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Perfect 3-Colorings on 4-Regular Graph of Order 8

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Abstract— We study the perfect 3-colorings (also known as the equitable partitions into three parts) on 4-regular graphs of order 8. A perfect n-coloring of a graph is a partition of its vertex set into n parts $A_1, A_2, ..., A_n$ such that for all p, q ϵ {1, 2,..., n}, each vertex of A_p is adjacent to a_{pq} number of vertices of A_q. The matrix $A = (a_{pq})_{n\times n}$ is called quotient matrix or parameter matrix. The concept of a perfect coloring generalizes the concept of completely regular code introduced by P. Delsarte. In particular, we classify all the realizable parameter matrices of perfect 3-colorings on 4-regular graphs of order 8.

Keywords— Perfect colorings; equitable partition; regular graph.

I. INTRODUCTION

We consider only undirected finite simple graphs. Let G be a connected graph then we define $x, y \in V(G)$, $d(x, y) := dist(x, y)$ in G (i.e the minimum number of edges in a path joining x and y in G). The diameter of G, $\text{diam}(G) = max_{x,y \in V(G)}\, d(x, y) = r$ (say).

For $X \subseteq V(G)$, the induced subgraph $G[X]$ is a graph with vertex set X and edge set $E(G[X]) = \{e = (x, y) \in E(G) : x, y \in X\}$. For $x \in V(G)$, $G_i(x) = \{y \in V(G): d(x, y) = i\}$, where $i \in \{1, 2, ..., r\}$ and $G_{-1}(x) = G_{r+1}(x) = \varphi$. We will write $G(x)$ instead of $G_1(x)$.

A connected graph G with diameter r is called distance-regular graph if there exist integers x_i, y_i, z_i, where $i \in \{1, 2,...,r\}$ such that for every $x, y \in V(G)$ and $d(x, y) = i$, and z_i neighbors of x in $G_{i-1}(y)$ and y_i neighbors of x in $G_{i+1}(y)$ and $x_i = y_0 - y_i - z_i$. The numbers x_i, y_i, z_i, where $i \in \{1, 2,..., r\}$ are called the intersection number and the array $\{y_0, y_1,..., y_{r-1}; z_1, ..., z_r\}$ is called the intersection array of the distance-regular graph G.

For a graph G and a positive integer n, the mapping $T : V(G) \rightarrow \{1, 2, ..., n\}$ is called a perfect n-coloring with matrix $A = (a_{ij})$, where $i, j \in \{1, 2, ..., n\}$, if it is surjective and for all i, j for every vertex of color i, the number of its neighbors of color j is equals to a_{ij}. The matrix A is called the parameter matrix or quotient matrix of a perfect coloring. In other words perfect n coloring is the equitable partitions of the vertex set into n disjoint parts.

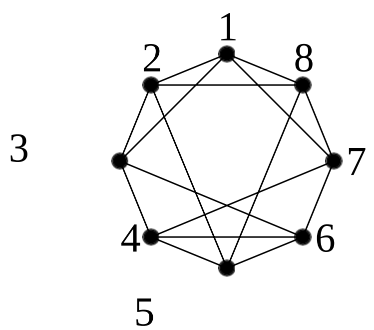
A non empty set $C \subseteq V(G)$ is called a code. Elements of C are called codewords. The distance of $x \in V(G)$ from C is $d(x, C) := \min\{d(x, y): y \in C\}$ and the covering radius $\rho_C := max_{x \in V(G)} d(x, C)$ of C. A code C gives a natural partition of $V(G)$ and the partition is $\sqcap = \{G_0(C), G_1(C),..., G_{\rho_C}(C)\}$. For $x \in V(G)$, $\delta_i(x, C) := |G_i(x) \cap C|$ is called the outer distribution numbers of C, where $i \in \{1, 2, ..., r\}$. A code C in the distance-regular graph G is called completely regular code if $\delta_i(x, C)$ only depends on i and $d(x, C)$. Note that a code C is completely regular iff $\sqcap$ is perfect $(\rho_C + 1)$-coloring, see [2]. So perfect coloring is a generalization of completely regular codes.

The existence of completely regular codes in graphs is a historical problem in mathematics. In 1973, Delsarte [4] conjectured the non-existence of nontrivial perfect codes in Johnson graphs. Therefore, some effort has been made on enumerating the parameter matrices of some Johnson graphs, including $J(4, 2)$, $J(5, 2)$, $J(6, 2)$, $J(6, 3)$, $J(7, 3)$, $J(8, 3)$, $J(8, 4)$ and $J(v, 3)$ (v odd) (see [1], [12], [13], [14]). Fon-Der-Flass enumerated the parameter matrices (perfect 2-colorings) of n-dimensional hypercube Q_n for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n- dimensional cube with a given parameter matrix (see [7], [8], [9]). Aleiyan and

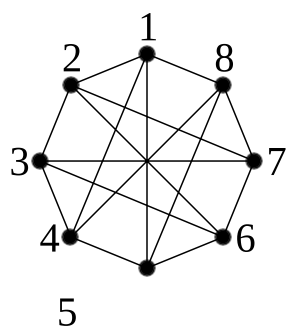
Meherbani [10] obtained perfect 3- colorings of cubic graphs on 10 vertices. M. Alaeiyan and H. Karami [11] obtained perfect 2-colorings of generalised Petersen graph. In this paper we discuss about perfect 3-colorings on 4-regular graphs of order 8.

II. PRELIMINARIES

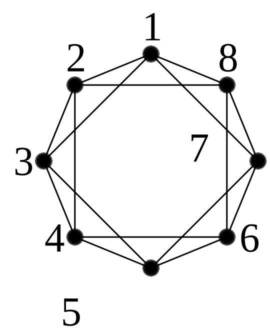
In [6], it is shown that the number of connected 4-regular graphs with 8 vertices is 6. The graphs are given below:



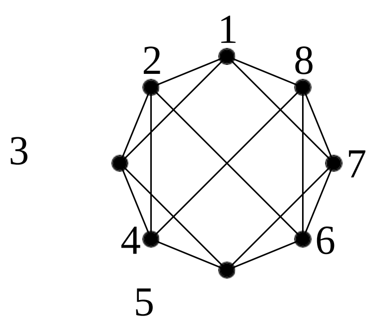
G_1



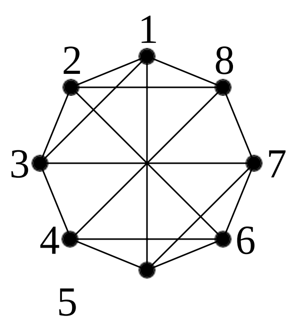
G_2



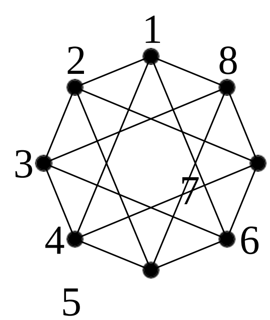
G_3



G_4



G_5



G_6

For perfect 3-colorings, $n = 3$. We called first color white, second color black and third color red. We generally denote a parameter matrix by $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

We consider all perfect 3-colorings up to renaming the colors, i.e. identify the perfect 3-color with the matrices

$$\begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix};\ \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix};\ \begin{bmatrix} a & c & b \\ g & i & h \\ d & f & e \end{bmatrix};\ \begin{bmatrix} e & f & d \\ h & i & g \\ b & c & a \end{bmatrix};\ \begin{bmatrix} i & g & h \\ c & a & b \\ f & d & e \end{bmatrix} \qquad \text{....................}(I)$$

Obtained by switching the colors with the original coloring.

The simplest necessary condition for the existences of perfect 3-colorings of 4-regular connected graph with the matrix A is

$$a + b + c = d + e + f = g + h + i = 4. \qquad \text{..................(II)}$$

Note that the diagonal elements a, e, i of parameter matrix A could not be 4 (as degree of the degree regular graph is 4).

We mean by eigenvalue of a graph is the eigenvalue of the adjacency matrix of this graph.

III. MAIN RESULT

Proposition 1: If T is perfect coloring of a graph G in n colors then any eigenvalue of T is an eigen value of G. (see [3])

Now, without lost of generality, we can assume that $|W| \leq |B| \leq |R|$, where W, B, R represents white, black, red color respectively.

Proposition 2: Let T is perfect 3-coloring of a graph G with the parameter matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Then
1. $|W|b = |B|d$
2. $|W|c = |R|g$
3. $|B|f = |R|h$.

Note that $|W| + |B| + |R| = |V(G)| = 8$ and parameter matrix is symmetric with respect to 0 (i.e if $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$).

Lemma 1.1: Let G be connected 4-regular graph with 8 vertices. And $|W| = 1$, $|B| = 1$, $|R| = 6$ then G has no perfect 3-coloring.

Proof: From proposition 2 we have $b = d$, $c = 6g$, $f = 6h$. $|W| = 1$ gives $a = 0$ and $|B| = 1$ gives $e = 0$. As $c = 6g$, $0 \leq c \leq 4$ and $0 \leq g \leq 4$ gives $g = 0$ which imply $c = 0$. So from condition (II) we get $b = 4$. Similarly $d = 4$, $f = 0$, $h = 0$, $i = 4$. So the parameter matrix can only be $\begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Which represent one white vertex adjacent to four black vertices. But there is only one black vertex. So this parameter matrix is not possible. Therefore G has no perfect 3-colorings.

Lemma 1.2: Let G be connected 4-regular graph with 8 vertices. And $|W| = 1$, $|B| = 2$, $|R| = 5$ then G has no perfect 3-coloring.

Proof: similar as Lemma 1.1.

Lemma 1.3: Let G be connected 4-regular graph with 8 vertices. If T is a perfect 3-colorings with the matrix A and $|W| = 1$, $|B| = 3$, $|R| = 4$, then A should be $\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 1 & 3 & 0 \end{bmatrix}$.

Proof: From proposition 2 we have $b = 3d$, $c = 4g$, $3f = 4h$. $|W| = 1$ gives $a = 0$. As $b = 3d$, $0 \leq b \leq 4$ then $d = 0$ or 1. For $d = 0$ we get $b = 0$. So by (II) $c = 4$ which gives $g = 1$. Now from $3f = 4h$ and $0 \leq f \leq 4$ then $h = 0$ or 3. When $h = 0$ then $f = 0$ and $h = 3$ gives $f = 4$. So the possible parameter matrices are $\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 1 & 3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{bmatrix}$. For second matrix e = 4 which is not possible as $|B| = 3$.

For $d = 1$ we get $b = 3$. So by (II) $c = 1$ this cannot possible as $c = 4g$. So A is the only possible parameter matrix.

Lemma 1.4: Let G be connected 4-regular graph with 8 vertices. If T is a perfect 3-coloring with the matrix A and $|W| = 2$, $|B| = 2$, $|R| = 4$, then A should be one of the followings $\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 2 & 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$.

Lemma 1.5: Let G be connected 4-regular graph with 8 vertices. If T is a perfect 3-coloring with the matrix A and $|W| = 2$, $|B| = 3$, $|R| = 3$, then A should be one of the followings $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

Proof: similar as above.

Note that, we can obtain the second matrix by switching the colors of the first matrix. So we ignore the second one.

So all possible perfect 3-colorings on connected 4-regular graph with 8 vertices are $A_1 = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 1 & 3 & 0 \end{bmatrix}$,

$A_2 = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 2 & 2 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$, $A_4 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ and $A_5 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix}$.

Now we list all the eigenvalues of A_1, A_2, A_3, A_4 and A_5 in the following table:

Matrix	λ_1	λ_2	λ_3
A_1	−4	0	4
A_2	−4	0	4
A_3	−2	0	4
A_4	0	0	4
A_5	−2.56	1.56	4

And all the eigenvalues of the graphs G_1, G_2, G_3, G_4, G_5 and G_6 are listed below:

Graph	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
G_1	−2.56	−1.62	−1.62	−1	0.62	0.62	1.56	4
G_2	−2	−2	−2	0	0	0	2	4
G_3	−2	−2	−1.41	−1.41	0	1.41	1.41	4
G_4	−2.73	−2	−1.41	0	0	0.73	1.41	4
G_5	−3.24	−2	0	0	0	0	1.24	4
G_6	−4	0	0	0	0	0	0	4

Now by proposition (1) the possible parameter matrices of the above graphs are listed below:

Graph	A_1	A_2	A_3	A_4	A_5
G_1	×	×	×	×	✓
G_2	×	×	✓	✓	×
G_3	×	×	✓	✓	×
G_4	×	×	✓	✓	×
G_5	×	×	✓	✓	×
G_6	✓	✓	×	✓	×

Theorem: The parameter matrices of the connected 4-regular graph of order 8 are listed below:

Graph	A_1	A_2	A_3	A_4	A_5
G_1	×	×	×	×	✓
G_2	×	×	✓	✓	×
G_3	×	×	✓	×	×
G_4	×	×	✓	✓	×
G_5	×	×	✓	✓	×
G_6	✓	✓	×	✓	×

Proof: • We know that A_5 is the only possible parameter matrix for G_1. Now we consider mapping $T_{1,5}$ as

$T_{1,5}(4) = T_{1,5}(6) = 1$

$T_{1,5}(1) = T_{1,5}(2) = T_{1,5}(8) = 2$

$T_{1,5}(3) = T_{1,5}(5) = T_{1,5}(7) = 3.$

It is clear that $T_{1,5}$ is the perfect 3-colorings of G_1 with the parameter matrix A_5.

- We know that A_3 and A_4 are the only possible parameter matrices for G_2. Now we consider mapping $T_{2,3}$ and $T_{2,4}$ as

$T_{2,3}(6) = T_{2,3}(8) = 1$

$T_{2,3}(5) = T_{2,3}(7) = 2$

$T_{2,3}(1) = T_{2,3}(2) = T_{2,3}(3) = T_{2,3}(4) = 3.$

$T_{2,4}(7) = T_{2,4}(8) = 1$

$T_{2,4}(5) = T_{2,4}(6) = 2$

$T_{2,4}(1) = T_{2,4}(2) = T_{2,4}(3) = T_{2,4}(4) = 3.$

It is clear that $T_{2,3}$ and $T_{2,4}$ are the perfect 3-colorings of G_2 with the parameter matrices A_3 and

A_4 respectively.

- We know that A_3 and A_4 are the only possible parameter matrices for G_3. Now we consider mapping $T_{3,3}$ as

$T_{3,3}(4) = T_{3,3}(8) = 1$

$T_{3,3}(2) = T_{3,3}(6) = 2$

$T_{3,3}(1) = T_{3,3}(3) = T_{3,3}(5) = T_{3,3}(7) = 3.$

It is clear that $T_{3,3}$ is the perfect 3-colorings of G_3 with the parameter matrix A_3.

claim: A_4 cannot be parameter matrix for G_3. A_4 can be parameter matrix when

$|W| = 2$, $|B| = 2$, $|R| = 4$. As in A_4, $i = 2$ gives four red vartices form a rectangle. In G_3 there are only two rectangles (1,3,5,7) and (2,4,6,8). For each of this rectangle we cannot color by white and black such that it satisfy A_4.

- We know that A_3 and A_4 are the only possible parameter matrices for G_4. Now we consider mapping $T_{4,3}$ and $T_{4,4}$ as

$T_{4,3}(3) = T_{4,3}(7) = 1$

$T_{4,3}(1) = T_{4,3}(5) = 2$

$T_{4,3}(2) = T_{4,3}(4) = T_{4,3}(6) = T_{4,3}(8) = 3.$

$T_{4,4}(2) = T_{4,4}(6) = 1$

$T_{4,4}(4) = T_{4,4}(8) = 2$

$T_{4,4}(1) = T_{4,4}(3) = T_{4,4}(5) = T_{4,4}(7) = 3.$

It is clear that $T_{4,3}$ and $T_{4,4}$ are the perfect 3-colorings of G_4 with the parameter matrices

A_3 and A_4 respectively.

- We know that A_3 and A_4 are the only possible parameter matrices for G_5. Now we consider mapping $T_{5,3}$ and $T_{5,4}$ as

$T_{5,3}(1) = T_{5,3}(6) = 1$

$T_{5,3}(2) = T_{5,3}(5) = 2$

$T_{5,3}(3) = T_{5,3}(4) = T_{5,3}(7) = T_{5,3}(8) = 3.$

$T_{5,4}(3) = T_{5,4}(7) = 1$

$T_{5,4}(4) = T_{5,4}(8) = 2$

$T_{5,4}(1) = T_{5,4}(2) = T_{5,4}(5) = T_{5,4}(6) = 3.$

It is clear that $T_{5,3}$ and $T_{5,4}$ are the perfect 3-colorings of G_5 with the parameter matricesA_3 and A_4 respectively.

- We know that A_1, A_2 and A_4 are the only possible parameter matrices for G_6. Now we consider mapping $T_{6,1}$, $T_{6,2}$ and $T_{6,4}$ as

$T_{6,1}(2) = 1$

$T_{6,1}(4) = T_{6,1}(6) = T_{6,1}(8) = 2$

$T_{6,1}(1) = T_{6,1}(3) = T_{6,1}(5) = T_{6,1}(7) = 3.$

$T_{6,2}(2) = T_{6,2}(4) = 1$

$T_{6,2}(6) = T_{6,2}(8) = 2$

$T_{6,2}(1) = T_{6,2}(3) = T_{6,2}(5) = T_{6,2}(7) = 3.$

$T_{6,4}(5) = T_{6,4}(6) = 1$

$T_{6,4}(7) = T_{6,4}(8) = 2$

$T_{6,4}(1) = T_{6,4}(2) = T_{6,4}(3) = T_{6,4}(4) = 3.$

It is clear that $T_{6,1}$, $T_{6,2}$ and $T_{6,4}$ are the perfect 3-colorings of G_6 with the parameter matrices A_1, A_2 and A_4 respectively.

IV. CONCLUTION

In this article, we study perfect 3- colorings of 4-regular graph of order 8. Here we conclude that only parameter matrix for G_1 is A_5; G_2 are A_3, A_4; G_3 is A_3; G_4 are A_3, A_4; and G_5 are A_1, A_2, A_4.

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