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# Weak $X$-essential Ideals

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***Abstract***- $Weak\ X-essential$ ideals are introduced in this paper which are generalizations of $essential$ ideals and $X-essential$ ideals. Some properties of weak $X-essential$ ideals are investigated. In particular, we proved that the property of weak $X-essential$ is preserved under finite intersection, inverse image and factor rings. We also found out conditions on Noetherian rings which ensure that direct sums of $weak\ X-essential$ ideals are weak $essential$ ideals and vice versa.

***Keywords*** − essential submodule, X-essential submodule, weak X-essential ideal.

## I. INTRODUCTION

All rings are commutative with identity unless mentioned otherwise. The notion of essential submodules were first introduced by Johnson [1] way back in 1951 and the name was given by Eckmann and Schopf in 1953[2]. We begin by recalling some of the definitions. A module $N$ is said to be $essential$(or large) [1] in an $R$ module $M$, abbreviated $N \trianglelefteq M$, in case for any submodule $L$ of $M$, whenever $N \cap L = 0$ we have $L = 0$. In this paper we generalize the concept of $X-essential$ to the concept of $weak\ X-essential$. But in order for the definition to make sense, we introduce $weak\ X-essential$ ideals which generalise $essential$ and $X-essential$ [4] on ideals. Let $R$ be a ring and $X, J$ be ideals of $R$. An ideal $I$ of $R$ contained in $J$ is called $weak\ X-essential$ in $J$(written as $I \trianglelefteq_{X-weak} J$) if for each ideal μ contained in $J$, $\mu \cap I \subseteq X$ we have $\mu^n \subseteq X$ for some $n \in \mathbb{N}$. The purpose of this paper is to investigate the properties of weak $X-essential$ ideals.

**Definition 1.1**: (S. Safaeeyan and N. Saboori Shirazi [4]) Let $R$ be a ring. Let $X$ and $J$ be ideals of $R$. An ideal $I$ of $R$ contained in $J$ is called $X-\ essential$ in $J$ (written $I \trianglelefteq_X J$) if for each ideal μ contained in $J$, $\mu \cap I = X$ implies $\mu \subseteq X$.

**Definition 1.2**: Let $R$ be a ring. Let $X$ and $J$ be ideals of $R$. An ideal $I$ of $R$ contained in $J$ is called $weak\ X-essential$ in $J$ (written $I \trianglelefteq_{X-weak} J$) if for each ideal μ contained in $J$, $\mu \cap I \subseteq X$ implies $\mu^n \subseteq X$ for some $n \in \mathbb{N}$.

Note that $I \trianglelefteq_{X-weak} J$ is also equivalent to saying for every ideal μ with $\mu^n \nsubseteq X$ we have $\mu \cap I \nsubseteq X$.

**Definition 1.3**: Let $I, X$ be (left)ideals of $R$. We say that $I$ is $weak\ X-essential$ if it is $weak\ X-essential$ in $R$.

## II. X-ESSENTIAL AND WEAK X-ESSENTIAL

**Proposition 2.1:** Let $X \subseteq I,\ J$ be sub-ideals of $K$ and suppose that $J$ is maximal with respect to the property $I \cap J = X$. Then $I + J \trianglelefteq_X K$.

Proof: Let μ be sub ideal of $K$ such that $(I + J) \cap \mu \subseteq X$, then $I \cap (J + \mu) \subseteq X$. But $X = I \cap J \subseteq I \cap (J + \mu) \subseteq X$ so $I \cap (J + \mu) = X$. By maximality of $J$ we have $J + \mu = J$, i.e., $\mu \subseteq J$ and therefore as $(I + J) \cap \mu \subseteq X$ we have $\mu \subseteq X$.

**Proposition 2.2:** Let $I, J \subseteq K, X$ be ideals of $R$. Then

1. For $I \subseteq J \subseteq K$ we have $I \trianglelefteq_{X-weak} J$ and $J \trianglelefteq_{X-weak} K$ if $I \trianglelefteq_{X-weak} K$.
2. $I \trianglelefteq_{X-weak} K$ and $J \trianglelefteq_{X-weak} K$ if $I \cap J \trianglelefteq_{X-weak} K$.

**Proof:** Follows easily from definition.

**Lemma 2.3:** Let $X \subseteq I \subseteq J$ be ideals of $R$. Then $I \trianglelefteq_{X-weak} J$ if and only if $\frac{I}{X} \trianglelefteq_{X-weak} \frac{J}{X}$.

**Proof:** Suppose $I \trianglelefteq_{X-weak} J$. Let $\frac{\mu}{X}$ be an ideal of $\frac{R}{X}$ contained in $\frac{J}{X}$ such that $\frac{\mu}{X} \cap \frac{I}{X} = 0$, hence $\mu \cap I = X$. Now since $I \trianglelefteq_{X-weak} J$, we have $\mu^n = X$ and hence $\left(\frac{\mu}{X}\right)^n = 0$. Therefore $\frac{I}{X} \trianglelefteq_{X-weak} \frac{J}{X}$.

 

Conversely, let μ be an ideal of $R$ contained in $J$ such that $\mu \cap I \subseteq X$, then $\frac{\mu+X}{X} \cap \frac{I}{X} = 0$. By hypothesis we get that $\left(\frac{\mu+X}{X}\right)^n = 0$ for some $n \in \mathbb{N}$. Therefore $(\mu + X)^n = X$, i.e., $I \trianglelefteq_{X-weak} J$.

**Proposition 2.4:** Let $R$ be a ring and $I, J, X$ be ideals of $R$ such that $I, X \subseteq J$. Then $\frac{I+X}{X} \trianglelefteq_{0-weak} \frac{J}{X}$ if $I \trianglelefteq_{X-weak} J$.
**Proof:** Since, $I \trianglelefteq_{X-weak} J$ and $I \leq I + X \leq J$ we have by Proposition 2.2, $I + X \trianglelefteq_{X-weak} J$. Therefore by Lemma 2.3 it follows that $\frac{I+X}{X} \trianglelefteq_{0-weak} \frac{J}{X}$ if $I \trianglelefteq_{X-weak} J$.

If $R$ is a commutative ring and $I$ an ideal, then the radical of $I$ is an ideal of $R$ such that an element $x$ is in radical of $I$ if some power of $x$ is in $I$. It is denoted by $Rad(I)$.

**Lemma 2.5:** Let $R$ be a commutative noetherian ring. Let $J$ be an ideal of $R$ and $I, X$ be sub-ideals of $J$. Then the following are equivalent:

1. $I \trianglelefteq_{X-weak} J$.
2. For every $a \in J \setminus Rad(X)$, there exists $r \in R$ such that $ra \in I \setminus X$.

**Proof:** (1) ⟹ (2): Let $a \in J \setminus Rad(X)$. Since $I \trianglelefteq_{X-weak} J$ we have $I \cap aR \nsubseteq X$. Therefore, there exists $r \in R$ such that $ar \in I \setminus X$.
(2) ⟹ (1): Let μ be an ideal contained in $J$ such that $\mu \cap I \subseteq X$. By definition, in order to show $I \trianglelefteq_{X-weak} J$, we have to show that $\mu^n \subseteq X$ for some $n \in N$. Claim that $\mu \in Rad(X)$. If claim is false, then there exists $a \in \mu \setminus Rad(X)$, by (2), there exists $r \in R$ such that $ra \in I \setminus X$. But $ra \in \mu \cap I \subseteq X$, which is a contradiction. Hence the claim. Since $R$ is a commutative noetherian ring, it can be proved that $\mu^n \subseteq X$ for some $\mu \in \mathbb{N}$.

**Proposition 2.6:** Let $R$ be a commutative noetherian ring. Let $J$ be an ideal of $R$ and $I, X$ be sub-ideals of $J$. Suppose that for each $a \in J$, $(I : a) \trianglelefteq_{(X:a)-weak} R$, then $I \trianglelefteq_{X-weak} J$.
**Proof:** Let $a \in J \setminus Rad(X)$. By hypothesis, $(I : a) \trianglelefteq_{(X:a)-weak} R$, then by Lemma 2.5 there exists $r \in R$ such that $ra \in (I: a) \setminus (X: a)$. Therefore, $(ra)a \in I \setminus X$. Thus showing that $I \trianglelefteq_{X-weak} J$.

**Corollary 2.7:** Let $I, J$ be ideals of a commutative noetherian ring $R$, such that $I \subseteq J$. Then $I \trianglelefteq_{0-weak} J$ if $(I : a) \trianglelefteq_{ann(a)-weak} R$ , for all $a \in J$.

**Proposition 2.8:** Let $I, J$ be ideals over a commutative noetherian ring $R$ and $P$ be a prime ideal. Then for each $a \in J \setminus P$, the following are equivalent:

1. $(I: a) \trianglelefteq_{P-weak} R$.
2. $I \trianglelefteq_{P-weak} J$.
3. $(I: a) \trianglelefteq_{P} R$.

**Proof:** $(i) \Longrightarrow (ii)$ : Suppose for each $a \in J$, $(I: a) \trianglelefteq_{P-weak} R$. Let $a \in J \setminus Rad(P)$. Since $P$ is prime $P = Rad(P)$ then $a \in J \setminus P$, therefore by assumption, we have $(I: a) \trianglelefteq_{P-weak} R$. Also note that $P = (P : a)$. Therefore $(I: a) \trianglelefteq_{(P:a)-weak} R$. Hence by Proposition 2.6, $I \trianglelefteq_{P-weak} J$.
$(ii) \Longrightarrow (i)$ : Suppose $I \trianglelefteq_{P-weak} J$. Let $a \in J \setminus P$, since $P$ is prime we have $a \notin Rad(P)$. Therefore by Lemma 2.5 there exists $r \in R$ such that $ra \in I \setminus P$, i.e., $(I : a) \nsubseteq P$. Now let μ be an ideal such that, $\mu \cap (I: a) \subseteq P$, then $\mu(I: a) \subseteq \mu \cap (I: a) \subseteq P$. But as $P$ is prime, we have $(I : a) \nsubseteq P$, therefore $\mu \subseteq P$. Hence, $(I: a) \trianglelefteq_{P-weak} R$.
$(i) or (ii) \Longleftrightarrow (iii)$ : clear.

A set of ideals $\{I_i\}_{i=1}^n$ of a ring $R$ is said to be independent if $I_j \cap \sum_{i=1, i\neq j}^n I_i = 0$ for all $j = 1,2, \dots, n$.

**Proposition 2.9:** Let $\{I_i\}_{i=1}^n$ be a set of independent ideals and $\{J_i\}_{i=1}^n$ another set of independent ideals over a commutative noetherian ring $R$. Let $X$ be an ideal of $R$ such that $I_i \trianglelefteq_{X-weak} J_i$ for each $i \in \{1,2, \dots, n\}$, then $\bigoplus_{i=1}^n I_i \trianglelefteq_{X-weak} \bigoplus_{i=1}^n J_i$.
**Proof:** Let $a \in \bigoplus_{i=1}^n J_i \setminus Rad(X)$, then $a = a_1 + a_2 + \cdots + a_n$ where each $a_i \in J_i$ for all $i = 1, 2, \dots n$. Therefore there exists at least one $i \in \{1,2, \dots, n\}$ such that $a_i \notin Rad(X)$. With out any loss $a_1 \notin Rad(X)$. Then $a_1 a = a_1^2 \in J_1 \setminus Rad(X)$. Since $I_1 \trianglelefteq_{X-weak} J_1$, by Lemma 2.5, there exists $r \in R$ such that $ra_1 a \in I_1 \setminus X$. Therefore, $ra_1 a \bigoplus_{i=1}^n I_i \setminus X$. Hence $\bigoplus_{i=1}^n I_i \trianglelefteq_{X-weak} \bigoplus_{i=1}^n J_i$.

 

**Proposition 2.10:** Let $\{J_i\}_{i=1}^{n}$ be set of independent ideals and $\{I_i\}_{i=1}^{n}$ be another set of ideals over a commutative noetherian ring $R$ such that $I_i \subseteq J_i$ for all $i = 1, 2, \ldots n$. Let $X$ be an ideal of $R$ then $I_i \trianglelefteq_{X-weak} J_i$ if and only if $\bigoplus_{i=1}^{n} I_i \trianglelefteq_{X-weak} \bigoplus_{i=1}^{n} J_i$.
**Proof:** The direct part is done in the previous proposition.

Conversely, let $a_1 \in J_1 \setminus Rad(X)$. Then $a_1 \bigoplus_{i=1}^{n} J_i$. Therefore by Lemma 2.5, there exists $r \in R$ such that $ra_1 \bigoplus_{i=1}^{n} I_i \setminus X$. Then $ra_1 \in I_1 \setminus X$, since $\{J_i\}_{i=1}^{n}$ is independent set and $I_i \subseteq J_i$ for all $i = 1, 2, \ldots n$. Hence $I_1 \trianglelefteq_{X-weak} J_1$. Therefore, it follows that $I_i \trianglelefteq_{X-weak} J_i$ for every $i = 1, 2, \ldots n$.

**Proposition 2.11:** Let $X$ be an ideal of a commutative noetherian ring $R$. Let $\{J_i\}_{i=1}^{n}$ be the set of ideals satisfying $J_i \cap \sum_{j\neq i, j=1}^{n} J_j \subseteq X$ and $\{I_i\}_{i=1}^{n}$ be another set of ideals of $R$ such that $I_i \subseteq J_i$ and $X \cap J_i \subseteq I_i$ for all $i = 1, 2, \ldots n$. Then for all $i = 1, 2, \ldots n$, $I_i \trianglelefteq_{X-weak} J_i$ if and only if $\sum_{i=1}^{n} I_i \trianglelefteq_{X-weak} \sum_{i=1}^{n} J_i$.
**Proof:** Suppose $I_i \trianglelefteq_{X-weak} J_i$ for all $i = 1, 2, \ldots n$. Let $a \in \sum_{i=1}^{n} J_i \setminus Rad(X)$, then $a = a_1 + a_2 + \cdots + a_n$ where $a_i \in J_i$ for all $i = 1, 2, \ldots n$. Therefore, there exists at least one $i \in \{1,2, \ldots, n\}$ such that $a_i \notin Rad(X)$. Without any loss, let $a_1 \notin Rad(X)$. Clearly, since $J_1 \cap\in \sum_{j=2}^{n} J_j \subseteq X$, it follows that $a_1 a \in J_1 \setminus Rad(X)$. Since $I_1 \trianglelefteq_{X-weak} J_1$, by Lemma 2.5, there exists $r \in R$ such that $ra_1 a \in I_1 \setminus X$. Therefore, $ra_1 a \in \sum_{i=1}^{n} I_i \setminus X$. Hence, $\sum_{i=1}^{n} I_i \trianglelefteq_{X-weak} \sum_{i=1}^{n} J_i$.

Conversely, let $a_1 \in J_1 \setminus Rad(X)$ then $a_1 \in \sum_{i=1}^{n} J_i \setminus Rad(X)$. Therefore by Lemma 2.5, since $\sum_{i=1}^{n} I_i \trianglelefteq_{X-weak} \sum_{i=1}^{n} J_i$, there exists $r \in R$ such that $ra_1 \in \sum_{i=1}^{n} I_i \setminus X$. Then $ra_1 \in I_1 \setminus X$, since $J_i \cap \sum_{j\neq i, j=1}^{n} J_j \subseteq X$, $I_i \subseteq J_i$ and $X \cap J_i \subseteq I_i$ for all $i = 1, 2, \ldots n$. Hence $I_1 \trianglelefteq_{X-weak} J_1$. Therefore it follows that $I_i \trianglelefteq_{X-weak} J_i$ for every $i = 1, 2, \ldots n$.

**Proposition 2.12:** Let $R$ be a commutative noetherian ring. Let $X_1 \subseteq I_1 \subseteq J_1$ and $X_2 \subseteq I_2 \subseteq J_2$ be ideals of $R$ satisfying $X_1 \cap X_2 = J_1 \cap J_2$. Then $I_1 + I_2 \trianglelefteq_{X_1+X_2-weak} J_1 + J_2$ if and only if $I_1 \trianglelefteq_{X_1-weak} J_1$ and $I_2 \trianglelefteq_{X_2-weak} J_2$.
**Proof:** Suppose $I_1 + I_2 \trianglelefteq_{X_1+X_2-weak} J_1 + J_2$. Let $\mu$ be an ideal of $R$ contained in $J_1$ such that $\mu \cap I_1 \subseteq X_1$, then $\mu \cap (I_1 + I_2) \subseteq X_1 + X_2$. By assumption, we have $\mu^n \subseteq X_1 + X_2$ for some $n \epsilon \mathbb{N}$, which can be easily proved that $\mu^n \subseteq X_1$. Hence, $I_1 \trianglelefteq_{X_1-weak} J_1$ and similarly $I_2 \trianglelefteq_{X_2-weak} J_2$.
Conversely, $I_1 \trianglelefteq_{X_1-weak} J_1$ and $I_2 \trianglelefteq_{X_2-weak} J_2$. Let $a_1 \in J_1$, $a_2 \in J_2$ such that $a_1 + a_2 \in J_1 + J_2 \setminus Rad(X_1 + X_2)$, then either $a_1 \notin Rad(X_1 + X_2)$ or $a_2 \notin Rad(X_1 + X_2))$. With out any loss, $a_1 \notin Rad(X_1 + X_2)$. As $X_1 \cap X_2 = J_1 \cap J_2$ we can easily verify that $a_1(a_1+a_2) \notin Rad(X_1 + X_2)$ then $a_1(a_1+a_2) \notin Rad(X_1)$. Also note that $a_1(a_1+a_2) \in J \setminus Rad(X_1)$, therefore by Lemma 2.5 since $I_1 \trianglelefteq_{X-weak} J_1$, there exist $r \in R$ such that $ra_1(a_1+a_2) \in I_1 \setminus X_1$. Again by using $X_1 \cap X_2 = J_1 \cap J_2$, we get that $ra_1(a_1+a_2) \notin X_1 + X_2$. Therefore $ra_1(a_1+a_2) \in I_1 + I_1 \setminus X_1 + X_2$. Hence by Lemma 2.5, we have $I_1 + I_2 \trianglelefteq_{X_1+X_2-weak} J_1 + J_2$.

**Proposition 2.13:** Let $R$ be a commutative ring. Let $I, J, X$ be ideals of $R$ such that $X \subseteq J$ and $f \in Hom(I, J)$. Then $Imf \trianglelefteq_{X-weak} J$ if and only if for each $h \in Hom(J, .)$, $kerh \cap Imf \subseteq X$ we have $(Kerh)^n \subseteq X$ for some $n \in N$.
**Proof:** The direct part is clear. Conversely, let $\mu$ be an ideal of $R$ containing in $J$ such that $Imf \cap \mu \subseteq X$. Now $h : J \longrightarrow \frac{J}{\mu}$ by $h(x) = x + \mu$ for all $x \epsilon J$, then clearly $h \epsilon Hom\left(J, \frac{J}{\mu}\right)$ with $Kerh = \mu$. Therefore we see that, $Imf \cap Kerh \subseteq X$, by hypothesis we get, $\mu^n = (Kerh)^n \subseteq X$ for some $n \in N$. Therefore $Imf \trianglelefteq_{X-weak} J$.
**Lemma 2.14:** Let $I, J, K, X$ be ideals of a commutative ring $R$. Let $f : R \longrightarrow R$ be ring homomorphism such that $f^{-1}(K) \subseteq I$. Then $f^{-1}(K) \trianglelefteq_{f^{-1}(X)-weak} I$ if $K \trianglelefteq_{X-weak} J$.
**Proof:** Let $\mu$ be an ideal of $R$ contain in $I$ such that $\mu \cap f^{-1}(K) \subseteq f^{-1}(X)$. Then clearly $f(\mu) \cap K \subseteq X$ and by hypothesis we get $[f(\mu)]^n \subseteq X$ for some $n \in N$. Since $f$ is a homomorphism $\mu^n \subseteq f^{-1}(X)$. Therefore $f^{-1}(K) \trianglelefteq_{f^{-1}(X)-weak} I$.

**Corollary 2.15:** Let $I, J, K$ be ideals of a commutative ring $R$ such that $K \subseteq J$ and $f : R \longrightarrow R$ be ring homomorphism. Suppose that $f^{-1}(K) \subseteq I$, then if $K \trianglelefteq_{0-weak} J$, we have $f^{-1}(K) \trianglelefteq_{Kerf-weak} I$. Moreover, if $f$ is an epimorphism, then $K \trianglelefteq_{0-weak} J$ if and only if $f^{-1}(K) \trianglelefteq_{Kerf-weak} I$.
**Proof:** Suppose $K \trianglelefteq_{0-weak} J$. By Lemma 2.14, we have $f^{-1}(K) \trianglelefteq_{f^{-1}(0)-weak} I$. But $f^{-1}(0) = Kerf$, therefore $f^{-1}(K) \trianglelefteq_{Kerf-weak} I$.

If $f$ is an epimorphism, the direct part is done above. Conversely, let $\mu$ be an ideal of $R$ contained in $J$ such that $\mu \cap K = 0$. Then $f^{-1}(K) \cap f^{-1}(\mu) \subseteq Kerf$. Since $f^{-1}(K) \trianglelefteq_{Kerf-weak} I$ we have $[f^{-1}(\mu)]^n \subseteq Kerf$ for some $n \in N$. Again since $f$ is an epimorphism we have $\mu^n = 0$. Therefore $K \trianglelefteq_{0-weak} J$.

**Proposition 2.16:** Let $R$ be a commutative noetherian ring, $F = \{1,2 \ldots, n\}$ and for every $i \epsilon F$, $I_i$ are non-zero independent ideals of $R$. Let $I = \bigoplus_{i\in F} I_i$, then for every non-empty subset $F'$ of $F$ we have $\bigoplus_{i\in F} I_i \trianglelefteq_{X-weak} I$ where $X = \bigoplus_{i\in F\setminus F'} I_i$.

 

**Proof:** Let $a \in I \setminus Rad(X)$. Then $a = a_1 + a_2 + \cdots + a_n$ where $a_i \in I_i$ for every $i \in F$. Since $a \notin Rad(X)$, there exists $a_i \in I_i$ for some $i \in F$ such that $a \notin Rad(X)$, therefore $i \in F'$. Taking $r = a_i$ we get $ra = a_i^2 \in I_i \setminus X$. Therefore $ra = \bigoplus_{i \in F'} I_i \setminus X$. Hence $\bigoplus_{i \in F'} I_i \trianglelefteq_{X-weak} I$.

**Proposition 2.17:** Let $R$ be a commutative ring. Let $X$ be the nil-radical of $R$. Then for an ideal $I$ with $X \subseteq I$ we have $I \trianglelefteq_{X-weak} R$ if and only if $\frac{I}{X} \trianglelefteq \frac{R}{X}$.

**Proof:** Suppose $I \trianglelefteq_{X-weak} R$. Let $\frac{\mu}{X}$ be an ideal of $R$ such that $\frac{\mu}{X} \cap \frac{I}{X} = 0$ then $\mu \cap I = X$. By hypothesis, $\mu^n \subseteq X$ for some $n \in \mathbb{N}$. Since $X$ is nil-radical, therefore it follows that $\mu \subseteq X$. Thus $\mu = X$ and therefore $\frac{\mu}{X} = 0$. Hence $\frac{I}{X} \trianglelefteq \frac{R}{X}$.

Conversely, suppose $\frac{I}{X} \trianglelefteq \frac{R}{X}$ . Let $\mu$ be an ideal such that $\mu \cap I \subseteq X$, then $\frac{\mu+X}{X} \cap \frac{I}{X} = 0$. By assumption, $\frac{\mu+X}{X} = 0$, therefore $\mu + X = X$. Then $\mu \subseteq X$. Hence $I \trianglelefteq_{X-weak} R$.

**Proposition 2.18:** Let $I, J \subseteq K$ be ideals of a commutative ring $R$ and $X$ its nil-radical (or instead we can take the largest nil-ideal contain in $K$). Then

1. $I \trianglelefteq_{X-weak} K$ and $J \trianglelefteq_{X-weak} K$ if and only if $I \cap J \trianglelefteq_{X-weak} K$.
2. Let $I \subseteq J \subseteq K$. Then $I \trianglelefteq_{X-weak} J$ and $J \trianglelefteq_{X-weak} K$ if and only if $I \trianglelefteq_{X-weak} K$.

**Proof:** Proof follows easily from definition.

**Proposition 2.19:** Let $R$ be a commutative ring and $X$ its nil-radical (or instead we can take the largest nil-ideal contain in $J$). Let $J$ be an ideal of $R$ and $I$ be sub-ideal of $J$. Then the following are equivalent:

1. $I \trianglelefteq_{X-weak} J$.
2. For every $a \in J \setminus X$, there exists $r \in R$ such that $ra \in I \setminus X$.
3. For each $a \in J \setminus X$, $(I : a) \trianglelefteq_{(X:a)-weak} R$.

**Proof:** Similar to Lemma 2.5 and Proposition 2.6.

**Proposition 2.20:** Let $I_1, I_2, J_1, J_2$ be ideals of a commutative ring $R$ and $X$ its nil-radical. If $I_1 \trianglelefteq_{X-weak} J_1$ and $I_2 \trianglelefteq_{X-weak} J_2$, then $I_1 \cap I_2 \trianglelefteq_{X-weak} J_1 \cap J_2$.

**Proof:** Let $\mu$ be a sub-ideal of $J_1 \cap J_2$ such that $\mu \cap (I_1 \cap I_2) \subseteq X$. Then as $X$ is nil-ideal and $I_2 \trianglelefteq_{X-weak} J_2$, we have $\mu \cap I_1 \subseteq X$. Also since $I_1 \trianglelefteq_{X-weak} J_1$ we get that $\mu \subseteq X$.

**Proposition 2.21:** Let $I, J, K$ be ideals of a commutative ring $R, X$ the nil-radical and $f : J \longrightarrow K$ be homomorphism. If $I \trianglelefteq_{X-weak} K$, then $f^{-1}(I) \trianglelefteq_{f^{-1}(X)-weak} J$, infact $f^{-1}(I) \trianglelefteq_{f^{-1}(X)} J$.

**Proof:** Let $\mu$ be a sub-ideal of $J$ satisfying $\mu \cap f^{-1}(I) \subseteq f^{-1}(X)$, then $I \cap f(\mu) \subseteq X$. But as $I \trianglelefteq_{X-weak} K$ and $X$ is nil-ideal, we have $f(\mu) \subseteq X$. Therefore $\mu \subseteq f^{-1}(X)$. Hence $f^{-1}(I) \trianglelefteq_{f^{-1}(X)-weak} J$.

## III. SOME EXAMPLES

**Example 3.1:** An example of ideals which are weak 0-essential but not essential. Take $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in \mathbb{Q} \right\}$.

Then ideals of $R$ are: $I_1 = R, I_2 = 0, I_3 = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid c \in \mathbb{Q} \right\}$, $I_4 = \left\{ \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid b \in \mathbb{Q} \right\}$, $I_5 = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid b, c \in \mathbb{Q} \right\}$, $I_6 = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \mid c, d \in \mathbb{Q} \right\}$, $I_7 = \left\{ \begin{pmatrix} 0 & b & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid b \in \mathbb{Q} \right\}$, $I_8 = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \mid b, c, d \in \mathbb{Q} \right\}$, $I_9 = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid b, c \in \mathbb{Q} \right\}$.

Here $I_3$ and $I_4$ are weak 0-essential but not essential.

**Example 3.2:**

1. If $R = \frac{\mathbb{Z}}{p^2q^2\mathbb{Z}}, I = \frac{p^2\mathbb{Z}}{p^2q^2\mathbb{Z}}, J = \frac{p\mathbb{Z}}{p^2q^2\mathbb{Z}}, X = \frac{p^2q\mathbb{Z}}{p^2q^2\mathbb{Z}}$ where $p, q$ are distinct primes. Then $I \trianglelefteq_{X-weak} J$ and also $I \trianglelefteq_{0-weak} J$ but not $X$-essential in $J$.
2. Let $R = \frac{\mathbb{Z}}{p^2qr\mathbb{Z}}$, $I = \frac{p^2\mathbb{Z}}{p^2qr\mathbb{Z}}$, $J = \frac{p\mathbb{Z}}{p^2qr\mathbb{Z}}$, $X = \frac{p^2q\mathbb{Z}}{p^2qr\mathbb{Z}}$, where $p, q, r$ are distinct primes. Then $I \trianglelefteq_{X-weak} J$ but $I$ is neither weak 0-essential nor $X$-essential in $J$.

 

3. Let $R = \frac{\mathbb{Z}}{p^2 a\mathbb{Z}}$ where $p$ is a prime number, $a \in N$ is a natural number not divisible by $p$. If $a$ is composite and $q$ is a prime number dividing $a$ then take $I = \frac{p^2\mathbb{Z}}{p^2 a\mathbb{Z}}, J = \frac{p\mathbb{Z}}{p^2 a\mathbb{Z}}, X = \frac{p^2 q\mathbb{Z}}{p^2 a\mathbb{Z}}$ . Then $I \trianglelefteq_{X-weak} J$ but $I$ is neither weak 0-essential nor $X$-essential in $J$.
4. Let $R = \frac{\mathbb{Z}}{36\mathbb{Z}}, I = \frac{2\mathbb{Z}}{36\mathbb{Z}}, X = \frac{6\mathbb{Z}}{36\mathbb{Z}}$ . Then $I \trianglelefteq_{X-weak} R$ and $I$ is also $X$-essential but $I \ntrianglelefteq_{0-weak} R$.
5. Let $= \frac{\mathbb{Z}}{36\mathbb{Z}}, I = \frac{2\mathbb{Z}}{36\mathbb{Z}}, X = \frac{9\mathbb{Z}}{36\mathbb{Z}}$. Then $I \trianglelefteq_{0-weak} R$ but $I \ntrianglelefteq_{X-weak} R$ and $I$ is not $X$-essential.

**Example 3.3:** For every $m, n \in \mathbb{Z}$ we have $m\mathbb{Z} \trianglelefteq_{n\mathbb{Z}-weak} (m\mathbb{Z} + n\mathbb{Z})$. In fact $m\mathbb{Z} \trianglelefteq_{n\mathbb{Z}} (m\mathbb{Z} + n\mathbb{Z})$.

## IV. CONCLUSIONS

With the generalisation of $X - essential$ and $essential$ ideals, we have found out that when $R$ is a noetherian ring, $I \trianglelefteq_{X-weak} J$ if and only if for every $a \in J \setminus Rad(X)$, there exists $r \in R$ such that $ra \in I \setminus X$. This property helps us in determining if the ideal $I$ is weak $X - essential$ in $J$ without the use of the definition, in other words without using any ideal $\mu$, instead we only needed to focus on an element $a \in J \setminus Rad(X)$. We have proved so many results on this paper with the help of this property. We also proved that if the ideal $X$ is nilradical of the ring $R$ with $X$ containing in $I$, then $I \trianglelefteq_{X-weak} R$ if and only if $\frac{I}{X} \trianglelefteq \frac{R}{X}$. Most results in this paper are based on a ring $R$ which is assuming to be noetherian. So there are still questions to discuss and results to be found out for a ring $R$ that is not necessarily be noetherian.

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