$	x_0	NM	HM	NSM	BM3	MBM4
$f_1(x)$	–0.2 0.3	1.9116 (e–18) 1.1277 (e–21)	8.4084 (e–19) 1.9599 (e–22)	1.1234 (e–15) 2.1608 (e–18)	4.1542 (e–31) 2.7504 (e–38)	2.0729 (e–33) 6.1679 (e–38)
$f_2(x)$	4.0 4.5	1.2322 (e–17) 3.1056 (e–27)	1.1156 (e–18) 3.2560 (e–26)	5.8707 (e–15) 4.4483 (e–23)	4.5871 (e–34) 1.8077 (e–37)	1.6572 (e–29) 1.8884 (e–50)
$f_3(x)$	0.1 1.5	2.3464 (e–23) 3.1900 (e–32)	1.2697 (e–16) 1.8100 (e–17)	1.7984 (e–19) 7.5471 (e–27)	1.6151 (e–20) 7.3841 (e–18)	7.2436 (e–35) 1.4830 (e–49)
$f_4(x)$	1.0 2.0	2.2179 (e–22) 1.2356 (e–19)	9.0968 (e–21) 5.3942 (e–18)	6.1217 (e–19) 1.2533 (e–16)	3.0899 (e–29) 1.1805 (e–22)	3.3680 (e–38) 4.4865 (e–35)
$f_5(x)$	– 1.5 0.0	2.3956 (e–33) 4.3887 (e–33)	4.0291 (e–15) 3.0145 (e–09)	6.7780 (e–31) 1.7777 (e–24)	5.2077 (e–14) 2.0975 (e–08)	7.7383 (e–44) 5.3423 (e–39)
$f_6(x)$	1.2 2.0	3.2750 (e–24) 1.0784 (e–16)	4.9165 (e–22) 1.8755 (e–13)	1.7005 (e–20) 3.7902 (e–14)	8.7910 (e–30) 1.4162 (e–15)	2.3648 (e–41) 4.0043 (e–28)

Table 3 and 4 shown that the proposed method (MBM4) has the lowest absolute value of $f(x_n)$ and relatif error for all tested real functions. It is clearly that the accuration of the iterative method in (27) – (28) is better than others.

V. CONCLUSION AND FUTURE SCOPE

This research work have developed a new fourth-order convergence method for solving nonlinear equation $\theta = -1$, $\lambda = 3\beta + 2$, $\beta \in \Re$. The method requires two evaluation of functions and one its first derivative per iteration with the efficiency index equal to $4^{1/3} \approx 1.5874$. The numerical results show that the proposed method has better performance as compared with the other methods. Therefore, the results of this study provide a new contribution in computational sciences area.

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