International Journal of Scientific Research in Mathematical and Statistical Sciences
**Research Paper**
Vol.6, Issue.4, pp.108-112, August (2019)
**E-ISSN:** 2348-4519
*DOI: https://doi.org/10.26438/ijsrmss/v6i4.108112*

# On Certain Special Vector Fields in a Finsler Space II

**P.K. Dwivedi[1*], S.C. Rastogi[2], A.K. Dwivedi[3]**

[1]Applied Sciences (Maths.), Ambalika Institute of Management & Technology, Lucknow, U.P., India
[2] Professor (retd.), Lucknow, U.P., India
[3]Applied Sciences (Maths.), CIPET, Lucknow, U.P., India

**Available online at: www.isroset.org**

Received: 25/Apr/2019, Accepted: 20/Aug/2019, Online: 31/Aug/2019

***Abstract-*** Vector fields of different kind have been playing interesting part in the study of both Tensor Analysis and Differential Geometry of Finsler Spaces of two and three- dimensions. In this paper an attempt has been made to define several new special vector fields in a Finsler space of three- dimensions. We have studied their properties and their applications in the study of curvature properties with respect to first curvature tensor in a Finsler space of three- dimensions.

***Keywords-*** Vector fields, Finsler space, Three dimensions, Curvature Tensors.

## I. INTRODUCTION

The importance of the study of vector fields lies in the fact that vector fields of different kinds have been used in the study of several applications in general in Differential Geometry and in particular in Tensor Analysis. This work is in a way extension of the work done by Rastogi and Bajpai [1] in case of a two- dimensional and three-dimensional Finsler space, where they defined and studied a special vector field of first kind. In an earlier paper the present authors have defined and studied some of the vector fields in a two- dimensional Finsler space [2]. In this paper this study has been carried out and we have defined and studied special vector fields of six kinds in a Finsler space of three-dimensions. The study of these vector fields gives an insight into curvature properties in a three-dimensional Finsler space, which are important for further study in Finsler spaces. Before the start of this study some preliminaries based on Finsler space of three-dimensions are necessary and are given as follows:

Let $F^3$, be a Finsler space of three-dimensions with metric Function $L(x,y)$, metric tensor $g_{ij} = l_i\, l_j + m_i\, m_j + n_i\, n_j$, angular metric tensor $h_{ij} = m_i\, m_j + n_i\, n_j$, where $l_i$, $m_i$ and $n_i$ are vectors of three-dimensional orthonormal frame and are mutually Orthogonal such that $l_i = \Delta_i\, L = \partial L/\partial y^i$, for a metric function $L(x,y)$, Matsumoto [3]. The torsion tensor $A_{ijk} = L\, C_{ijk} = (L/2)\Delta_k\, g_{ij}$. The h-and v-covariant derivatives of a tensor field $T^i_j$ are defined as

$$T^i_{j/k} = \partial_k\, T^i_j - N^m_k\, \Delta_m\, T^i_j + T^m_j\, F^i_{mk} - T^i_m\, F^m_{jk} \qquad (1.1)$$

and

$$T^i_{j//k} = \Delta_k\, T^i_j + T^m_j\, C^i_{mk} - T^i_m\, C^m_{jk} \qquad (1.2)$$

where $\partial_k = \partial/\partial x^k$, $C^i_{jk}$ is the torsion tensor and $F^i_{jk}$ is the connection parameter Rund [4].

Corresponding to h- and v-covariant derivatives, in $F^3$, we have

$$l_{i/j} = 0,\; m_{i/j} = n_i\, h_j,\; n_{i/j} = -\, m_i\, h_j \qquad (1.3)$$

and

$$l_{i//j} = L^{-1}\, h_{ij},\; m_{i//j} = L^{-1}(-l_i\, m_j + n_i\, v_j),$$

$$n_{i//j} = -L^{-1}(l_i\, n_j + m_i\, v_j) \qquad (1.4)$$

where $h_j$ and $v_j$ are respectively h- and v-connection vectors in $F^3$.

In $F^3$, the torsion tensor $C_{ijk}$ is given as

$$C_{ijk} = C_{(1)}\, m_i\, m_j\, m_k - \sum_{(I,j,k)} \{C_{(2)}\, m_i\, m_j\, n_k + C_{(3)}\, m_i\, n_j\, n_k)$$
$$+ C_{(2)}\, n_i\, n_j\, n_k$$
(1.5),

Where $\sum_{(I,j,k)} \{\ \ \}$, means cyclic permutation of the terms inside the curly bracket.

Corresponding to covariant differentiation given by equation (1.1), we have following commutation formula

$$T^h_{i/j/k} - T^h_{i/k/j} = T^r_i\, R^h_{rjk} - T^h_r\, R^r_{ijk} - T^h_{i/r}\, R^r_{jk} \qquad (1.6)$$

where $R^r_{jkh}$ is first curvature tensor Rund [4].

In second section we have defined and studied some properties of special vector fields of second, third and fourth kind. In third section we have defined and studied some properties of special vector fields of fifth, sixth and seventh kind. The last section, i.e., section four deals with the study of curvature properties of Finsler space of three-dimensions based on first curvature tensor.

## II. SPECIAL VECTOR FIELDS OF SECOND, THIRD AND FOURTH KIND.

In $F^3$, let us assume that the vector field $X^i(x)$ is given by

$$X^i(x) = A\, l^i + B\, m^i + D\, n^i \tag{2.1}$$

where A, B and D are scalars such that $X^i\, l_i = A$, $X^i\, m_i = B$ and $X^i\, n_i = D$.

Using $X^i_{/j} = -\delta^i_j$, with the help of equation (2.1), we can obtain

$$A_{/j} = - l_j,\ B_{/j} = D\, h_j - m_j,\ D_{/j} = - B\, h_j - n_j \tag{2.2}$$

$$A_{//j} = L^{-1}(B\, m_j + D\, n_j),\ B_{//j} = (C_{(1)}\, B - C_{(2)}\, D - L^{-1}\, A) m_j + (C_{(3)}\, D - C_{(2)}\, B) n_j + L^{-1}\, D\, v_j,$$

$$D_{//j} = (C_{(3)}\, D - C_{(2)}\, B)\, m_j + (C_{(3)}\, B + C_{(2)}\, D - L^{-1} A)\, n_j - L^{-1}\, B\, v_j \tag{2.3}$$

Now we shall give following:

**Def. 2.1.** A vector field $X^i(x)$ in $F^3$, satisfying $X^i_{/j} = - \delta^i_j$ and $X^i\, S_{ij} = \varphi_j$, where $\varphi_j$ is a non-zero vector field in $F^3$ and $S_{ij} = l_i\, m_j + l_j\, m_i$, shall be called a special vector field of second kind.

From def. 2.1, we can obtain

$$\varphi_j = A\, m_j + B\, l_j \tag{2.4}$$

which yields

$$\varphi_{j/k} = (D\, l_j - A\, n_j)\, h_k - S_{jk} \tag{2.5}$$

and

$$\varphi_{j//k} = L^{-1}\{B\, h_{jk} + A(-l_j\, m_k + n_j\, v_k) + (B\, m_k + D\, n_k)\, m_j\} + l_j\{(C_{(1)}\, B - C_{(2)}\, D - L^{-1}\, A)\, m_k + (C_{(3)}\, D - C_{(2)}\, B)\, n_k + L^{-1}\, D\, v_k\} \tag{2.6}$$

From equations (2.5) and (2.6), we can get

$$\varphi_{j/k} - \varphi_{k/j} = A(h_j\, n_k - h_k\, n_j) + D(h_k\, l_j - h_j\, l_k) \tag{2.7}$$

and

$$\varphi_{j//k} - \varphi_{k//j} = L^{-1}\, D\, (Y_{jk} + l_j\, v_k - l_k\, v_j) + V_{jk}\, (C_{(1)}\, B - C_{(2)}\, D - 2\, L^{-1} A) + W_{jk}\, (C_{(3)}\, D - C_{(2)}\, B) \tag{2.8}$$

where

$$Y_{jk} = m_j\, n_k - m_k\, n_j,\ V_{jk} = l_j\, m_k - l_k\, m_j,$$

$$W_{jk} = l_j\, n_k - l_k\, n_j \tag{2.9}.$$

Hence:

Theorem 2.1. **In a three-dimensional Finsler space $F^3$, a special vector field of second kind $X^i(x)$ is such that both $\varphi_{j/k}$ and $\varphi_{j//k}$ are non-symmetric in j and k and satisfy equations (2.7) and (2.8) respectively.**

**Def. 2.2.** A vector field $X^i(x)$ in a three-dimensional Finsler space $F^3$, satisfying $X^i_{/j} = - \delta^i_j$ and $X^i\, T_{ij} = \psi_j$, where $T_{ij} = l_i\, n_j + l_j\, n_i$ and $\psi_j$ is a non-zero vector field in $F^3$, shall be called a special vector field of third kind.

From Def. 2.2., we can obtain

$$\psi_J = A\, n_j + D\, l_j, \tag{2.10}$$

which yields

$$\psi_{j/k} = - T_{jk} - h_k\, \varphi_j \tag{2.11}$$

and

$$\psi_{j//k} = L^{-1}\{D\, h_{jk} + (B\, m_k + D\, n_k) n_j - A\, (l_j\, n_k + m_j\, v_k)\} + l_j\{(C_{(3)}\, B + C_{(2)}\, D - L^{-1}\, A)\, n_k + (C_{(3)}\, D - C_{(2)}\, B)\, m_k - L^{-1} B\, v_k\} \tag{2.12}.$$

From these equations we can obtain

$$\Psi_{j/k} - \psi_{k/j} = (h_j\, \varphi_k - h_k\, \varphi_j) \tag{2.13}$$

and

$$\psi_{j//k} - \psi_{k//j} = L^{-1}\{B\, Y_{kj} - A\, (m_j\, v_k - m_k\, v_j) - B\, (l_j\, v_k - l_k\, v_j) - 2\, A\, W_{jk}\} + (C_{(3)}\, D - C_{(2)}\, B)\, V_{jk} + (C_{(3)}\, B + C_{(2)}\, D)\, W_{jk} \tag{2.14}$$

Hence:

Theorem 2.2. **In a three-dimensional Finsler space $F^3$, a special vector field of third kind $X^i(x)$, is such that both $\psi_{j/k}$ and $\psi_{j//k}$ are non-symmetric in j and k and satisfy equations (2.13) and (2.14).**

**Def. 2.3.** A vector field $X^i(x)$ in $F^3$ satisfying $X^i_{/j} = - \delta^i_j$ and $X^i\, U_{ij} = \omega_j$, where $\omega_j$ is a non-zero vector field in $F^3$ and

 

$U_{ij} = m_i n_j + m_j n_i$, shall be called a special vector field of fourth kind.

From Def.2.3, we can obtain

$$\omega_{/j} = B n_j + D m_j \quad (2.15)$$

which yields

$$\omega_{j/k} = 2 (D n_j – B m_k) h_k – U_{jk} \quad (2.16)$$

and

$$\omega_{j//k} = L^{-1}\{2 v_k(D n_j – B m_j) – l_j \omega_k - A U_{jk}\}$$
$$+ (C_{(3)} D – C_{(2)} B) h_{jk} +(C_{(1)} B – C_{(2)} D) m_k n_j$$
$$+ (C_{(3)} B + C_{(2)} D) m_j n_k \quad (2.17)$$

From equations (2.16) and (2.17) we get

$$\omega_{j/k} – \omega_{k/j} = 2 \{ h_k(D n_j – B m_j) – h_j (D n_k - B m_k)\} \quad (2.18)$$

and

$$\omega_{j//k} – \omega_{k//j} = (C_{(3)} B + 2 C_{(2)} D – C_{(1)} B)(m_j n_k – m_k n_j)$$
$$– L^{-1}[ B W_{jk} + D V_{jk} – 2\{D(v_k n_j – v_j n_k)$$
$$+ B(v_j m_k - v_k m_j)\}] \quad (2.19)$$

Hence:

Theorem 2.3. **In $F^3$, a special vector field of fourth kind is such that both $\omega_{j/k}$ and $\omega_{j//k}$ are non-symmetric in j and k and satisfy equations (2.18) and (2.19) respectively.**

## III. SPECIAL VECTOR FIELDS OF FIFTH, SIXTH AND SEVENTH KIND.

**Def. 3.1.** A vector field $X^i(x)$ in $F^3$, satisfying $X^i_{/j} = - \delta^i_j$ and $X^i V_{ij} = \alpha_j$, where $\alpha_j$ is a non-zero vector field shall be called a special vector field of fifth kind.

From Def. 3.1, we can obtain

$$\alpha_j = A m_j – B l_j \quad (3.1)$$

which yields

$$\alpha_{j/k} = V_{jk} + h_k(A n_j – D l_j) \quad (3.2)$$

and

$$\alpha_{j//k} = L^{-1}\{(B m_k + D n_k) m_j – A(l_j m_k – n_j v_k) – B h_{jk}\}$$
$$-l_j\{(C_{(1)} B – C_{(2)} D – L^{-1}A) m_k$$
$$+ (C_{(3)} D – C_{(2)} B) n_k + L^{-1} D v_k\} \quad (3.3)$$

From equations (3.2) and (3.3), we can obtain

$$\alpha_{j/k} - \alpha_{k/j} = (A n_j – D l_j) h_k – (A n_k -D l_k)h_j + 2 V_{jk} \quad (3.4)$$

and

$$\alpha_{j//k} - \alpha_{k//j} = L^{-1}\{D (Y_{jk} + v_j l_k – v_k l_j) + A (n_j v_k – n_k v_j)\}$$
$$-(C_{(1)} B – C_{(2)} D) V_{jk}$$
$$– (C_{(3)} D – C_{(2)} B) W_{jk} \quad (3.5)$$

Also, from equations (2.5) and (3.2) together with (2.6) and (3.3), we can get

$$\alpha_{j/k} + \varphi_{j/k} = - 2 l_k m_j \quad (3.6)$$

and

$$\alpha_{j//k} + \varphi_{j//k} = L^{-1}[B h_{jk} + 2\{(B m_k + D n_k) m_j$$
$$– A(l_j m_k – n_j v_k)\}] \quad (3.7)$$

Hence:

Theorem 3.1. **In a three-dimensional Finsler space $F^3$, a special vector field $X^i(x)$ of fifth kind is such that both $\alpha_{j/k}$ and $\alpha_{j//k}$ are non-symmetric in j and k and satisfy equations (3.4), (3.5), (3.6) and (3.7).**

**Def. 3.2.** A vector field $X^i(x)$ in $F^3$, satisfying $X^i_{/j} = - \delta^i_j$ and $X^i W_{ij} = \beta_j$, where $\beta_j$ is a non-zero vector field in $F^3$, shall be called a special vector field of sixth kind.

In analogy to earlier calculation from Def. 3.2, we can get

$$\beta_j = A n_j – D l_j, \quad (3.8)$$

$$\beta_{j/k} = W_{jk} – h_k \alpha_j \quad (3.9)$$

$$\beta_{j//k} = L^{-1}\{(B m_k + D n_k) n_j- A(l_j n_k + m_j v_k) – D h_{jk}\}$$
$$-l_j\{(C_{(3)} D - C_{(2)} B) m_k$$
$$+ (C_{(3)} B + C_{(2)} D – L^{-1} A) n_k – L^{-1} B v_k\} \quad (3.10)$$

$$B_{j/k} – \beta_{k/j} = 2 W_{jk} + h_j(A m_k – B l_k)$$
$$– h_k(A m_j – B l_j) \quad (3.11)$$

$$\beta_{j//k} – \beta_{k//j} = L^{-1}\{B(l_j v_k - l_k v_j – Y_{jk}) – A(m_j v_k – m_k v_j)\}$$
$$-(C_{(3)} D – C_{(2)} B) V_{jk}$$
$$– (C_{(3)} B + C_{(2)} D) W_{jk} \quad (3.12)$$

Hence:

Theorem 3.2. **In a three-dimensional Finsler space $F^3$, a special vector field of sixth kind is such that both $\beta_{j/k}$ and $\beta_{j//k}$ are non-symmetric in j and k and satisfy equations (3.11) and (3.12).**

 

**Def. 3.3**. A vector field $X^i(x)$ in $F^3$, satisfying $X^i_{/j} = -\delta^i_j$ and $X^i Y_{ij} = \gamma_j$, where $\gamma_j$ is a non-zero vector field in $F^3$, shall be called a special vector field of seventh kind.

In this case we get

$$\gamma_j = B n_j - D m_j \tag{3.13}$$

$$\gamma_{j/k} = Y_{jk} \tag{3.14}$$

$$\gamma_{j//k} = (C_{(1)} B - C_{(2)} D) m_k n_j - (C_{(3)} B + C_{(2)} D) m_j n_k + L^{-1}\{A(m_j n_k - m_k n_j) - l_j \gamma_k\} + (C_{(3)} D - C_{(2)} B)(n_j n_k - m_j m_k) \tag{3.15}$$

$$\gamma_{j/k} - \gamma_{k/j} = 2 Y_{jk},\ \gamma_{j/k} + \gamma_{k/j} = 0, \tag{3.16}$$

$$\gamma_{j//k} - \gamma_{k//j} = m_j n_k (C_{(1)} B - C_{(2)} D - 2C_{(3)} B) - m_k n_j (C_{(2)} D - C_{(3)} B - 2C_{(1)} B) + L^{-1}\{3A (m_j n_k - m_k n_j) - (l_j \gamma_k - l_k \gamma_j)\} \tag{3.17}$$

Hence:

Theorem 3.3. **In a three-dimensional Finsler space $F^3$, a special vector field of seventh kind is such that $\gamma_{j/k}$ is skew-symmetric while $\gamma_{j//k}$ is non-symmetric in j and k and satisfy equations (3.16) and (3.17) respectively.**

## IV. SOME CURVATURE PROPERTIES.

From equation (2.5), we can get

$$\varphi_{j/k/r} = - h_r (l_j n_k + l_k n_j - A m_j h_k + B h_k l_j) + h_k (l_r n_j - l_j n_r) + h_{k/r}(D l_j - A n_j),$$

which yields

$$\varphi_{j/k/r} - \varphi_{j/r/k} = 2 n_j(l_r h_k - l_k h_r) + (D l_j - A n_j)(h_{k/r} - h_{r/k}) \tag{4.1}$$

To find the value of $(h_{k/r} - h_{r/k})$, we find h-derivative of $m_{j/k} = n_j h_k$, which gives

$$m_{j/k/r} - m_{j/r/k} = n_j (h_{k/r} - h_{r/k}) \tag{4.2}$$

Substituting value of Left- hand side of (4.2) with the help of equation (1.6) and multiplying the resulting equation by $n^j$, we get on simplification

$$h_{r/k} - h_{k/r} = n^j m_p R^p_{jkr} + L^{-1} v_p R^p_{kr} \tag{4.3}$$

If we find h-derivative of $n_{j/k} = - m_j h_k$ and do the similar calculation as for equation (4.3), we get

$$h_{k/r} - h_{r/k} = - m^i n_p R^p_{ijk} + L^{-1} v_p R^p_{kr} \tag{4.4}$$

Comparing equations (4.3) and (4.4) we get

$$R^p_{ijk} (m_p n^i + n_p m^i) = 0. \tag{4.5}$$

Hence:

Theorem 4.1. **In a three-dimensional Finsler space $F^3$, curvature tensor $R^p_{ijk}$ satisfies equation (4.5).**

It is known that in $F^3$, $R^p_{ijk}$ can be expressed as Matsumoto [3]

$$R^p_{ijk} = \mathcal{C}_{(j,k)}\{g_{ij} R^p_k + \delta^p_k(R_{ij} - (½)R g_{ij})\}, \tag{4.6}$$

therefore, from equations (4.5) and (4.6) on simplification we can obtain

$$R^p_k (m^k m_p + n^k n_p) = (1/2) R \tag{4.7}$$

Hence:

Theorem (4.2). **In a three-dimensional Finsler space $F^3$, tensor $R^p_k$ satisfies equation (4.7).**

Putting value of left-hand side of equation (4.1) with the help of equation (1.6), substituting value of $(h_{k/r} - h_{r/k})$ with the help of equation (4.3) and multiplying the resulting equation by $m^j$, we get on simplification

$$m^j \varphi_p R^p_{jkr} + L^{-1} (D n_p + 2 B m_p) R^p_{kr} = 0 \tag{4.8}$$

Hence:

Theorem 4.3. **In a three-dimensional Finsler space $F^n$, the curvature tensor $R^p_{ijk}$ for a special vector field of second kind satisfies equation (4.8).**

From equation (2.11), we can obtain the value of $\psi_{j/k/r} - \psi_{j/r/k}$, which after simplification leads to

$$R^p_{tkr}(\varphi_j m_p n^t + \psi_p \delta^t_j) + R^p_{kr}(L^{-1} \varphi_j v_p + \psi_{j//p}) = 0 \tag{4.9}$$

Hence:

Theorem 4.4. **In a three-dimensional Finsler space $F^3$, the curvature tensor $R^p_{ijk}$ for a special vector field of third kind satisfies equation (4.9).**

Similarly, from equation (2.16), we obtain the value of $\omega_{j/k/r} - \omega_{j/r/k}$, which on simplification leads to

$$R^p_{tkr} [2(B m_j - D n_j) n^t m_p + \omega_p \delta^t_j] + R^p_{kr} [2(B m_j - D n_j) L^{-1} v_p + \omega_{j//p}] = 0 \tag{4.10}$$

Hence:

 

Theorem 4.5. **In a three-dimensional Finsler space $F^3$, the curvature tensor $R^p_{ijk}$ for a special vector field of fourth kind satisfies equation (4.10).**

In case of definition (3.1), for a vector $\alpha_j$, we obtain value of $\alpha_{j/k/r}$- $\alpha_{j/r/k}$, which on simplification leads to

$$R^p_{tkr} (\alpha_p \delta^t_j - n^t m_p \beta_j) + R^p_{kr} (\alpha_{j//p} - L^{-1} v_p \beta_j) = 0 \quad (4.11)$$

Hence:

Theorem 4.6. **In a three-dimensional Finsler space $F^3$, the curvature tensor $R^p_{ijk}$ for a special vector field of fifth kind satisfies equation (4.11).**

In case of definition (3.2), for a vector $\beta_j$, we obtain value of $\beta_{j/k/r}$ - $\beta_{j/r/k}$, which on simplification leads to

$$R^p_{tkr} (\beta_p \delta^t_j - \alpha_j n^t m_p) + R^p_{kr} (\beta_{j//p} - L^{-1} v_p \alpha_j) = 0 \quad (4.12)$$

Theorem 4.7. **In a three-dimensional Finsler space $F^3$, the curvature tensor $R^p_{ijk}$ for a special vector field of sixth kind satisfies equation (4.12).**

In case of definition (3.3), for a vector $\gamma_j$, we can observe that $\gamma_{j/k/r} = 0$. Hence:

Theorem 4.8. **In a three-dimensional Finsler space $F^3$, tensor $\gamma_{j/k}$, for a special vector field of seventh kind satisfies $\gamma_{j/k/r} = 0$.**

In subsequent research work we propose to study properties of other curvature tensors, vis-a-vis different type of special vector fields studied in a three - dimensional Finsler space.

## REFERENCES


[1] S.C. Rastogi and P. Bajpai, "On certain special vector fields in a Finsler space", Acta Ciencia Indica, XLIII M no. 2, (2017), 149-152.

[2] P.K. Dwivedi, S.C. Rastogi and A.K. Dwivedi, "On certain special vector fields in a Finsler space I", IJAIR, Vol. 8, Issue 3, (2019), 1-6.

[3] M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, Kaiseisha Press, Otsu, Japan, 1986.

[4] H.Rund, The differential geometry of Finsler spaces, Springer-Verlag, 1959.


 