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Indicatrix Given by a Square Change of (*α*, *β*)-metric

Mohammad Rafee[1*], Avdhesh Kumar[2], G.C. Chaubey[3]

[1] Department of Mathematics, I.K. Gujral Punjab Technical University, Kapurthala, India
[2] Department of Mathematics, I.K. Gujral Punjab Technical University, Kapurthala, India
[3] Department of Mathematics, T.D.P.G. College, V.B.S. Purvanchal University, Jaunpur, India

Corresponding Author: mohd_rafee60@yahoo.com, Tel.: +91-9729332662

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Abstract: Characteristic properties of indicatrix has been studied by so many authors in different situations. In 1984, C. Shibata introduced the concept of *β*-change in Finsler geometry. Using the theory of *β*-change, we study the behavior of indicatrix given by square change of famous Z. Shen's square metric.

Keywords— Indicatrix, β-change, Square change, conformal change, Square metric.

I. INTRODUCTION

Starting from its emergence, the theory of Finsler space [13] is an important topic of research in Differential geometry. C. Shibata [16] introduced the concept of *β*-change in Finsler geometry. Later on, S.H. Abed [1] generalized the theory of *β*-change and introduced the concept of conformal *β*-change in Finsler geometry. The theory of indicatrix has been studied by so many authors [8, 11, 14]. In this paper, we study the behavior of indicatrix under square change of (*α*, *β*)-metric.

The paper is organized as follows: In section 2, we discuss the theory of *β*- change in particular square change. In section 3, we consider the indicatrices given by a particular *β*-change, known a square change and study its properties in detail.

II. PRELIMINARIES

Let M be a connected n-dimensional smooth manifold and let $TM = \bigcup_{x\in M} T_xM$ be its tangent bundle, where T_xM is the tangent space at x$\in M$. A generic point u in *TM* is denoted by $u = (x, y) \in TM$, where $y \in T_xM$. Locally, if $x = (x^i)$ is a local coordinate system on *M*, then $u =$

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad (2.1)$$

$$|s| \leq \text{b} < b_0$$

$$\phi(s) - s\phi'(s) > 0, |s| < b_0 \qquad (2.2)$$

$(x, y) = (x^i, y^i) \in TM$, where i=1, 2…n, and $y = y^i \frac{\partial}{\partial x^i}_x$.

Recall, that a Finsler metric on *M* is a function *F: TM* → [0, ∞) with properties:

(i) F is C^∞ on $\widetilde{TM}$:= *TM*\{0}

(ii) $F(x, \lambda y) = \lambda F(x, y)$ for any *λ* > 0

(iii) The hessian matrix $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite for all (*x*, *y*) $\in\widetilde{TM}$.

One of the most studied examples of Finsler metrics are the (*α*, *β*)-metrics, i.e., the metrics of the form $F = F(\alpha, \beta)$ where *F* is a positive 1-homogeneous function of two arguments *α* and *β*.

Here $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ a 1-form on *M*. These metrics have a lot of applications [2, 3, 6]. Following Shen, Finsler metric *F* can be written in the form $F = \alpha\phi\frac{\beta}{\alpha}$, where $\phi: I = [-b, b] \longrightarrow [0, \infty]$ is a C^∞-function, and the interval I can be chosen big enough such that $r \geq \frac{\beta}{\alpha}$, $\forall x \in M$ and $y \in T_xM$.

Lemma 2.1 (Shen's Lemma) The function $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, is a Finsler metric for any $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and any $\beta = b_i(x)y^i$ with $\| \beta_x \|_\alpha < b_0$ if and only if $\phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying the following conditions:

Classical examples of (α, β)-metrics are: the Randers metric, namely $F=\alpha+\beta$, Matsumoto metric, i.e., $F=\frac{\alpha^2}{\alpha-\beta}$, Z. Shen's square metric, i.e., $F=\frac{(\alpha+\beta)^2}{\alpha}$ and many others [9,12].

In particular, we are interested in square metric $F=\frac{(\alpha+\beta)^2}{\alpha}$. In this case, we have $F=\alpha\phi(s)$, where $\phi(s)=(1+s)^2$. Next, we discuss some definitions and results required for the next section.

Definition 2.1 *If there exist a covariant vector field P_i such that the hv-curvature tensor P_{hijk} of F^n can be written in the form*

$$P_{hijk}=P_hC_{ijk}-P_iC_{hjk},$$

then the Finsler space F^n $(n>2)$ is called P2- like.

For a P2-like Finsler space F^n $(n>2)$, we recall [10] the following theorem:

Theorem 2.1 *If F^n $(n>2)$ is a P2-like Finsler space, then its hv-curvature P_{hijk} or v-curvature tensor S_{hijk} vanishes, i.e. $S_{hijk}=0$.*

Definition 2.2 *If the third curvature tensor R_{hijk} of Cartan space can be expressed in the form*

$$R_{hijk}=g_{hj}F_{ik}+g_{ik}F_{hj}-g_{hk}F_{ij}-g_{ij}F_{hk},$$

where

$F_{ik}=\frac{1}{n-2}\left(R_{ik}-\frac{r}{2}g_{ik}\right), R_{hj}=R^m_{hjm}, r=\frac{1}{n-1}R^m_m$ *then the Finsler space F^n $(n>3)$ is called R3-like.*

For the (v)hv-torsion tensor P_{hijk} and the (h)hv torsion tensor C_{hij} we define $*P_{hij}=P_{hij}-\lambda C_{hij}$.

where the scalar λ is homogeneous of degree one in y^i and is given by $\frac{P_iC^i}{C_jC^i}$ for $C_j\neq 0$.

Definition 2.3 *If the torsion tensor $*P_{hij}=0$, then the Finsler space F^n $(n>2)$ is called a *P-Finsler space.*

Definition 2.4 [4] *If the (v)hv-torsion tensor $P_{hij}=0$, the the Finsler space F^n is called a Landsberg space.*

Definition 2.5 [6] *If the v-curvature tensor S_{hijk} of a non-Riemannian Finsler space F^n $(n>4)$ can be written in the form*

$$F^2S_{hijk}=h_{hj}M_{ik}+h_{ik}M_{hj}+h_{hk}M_{ij}-h_{ij}M_{hk},$$

where M_{ij} *is symmetric and indicatory tensor given by* $M_{ij}=\frac{1}{n-3}[S_{ij}-\frac{Sh_{ij}}{2(n-2)}]$ *, then it is called S4- like.*

Next, we recall the following theorems [17] for later use.

Theorem 2.2 *Every R3-like (non-Landsberg) *P-Finsler space F^n $(n>4)$ is S4- like.*

Theorem 2.3 *An R3-like Landsberg space F^n $(n>3)$ is a Finsler space satisfying $S_{hijk}=0$ or a Riemannian space of constant curvature.*

III. INDICATRICES GIVEN BY A SQUARE CHANGE

Let M be an n-dimensional connected, smooth manifold equipped with a Finsler metric F. Then for any x$\in M$, the tangent space T_xM can be regarded as an n-dimensional Riemannian space with the fundamental metric tensor $g_{ij}(x,y)$, where x= (x^i) is fixed. Then in terms of Cartan connection $C\Gamma$ of the Finsler space F^n the components C^i_{jk} of (h)hv-torsion tensor are Christoffel symbols of T_xM and the v-curvature tensor S^i_{hjk} is the Riemannian curvature tensor of T_xM. The indicatrix I_x at a point x is a hypersurface of the Riemannian space T_xM, defined by the equation $F(x,y)=1$, where x is fixed. Consequently, I_x is regarded as an $(n-1)$-dimensional Riemannian space.

Next, we consider the Z. Shen's square metric F=$\frac{(\alpha+\beta)^2}{F}$, and apply β-change on it.

Let us denote the new metric by $\bar{F}$, then

$$\bar{F}=\frac{(F+\beta)^2}{F}=f(F,\beta) \tag{3.1}$$

is called square change of (α,β)-metric, where $\beta=b_i(x)y^i$ is a non-zero 1-form on M.

Differentiating (3.1) with respect to F and β up to and including second order, we get the following relations:

$$f_1=\frac{\partial\bar{F}}{\partial F}=\frac{F^2-\beta^2}{F^2}, f_2=\frac{\partial\bar{F}}{\partial\beta}=\frac{2(F+\beta)}{F}, f_{11}=\frac{\partial^2\bar{F}}{\partial F^2}=\frac{2\beta^2}{F^3}, \quad f_{22}=\frac{\partial^2\bar{F}}{\partial\beta^2}=\frac{2}{F}, f_{12}=\frac{\partial^2\bar{F}}{\partial F\partial\beta}=\frac{-2\beta}{F^2} \tag{3.2}$$

Since $\bar{F}=f(F,\beta)$ is positively homogeneous of of degree one, therefore using Euler's theorem on homogeneous functions we have

$Ff_1 + f_2\beta = f = \bar{F},\ Ff_{12} + \beta f_{22} = 0,\ Ff_{11} + \beta f_{12} = 0$ (3.3)

For the later use, let us define p, q and q_0 as follows:

$p = \frac{ff_1}{F} = \frac{(F+\beta)^3(F-\beta)}{F^4}, q = ff_2 = \frac{2(F+\beta)^3}{F^2} q_0 = ff_{22} = \frac{2(F+\beta)^2}{F^2}$ (3.4)

Furthermore, $\bar{l}_i = \bar{F}_{y^i}$ gives

$\bar{l}_i = f_1 l_i + f_2 b_i$ (3.5)

Differentiating (3.5) with respect to y^j, we get the angular metric tensor $\bar{h}_{ij} = \bar{F}\dot{\partial}_i\dot{\partial}_j\bar{F}$ of $\bar{F}^n$

$$\bar{h}_{ij} = ph_{ij} + q_0 m_i m_j \quad (3.6)$$

where the covariant vector m_i is defined by

$$m_i = b_i - \frac{\beta y_i}{F^2}. \quad (3.7)$$

Here it is to be noted that m_i is a non-zero vector orthogonal to y^i. For, if $m_i = 0$, then $F^2 b_i - \beta y_i = 0$. Differentiation of this with respect to y^j gives $\beta g_{ij} - 2Fl_j b_i + b_j y_{i=0}$, which leads to a contradiction $g_{ij} - l_i l_j = 0$.

Now, from (3.3), (3.4), (3.5), (3.6) and (3.7) the fundamental metric tensor $\bar{g}_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j\bar{F}^2 = \bar{h}_{ij} + \bar{l}_i\bar{l}_j$ of $\bar{F}^n$ is given by

$$\bar{g}_{ij} = pg_{ij} + p_0 b_i b_j + p_{-1}(b_i y_j + b_j y_i) + p_{-2} y_i y_j \quad (3.8)$$

where we put

$$p_0 = q_0 + {f_2}^2 = \frac{6(F+\beta)^2}{F^2} q_{-1} = \frac{ff_{12}}{F} = \frac{-2\beta(F+\beta)^2}{F^4},$$

$$p_{-1} = q_{-1} + \frac{pf^2}{f} = \frac{2(F^3 - 3F\beta^2 - 2\beta^3)}{F^4},$$

$$q_{-2} = \frac{f(f_{11} - \frac{f_1}{F})}{F^2} = \frac{(F+\beta)^2(F^2+3\beta^2)}{F^6}, p_{-2} = q_{-2} + \frac{p^2}{f^2} = \frac{2(F+\beta)^2(F^2 - F\beta + 2\beta)^2}{F^6}$$

Next we have theorem:

Theorem 3.1 *If a Finsler space F^n $(n > 4)$ is a S4-like, then the Finsler space $\bar{F}^n$ obtained by square change is also S4-like.*

A tensor which is invariant under the square change, we call it S-invariant tensor.

The v-curvature S^*_{hijk} under square change, is given by

$$FS^*_{hijk} = S_{hijk} + \frac{1}{n-3} U_{jk}[h_{ij}S_{hk} + h_{nk}S_{ij} - Sh_{ij}h_{hk}], \quad (3.9)$$

where U_{jk} denotes the interchange of indices j, k and subtraction.

Assume that $S^*_{hijk} = S_{hijk}$, then we say that S^*_{hijk} is S-variant.

For a S4-like Finsler space, let us recall the following theorem [6].

Theorem 3.2 Let F^n $(n > 4)$ be a S4-like Finsler space. Then the indicatrix I_x is conformally flat.

If the v-curvature tensor S^*_{hijk} is S-invariant under square change, then we have the following theorem:

Theorem 3.3 A non-Riemannian Finsler Space F^n $(n > 4)$ is S4-like if and only if the S-invariant tensor S^*_{hijk} vanishes.

By theorem (2.1), equation (3.9), theorem (3.3), theorem (3.1) and theorem (3.2), we find the following.

Theorem 3.4 If F^n $(n > 4)$ is P2-like Finsler space, then the indicatrix $\bar{I}_x$ of $\bar{F}^n$ obtained from F^n by square change is conformally flat if $P_{hijk} \neq 0$.

From theorem 2.2, theorem 3.1, and theorem 3.2, we find the following:

Theorem 3.5 If F^n $(n > 4)$ is R3-like (non-Landsberg) *P-Finsler space, then the indicatrix $\bar{I}_x$ of $\bar{F}^n$ obtained from F^n by a square change is conformally flat.

From the theorem (2.3), equation (3.9), theorem (3.3), theorem (3.1) and theorem (3.2), we immediately find:

Theorem 3.6 Assume that F^n $(n > 4)$ is R3-like Landsberg space. Then the indicatrix $\bar{I}_x$ of $\bar{F}^n$, obtained from F^n by a square change is conformally flat, provided F^n is not a Riemannian space of constant curvature.

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