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Research Paper

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Exact Solutions of the (3+1) Generalized Fractional Nonlinear Wave Equation with Gas Bubbles

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Abstract—This article investigates exact desolate wave blend (solutions) for the fractional (3+1) generalized computational (nonlinear) wave equation (identification) with gas bubbles. Liquids with gas bubbles mainly arise in manifold or crowded applications like science, engineering, nature, and physics. We explored this model using some well-known ansatz techniques and the sine-cosine procedure. These procedure or methods yield different periodic and hyperbolic desalate wave blend (solutions). Moreover, solving the (3+1) Aspect (dimensional) generalized fractional nonlinear wave equation with gas bubbles is equivalent to solving many physical models, such as the (2+1)-dimensional Kadomtsev-Petviashvil model with gloomy despair, the (3+1)-dimensional Kadomtsev-Petviashvili model, the (3+1) dimensional(aspect) nonlinear waves with bubble liquid mixture, and other special cases of the considered model. Finally, we conspiracy both 2D and 3D as well as the curve plots to understand the physical application of the considered model using maple.

Keywords— Wave equation; gas bubbles; ansatz technique; sine-cosine method, desolate wave blend (solutions)

I. INTRODUCTION

The inquiry or exploration of computational (nonlinear) partial differential equation solutions is critical in understanding various physical fact or situation in many scientific and engineering applications. As a result, numerous logical and numerical technique (methods) have been used to tackle a variety of such problems, including the generalized Kudryashov[1], sine-cosine [2], sine-Gordon expansion, extended auxiliary equation [3], direct algebra [5], Sadar sub-equation [6], and the generalized Riccati methods, see [7-39] for more details. These methods heavily depend on wave transformation techniques. However, other analytical techniques didn't depend on the wave transforms approach, among which were the invariant subspace method [12, 13], Lie symmetry method [8, 11], reduction method [14, 15], etc.

The study of bubbling liquids and their applications in various disciplines of engineering and medical sciences has piqued the interest of numerous scholars for decades. Most bubbles with uniform radius are explained by a fourth-order linear partial differential equation for certain physical phenomena in isothermal bubbly liquids [17, 20].

Fundamental investigation (analysis) of the bubble dynamics (change) problem was made by Rayleigh and can be found [16]. Among the interesting models is the equation that described the liquid with the gas bubbles phenomena given as:

$$\left(q_t + g_1 q q_x + g_2 q_{xxx} + g_3 q_x\right)_x + g_4 q_{yy} + g_5 q_{zz} = 0, \qquad (1.1)$$

where q_t and qq_x perform a role in the evolution of time and the steepening of the wave where q is the wave amplitude, the bubble liquid dispersion, g_1, g_2, g_3, g_4, an d g_5 represent the bubble-liquid-nonlinearity, the bubble liquid-viscosity, the y transverse perturbation, and the z transverse perturbation. This is equation is known as the (3+1)-dimensional generalized nonlinear wave equation describing liquid with gas bubbles [27, 28]. Assigning $g_1 = g_2 = g_4 = 1, g_3 = g_5 = 0,$ the equation (1.1) reduces to the popular (2+1)-dimensional Kadomtsev-Petviashvil model with negative dispersion [30] given by:

$$\left(q_t + qq_x + q_{xxx}\right)_x + q_{yy} = 0. \qquad (1.2)$$

Also, if you assign $g_1 = -6, g_2 = 1, g_3 = 0, g_4 = g_5 = 3,$ the equation (1.1) reduces to the well-known (3+1)-dimensional (aspect) Kadomtsev-Petviashvili equation [29], the identify equations is given as:

$$\left(q_t - 6qq_x + q_{xxx}\right)_x + 3\left(q_{yy} + q_{zz}\right) = 0. \quad (1.3)$$

Again, if one set $g_1 = g_2 = 1, g_3 = 0, g_4 = g_5 = 0.5$ will obtain the liquid mixture equation given by

$$\left(q_t + qq_x + q_{xxx}\right)_x + 0.5\left(q_{yy} + q_{zz}\right) = 0. \quad (1.4)$$

This equation is known as the (3+1) –dimensional (aspect) nonlinear waves with bubble liquid solution (mixture) given in [31]. The (3+1)-dimensional generalized nonlinear wave equation describing liquid with gas bubbles is the generalization of the (2+1)-dimensional (aspect) Kadomtsev-Petviashvil model with negative dispersion, the (3+1)-dimensional(aspect) Kadomtsev-Petviashvili equation, and the (3+1) -dimensional (aspect) nonlinear waves with bubble liquid mixture. These equations are paramount in describing(report) different situation (phenomena) in mathematical physics and particular oceanic engineering.

II. RELATED WORK

Moreover, some studies on the logical and numerical blend (solution) of generalized nonlinear model (1.1) with gas bubbles have been investigated in the literature, for example, the bilinear formalism and soliton solutions using Hirota bilinear method [21], Assemble mixed rogue wave-stripe solitons and mixed lump-stripe solitons [23], the binary Bell polynomials obtaining the bilinear form of this model [25], and the solitons and lumps solution for the generalized nonlinear wave [26]. There are several fractional derivative operators in fractional calculus, such as the Caputo derivative, Grunwald derivative, Riemann-Liouville derivative, and so on. Although the research concentrate on the aforementioned derivatives has been difficult to apply in reality due to their limitations. Inspired by Khalil [32], who established a new fractional derivative, called Immitative Fractional (divided) Derivative, that is derived based on classical calculus and possesses semigroup, exponent, and identity features that are useful for solving the differential system. Scholars have just lately begun to focus on this field of study. The fractional derivative is without a doubt another method for improving the prediction performance of the (3+1)-dimensional(aspect)generalized nonlinear wave equation representing liquid with gas bubbles. Therefore, the main focus of this reseach is to obtain the desolate or lonely waves for the fractional (3+1)-dimensional(aspect) generalized nonlinear wave equation describing(report) liquids with gas bubbles using the sine-cosine method and other popular ansatz techniques.

Section 2 contains the definitions of some popular fractional derivatives, properties of the accodant derivative, and illustration of the used ansatz techniques procedure (methods). In the next section, the implementation of the method on the fractional (3+1)-dimensional generalized nonlinear wave equation report (describing) liquids with gas bubbles will lead to desolate wave solutions. In section 4, the conspiracy representation will be given for some achievable solutions.

III. THE DESCRIPTION (ILLUSTRATION) OF THE ACCORDANT FRACTIONAL DERIVATIVES AND THE METHOD

This section will start by defining the bulk common partial(fractional) derivative definitions, like: the Riemann-Liouville, Caputo, and Grunwald-Letnikov definitions [32].

Definition 1 (Riemann Liouville)

$$F_x^{\beta} q(x) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dx}\right)^n \int_0^x (x-z)^{n-\beta-1} q(z)dz, \quad n-1 < \beta \le n.$$

Definition 2 (Caputo)

$$F_x^{\beta} q(x) = \frac{1}{\Gamma(n-\beta)} \int_0^x (x-z)^{n-\beta-1} q^n(z)dz, \quad n-1 < \beta \le n.$$

Definition 3 (Grunwald-Letnikov)

$$F_{ax}^{\beta} q(x) = \lim_{m\to 0} m^{-\beta} \sum_{i\to 0}^{\frac{x-a}{m}} (-1)^i \begin{pmatrix} \beta \\ i \end{pmatrix} q(x-im).$$

Definition 4 (Accordant divided (fractional) derivative) consider a function $q:[0,\infty) \to \mathbb{R}$, then the Accordant divided (fractional) derivative of q order β is as follow

$$F_z^{\beta} q(z) = \lim_{\tau\to 0} \frac{q(z+\tau z^{1-\beta}) - q(z)}{\tau}, \text{ for all } z > 0, \beta \in (0,1].$$

Theorem 2.1[32]

Suppose $p(z), q(z)$ are $\beta-$ differentiable at a point $z > 0$ and $\beta \in (0,1]$. Then

1. $F_z^{\beta}\left(ap(z) + bq(z)\right) = a\left(F_z^{\beta} p(z)\right) + b\left(F_z^{\beta} q(z)\right)$, for all $a, b \in \mathbb{R}$.

2. $F_z^{\beta}\left(z^h\right) = hz^{h-\beta}$, for all $h \in \mathbb{R}$.

3. $F_z^{\beta}\left(\eta\right) = 0$, for all constant function $q(z) = \eta$.

4. $F_z^{\beta}\left(p(z)q(z)\right) = p(z)\left(F_z^{\beta} q(z)\right) + q(z)\left(F_z^{\beta} p(z)\right)$.

6. If in addition $p(z)$ is differentiable, then $\left(F_z^{\beta} p\right)(z) = z^{1-\beta} \frac{dp(z)}{dz}$.

Now, we will describe the method for the solution of the fractional (3+1)-dimensional (aspect) generalized nonlinear (discriminating) wave equation describing liquids with gas bubbles. In general, given nonlinear divided (fractional) PDE as follow:

$$0 < \beta 1, \beta 2 \le 1. \quad (2.1)$$

$$q(x,t) = U(\xi), \quad \xi = k\left(a\frac{x^{\beta}}{\beta} + b\frac{y^{\beta}}{\beta} + c\frac{z^{\beta}}{\beta} - v\frac{t^{\beta}}{\beta}\right), \quad (2.2)$$

Where $k, a, b, c,$ and v are persistant to be decide later. Moreover, for the first ansatz method, we assumed that the solution of the ODE (2.2) is given by:

$$U(\xi)=\sum_{i=0}^{N} d_i \Theta^i(\xi), \tag{2.3}$$

where d_i are persistant to be decide and N to be dictate by balancing the highest derivative with the elevated nonlinear terms. In addition, the periodic solution assumes the function $\Theta(\xi)$ to be a periodic function and vice versa to acquire a structure of algebraic identify. Solving the algebraic structure using maple or any other computational software to obtain the values of the unknown constants. However, for the sine and cosine method, we assumed that the blend of the fractional PDE is as follow:

$$U(\xi)=\gamma \sin^{\mu}\left(\vartheta\xi\right), \text{ or } U(\xi)=\gamma \cos^{\mu}\left(\vartheta\xi\right), \tag{2.4}$$

where $\gamma, \mu,$ and ϑ are constants to be determined. The derivatives of the solution (2.4) are given respectively as:

$$\begin{aligned}
&U(\xi)=\gamma \sin^{\mu}\left(\vartheta\xi\right),\\
&U^n(\xi)=\gamma^n \sin^{n\mu}\left(\vartheta\xi\right),\\
&\left(U^n(\xi)\right)_{\xi}=n\vartheta\mu\gamma^n \cos\left(\vartheta\xi\right)\sin^{n\mu-1}\left(\vartheta\xi\right),\\
&\left(U^n(\xi)\right)_{\xi\xi}=-n^2\vartheta^2\mu^2\gamma^n \sin^{n\mu}\left(\vartheta\xi\right)+n\vartheta^2\gamma^n\mu(n\mu-1)\sin^{n\mu-2}\left(\vartheta\xi\right),
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&U(\xi)=\gamma \cos^{\mu}\left(\vartheta\xi\right),\\
&U^n(\xi)=\gamma^n \cos^{n\mu}\left(\vartheta\xi\right),\\
&\left(U^n(\xi)\right)_{\xi}=-n\vartheta\mu\gamma^n \sin\left(\vartheta\xi\right)\cos^{n\mu-1}\left(\vartheta\xi\right),\\
&\left(U^n(\xi)\right)_{\xi\xi}=-n^2\vartheta^2\mu^2\gamma^n \cos^{n\mu}\left(\vartheta\xi\right)+n\vartheta^2\gamma^n\mu(n\mu-1)\cos^{n\mu-2}\left(\vartheta\xi\right),
\end{aligned} \tag{2.6}$$

and so on for the higher-order derivatives. When we substitute (2.5) or (2.6) into the ODE and steady the expressions of the sine or cosine function to generate an algebraic structure of linear equations. Getting the resulting system using computerized symbolic calculations namely maple or Mathematica to obtain the values of all the possible unknowns. The main advantage of these ansatz techniques is easy to apply to the most complicated PDEs models with less computational cost.

IV. RESULTS AND DISCUSSION (THE APPLICATION)

This section will present hyperbolic and periodic result for the fractional (3+1)-dimensional (aspect) generalized nonlinear (computational) wave equation of liquids with gas bubbles (1.1). Now, applying the definition of the accordant fractional derivative in (1.1) to obtain

$$F_x^{\beta}\left(F_t^{\beta}q+g_1F_x^{\beta}q^2+g_2F_x^{3\beta}q+g_3F_x^{\beta}q\right)+g_4F_{yy}^{2\beta}q+g_5F_{zz}^{2\beta}q=0. \tag{3.1}$$

Applying the wave transformation (2.2) and integrating the develop ordinary differential equation over again setting the integration constant as zero to have

$$\left(g_3a^2+g_4b^2+g_5c^2-va\right)U+g_1a^2U^2+g_3a^4k^2U^{"}=0. \tag{3.2}$$

We further balance U^2 with $U^{"}$ to get $N=2$ and suggest the following solution via (2.3) to get

$$U(\xi)=d_0+d_1\operatorname{sech}(\xi)+d_2\operatorname{sech}^2(\xi), \tag{3.3}$$

where $\Theta(\xi)=\operatorname{sech}(\xi)$ for the hyperbolic solution. By taking the derivative of (3.2) and identity the quantity of $\operatorname{sech}^i(\xi), i=0,1,2,...$ to obtain the following linear algebraic structure

$$\begin{cases}
g_5c^2d_1+g_2k^2a^4d_1+g_3a^2d_1+2g_1a^2d_0d_1+g_4b^2d_1-avd_1,\\
g_5c^2d_2+g_4b^2d_2+4g_2k^2a^4d_2-avd_2+g_1a^2d_0d_1^2+2g_1a^2d_0d_2+g_3a^2d_2=0,\\
2g_1a^2d_0d_2-2g_2k^2a^4d_2=0,\\
-6g_2k^2a^4d_2+g_1a^2d_2^2=0,\\
-d_0\left(-g_4b^2-g_3a^2-g_5c^2+va-g_1a^2d_0\right)=0.
\end{cases} \tag{3.4}$$

Solving the system (3.4) using maple to get
Case I

$$d_0=0,\ d_1=0,\ d_2=\frac{3\left(va-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1a^2},\ k=\sqrt{\frac{1}{4}\frac{av-g_3a^2-g_4b^2-g_5c^2}{g_2a^4}}. \tag{3.5}$$

Case II

$$d_0=\frac{\left(va-g_3a^2-g_4b^2-g_5c^2\right)}{g_1a^2},\ d_1=0,\ d_2=\frac{-3\left(va-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1a^2},\ k=\sqrt{\frac{1}{4}\frac{g_3a^2+g_4b^2+g_5c^2-av}{g_2a^4}}. \tag{3.6}$$

Substituting (2.7) into

$$q_1(x,y,z,t)=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)\operatorname{sech}\left(\sqrt{\frac{1}{4}\frac{av-g_3a^2-g_4b^2-g_5c^2}{g_2a^4}}\xi\right)^2}{2a^2g_1}, \tag{3.7}$$

$$q_2(x,y,z,t)=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)\operatorname{sech}\left(-\sqrt{\frac{1}{4}\frac{av-g_3a^2-g_4b^2-g_5c^2}{g_2a^4}}\xi\right)^2}{2a^2g_1}, \tag{2.9}$$

$$q_3(x,y,z,t)=\frac{\left(va-g_3a^2-g_4b^2-g_5c^2\right)}{g_1a^2}-\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)\operatorname{sech}\left(\sqrt{\frac{1}{4}\frac{g_3a^2+g_4b^2+g_5c^2-av}{g_2a^4}}\xi\right)^2}{2a^2g_1}, \tag{3.8}$$

$$q_4(x,y,z,t)=\frac{\left(va-g_3a^2-g_4b^2-g_5c^2\right)}{g_1a^2}-\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)\operatorname{sech}\left(-\sqrt{\frac{1}{4}\frac{g_3a^2+g_4b^2+g_5c^2-av}{g_2a^4}}\xi\right)^2}{2a^2g_1}.$$

Again setting $\Theta(\xi)=\sec(\xi)$ for the hyperbolic solution, by taking the derivative of (3.2) and identify the quantity of $\sec^i(\xi), i=0,1,2,...$ to obtain the following linear algebraic structure

$$\begin{cases} g_5c^2d_1-g_2k^2a^4d_1+g_3a^2d_1+2g_1a^2d_0d_1+g_4b^2d_1-avd_1, \\ g_5c^2d_2+g_4b^2d_2-4g_2k^2a^4d_2-avd_2+g_1a^2d_0d_1^2+2g_1a^2d_0d_2+g_3a^2d_2=0, \\ 2g_1a^2d_0d_2+2g_2k^2a^4d_2=0, \\ 6g_2k^2a^4d_2+g_1a^2d_2^2=0, \\ -d_0\left(-g_4b^2-g_3a^2-g_5c^2+va-g_1a^2d_0\right)=0. \end{cases} \tag{3.9}$$

Solving the system (3.9) using maple to get

Case I

$$d_0=0,\ d_1=0,\ d_2=\frac{3\left(va-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1a^2},\ k=\sqrt{-\frac{1}{4}\frac{av-g_3a^2-g_4b^2-g_5c^2}{g_2a^4}}. \tag{3.10}$$

Case II

$$d_0=\frac{\left(va-g_3a^2-g_4b^2-g_5c^2\right)}{g_1a^2},\ d_1=0,\ d_2=\frac{-3\left(va-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1a^2},$$

$$k=\sqrt{-\frac{1}{4}\frac{g_3a^2+g_4b^2+g_5c^2-av}{g_2a^4}}. \tag{3.11}$$

$$q_5(x,y,z,t)=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)\sec\left(\sqrt{-\frac{1}{4}\frac{av-g_3a^2-g_4b^2-g_5c^2}{g_2a^4}}\xi\right)^2}{2a^2g_1}, \tag{3.12}$$

$$q_6(x,y,z,t)=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)\sec\left(-\sqrt{-\frac{1}{4}\frac{av-g_3a^2-g_4b^2-g_5c^2}{g_2a^4}}\xi\right)^2}{2a^2g_1},$$

$$q_7(x,y,z,t)=\frac{\left(va-g_3a^2-g_4b^2-g_5c^2\right)}{g_1a^2}-\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)\sec\left(\sqrt{-\frac{1}{4}\frac{g_3a^2+g_4b^2+g_5c^2-av}{g_2a^4}}\xi\right)^2}{2a^2g_1}, \tag{3.13}$$

$$q_8(x,y,z,t)=\frac{\left(va-g_3a^2-g_4b^2-g_5c^2\right)}{g_1a^2}-\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)\sec\left(-\sqrt{-\frac{1}{4}\frac{g_3a^2+g_4b^2+g_5c^2-av}{g_2a^4}}\xi\right)^2}{2a^2g_1}.$$

$$q_8(x,y,z,t)=\frac{\left(va-g_3a^2-g_4b^2-g_5c^2\right)}{g_1a^2}-\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)\sec\left(-\sqrt{-\frac{1}{4}\frac{g_3a^2+g_4b^2+g_5c^2-av}{g_2a^4}}\xi\right)^2}{2a^2g_1}.$$

Now, for the sine-cosine method, we assumed the (3.2) has the result as follow:

$$U(\xi)=\gamma\sin^{\mu}\left(\vartheta\xi\right), \tag{3.14}$$

Substituting (3.14) and its derivative into (3.2) to obtain the following

$$\left(g_3a^2+g_4b^2+g_5c^2-va\right)\gamma\sin^{\mu}\left(\vartheta\xi\right)+g_1a^2\gamma^2\sin^{2\mu}\left(\vartheta\xi\right)+g_3a^4k^2\left(-\vartheta^2\mu^2\gamma\sin^{\mu}\left(\vartheta\xi\right)+\vartheta^2\gamma\mu(\mu-1)\sin^{\mu-2}\left(\vartheta\xi\right)\right)=0.$$

It is obvious that the equation (3.14) is satisfied if the following algebraic system is also satisfied

$$\begin{aligned}
&\mu-1\neq 0,\\
&\mu-2=2\mu,\\
&g_1a^2\gamma=-g_2k^2a^4\vartheta^2\mu(\mu-1),\\
&(g_3a^2+g_4b^2+g_5c^2-va)=\vartheta^2\mu^2g_2k^2a^2.
\end{aligned} \tag{3.15}$$

Solving the system (3.15) yields the following solutions

Case I

$$\mu=-2,\ \gamma=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1},\ \text{and}\ \vartheta=\sqrt{\frac{g_3a^2+g_4b^2+g_5c^2-va}{4g_2k^2a^2}}. \tag{3.16}$$

Case II

$$\mu=-2,\ \gamma=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1},\ \text{and}\ \vartheta=-\sqrt{\frac{g_3a^2+g_4b^2+g_5c^2-va}{4g_2k^2a^2}}. \tag{3.17}$$

For the case I and II we obtained the following solution $\frac{g_3a^2+g_4b^2+g_5c^2-va}{4g_2k^2a^2}<0,$

$$q_9(x,y,z,t)=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1}\sin^{-2}\left(\sqrt{\frac{g_3a^2+g_4b^2+g_5c^2-va}{4g_2k^2a^2}}\xi\right),$$

$$q_{10}(x,y,z,t)=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1}\sin^{-2}\left(-\sqrt{\frac{g_3a^2+g_4b^2+g_5c^2-va}{4g_2k^2a^2}}\xi\right),$$

$$q_{11}(x,y,z,t)=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1}\cos^{-2}\left(\sqrt{\frac{g_3a^2+g_4b^2+g_5c^2-va}{4g_2k^2a^2}}\xi\right),$$

$$q_{12}(x,y,z,t)=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1}\cos^{-2}\left(-\sqrt{\frac{g_3a^2+g_4b^2+g_5c^2-va}{4g_2k^2a^2}}\xi\right).$$

Similarly, for the case I and II we obtained the following solution for $\frac{g_3a^2+g_4b^2+g_5c^2-va}{4g_2k^2a^2}>0,$

$$q_{13}(x,y,z,t)=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1}\sinh^{-2}\left(\sqrt{\frac{g_3a^2+g_4b^2+g_5c^2-va}{4g_2k^2a^2}}\xi\right),$$

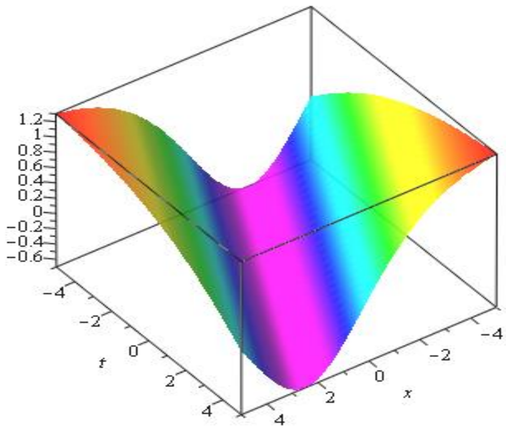
$$q_{14}(x,y,z,t)=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1}\sinh^{-2}\left(-\sqrt{\frac{g_3a^2+g_4b^2+g_5c^2-va}{4g_2k^2a^2}}\xi\right),$$

$$q_{15}(x,y,z,t)=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1}\cosh^{-2}\left(\sqrt{\frac{g_3a^2+g_4b^2+g_5c^2-va}{4g_2k^2a^2}}\xi\right),$$

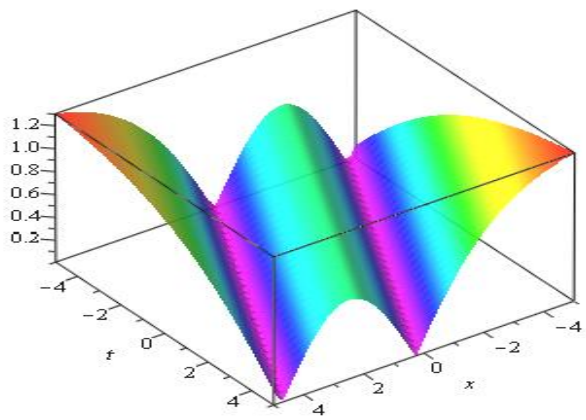
$$q_{16}(x,y,z,t)=\frac{3\left(av-g_3a^2-g_4b^2-g_5c^2\right)}{2g_1}\cosh^{-2}\left(-\sqrt{\frac{g_3a^2+g_4b^2+g_5c^2-va}{4g_2k^2a^2}}\xi\right).$$

V. PHYSICAL EXPLANATION

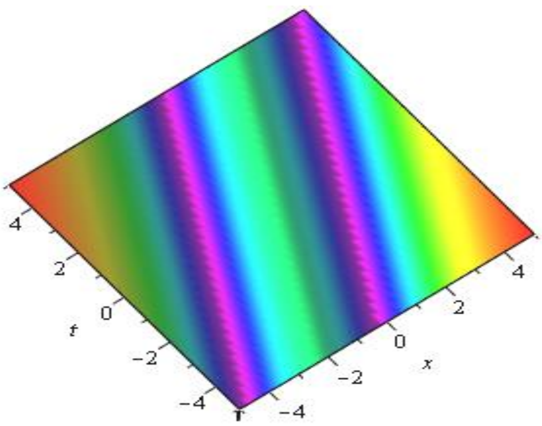
As we get, we conspired the 2D, 3D, and curves plots of the proposed solutions. The conspiracy exhibits some interesting features of the recovered solutions at different fractional (divided) orders using some appropriate values. For figures 1-3, we assigned $g_1=1, g_2=4, g_3=-4, g_4=0.1$ $g_5=3,\ a=0.2,\ b=0.2,\ c=0.2,\ p=0.1,\ y=z=0.$ Also, figures 4-9, we assigned $g_1=-1, g_2=2, g_3=-2, g_4=3$ $g_5=1, a=0.1, b=0.2, c=0.2, p=0.1, y=z=0.$ The figures below showed different forms of soliton structure such as dark, bright and singular solitons. Moreover, more soliton structures can be recovered by assigning appropriate values for the derived solutions.



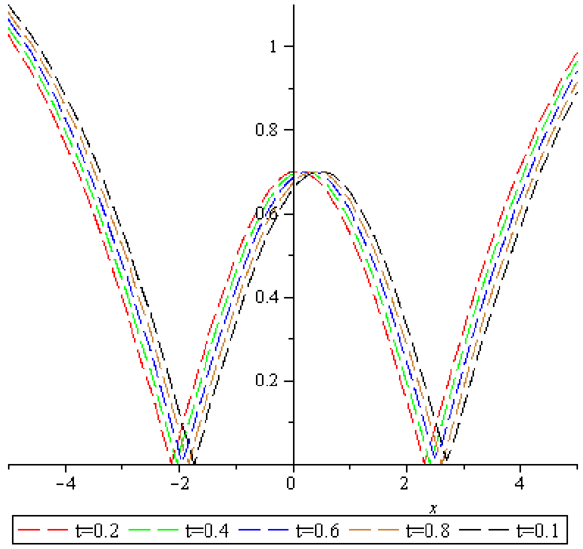
b. 3D conspiracy of $\mathrm{Re}\left(q_7(x,y,z,t)\right)$



a. 3D conspiracy of $\left|q_7(x,y,z,t)\right|$

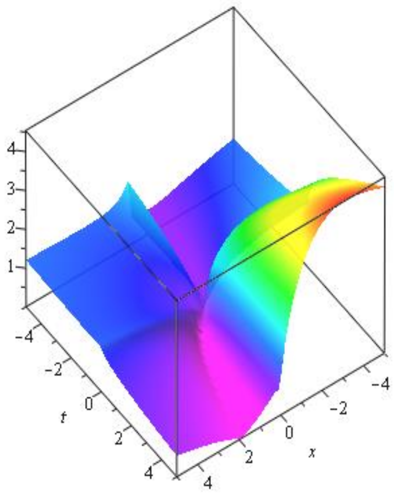


c. curves plot of $\left|q_7(x,y,z,t)\right|$

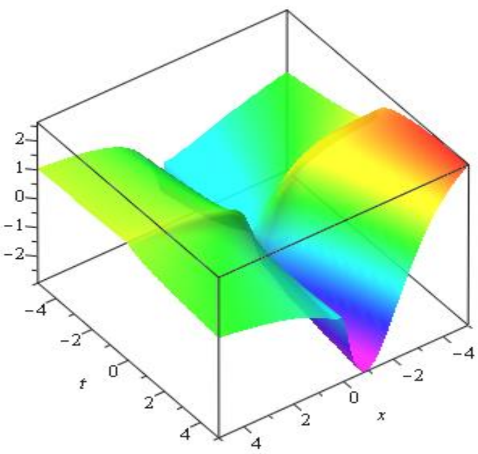


d. 2D conspiracy of $|q_7(x,y,z,t)|$

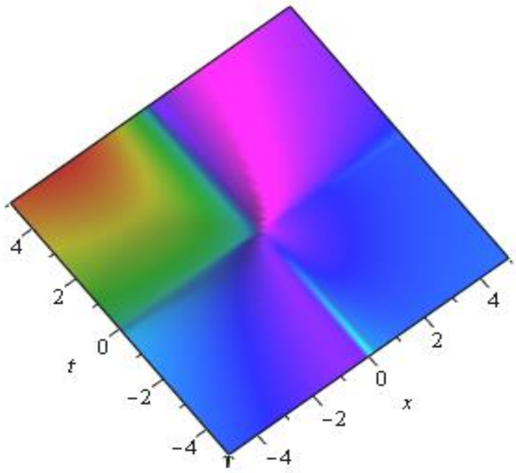
Figure 1: Some conspiracy of the solution $q_7(x,y,z,t)$ at $\beta = 1.$



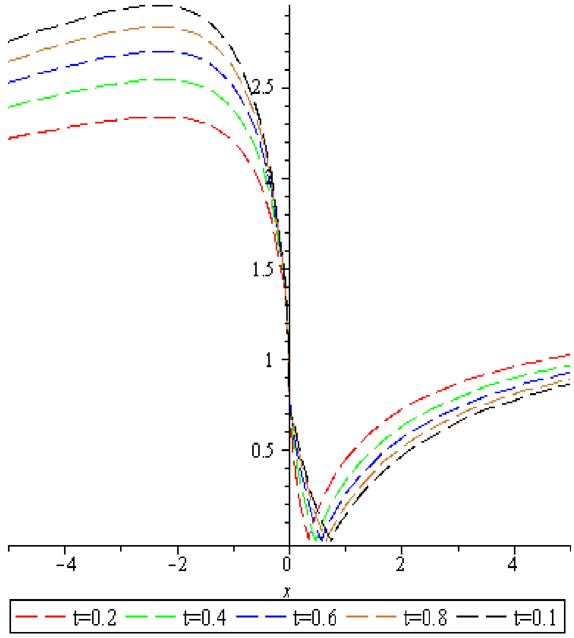
a. 3D conspiracy of $|q_7(x,y,z,t)|$



b. 3D conspiracy of $\mathrm{Re}(q_7(x,y,z,t))$

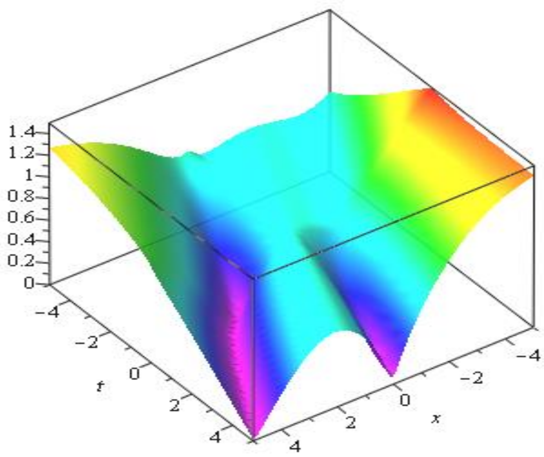


c. curve plot of $|q_7(x,y,z,t)|$

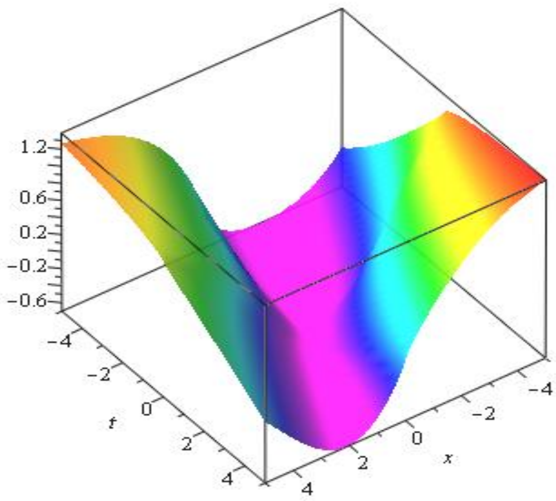


d. 2D conspiracy of $|q_7(x,y,z,t)|$

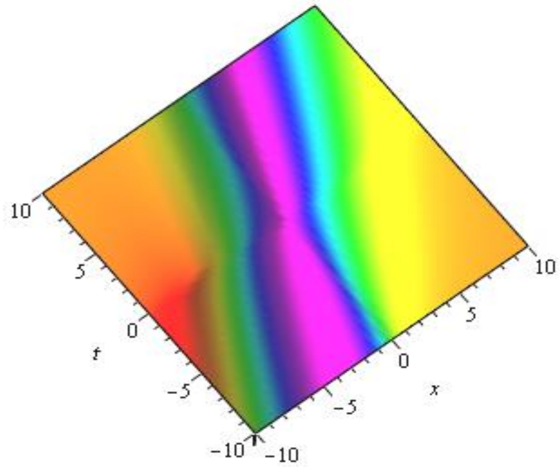
Figure 2: Some conspiracy of the solution $q_7(x,y,z,t)$ at $\beta = 0.4.$



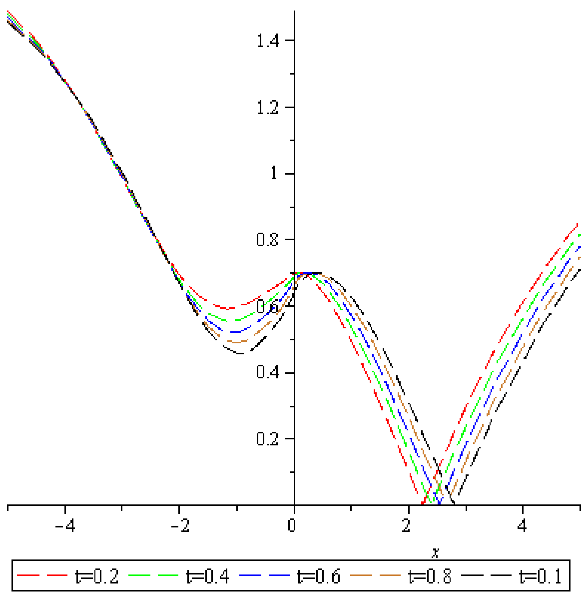
a. 3D conspiracy of $|q_7(x,y,z,t)|$



b. 3D conspiracy of $\mathrm{Re}\left(q_7(x,y,z,t)\right)$

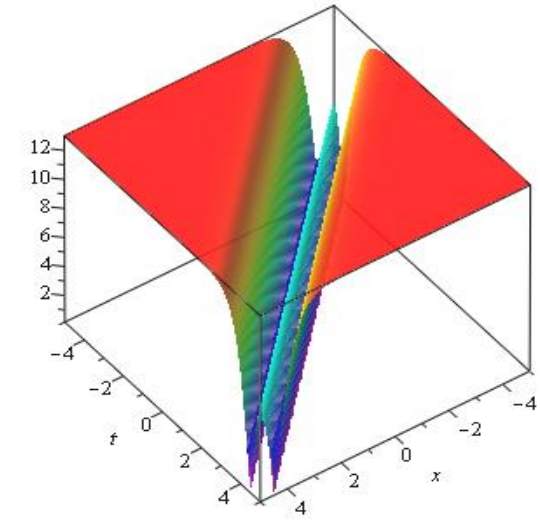


c. curve plot of $\left|q_7(x,y,z,t)\right|$

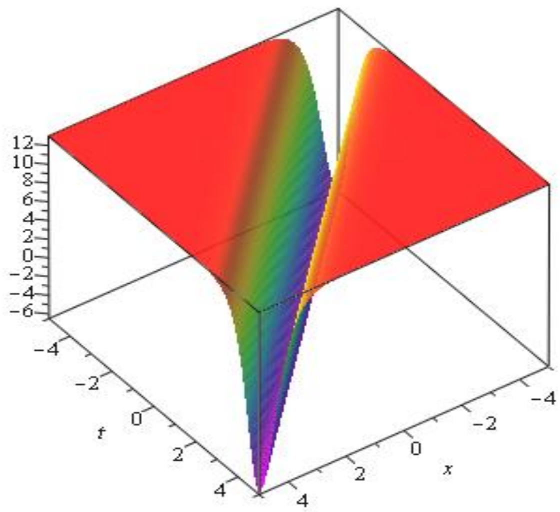


d. 2 D conspiracy of $\left|q_7(x,y,z,t)\right|$

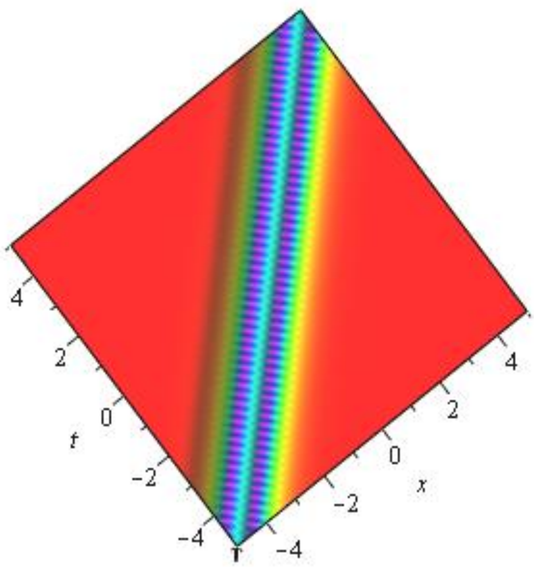
Figure 3: Some conspiracy of the solution $q_7(x,y,z,t)$ at $\beta = 0.4.$



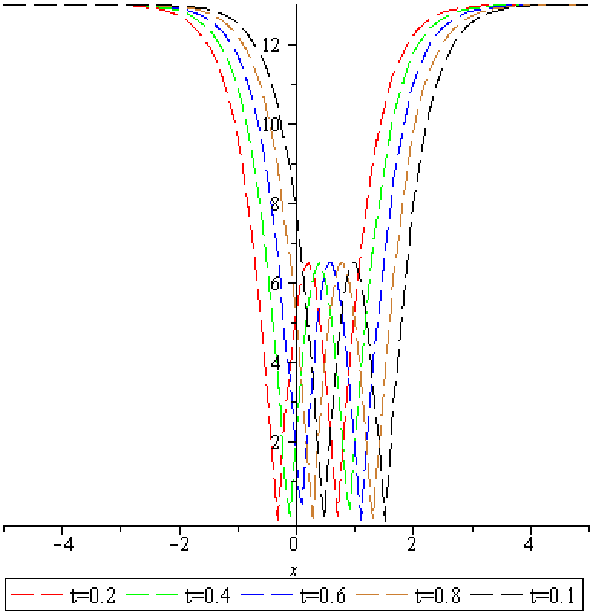
a. 3D conspiracy of $\left|q_6(x,y,z,t)\right|$



b. 3D conspiracy of $\mathrm{Re}\left(q_6(x,y,z,t)\right)$

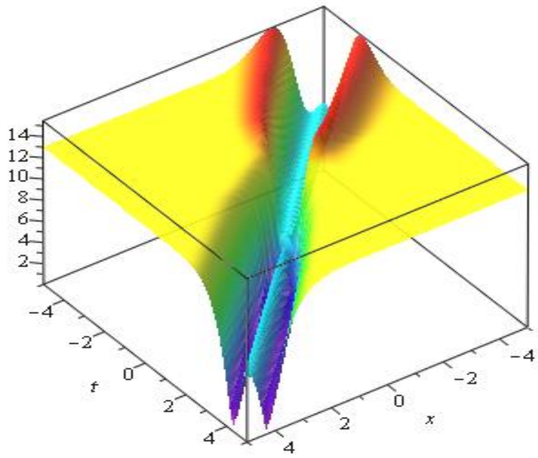


c. curve plot of $\left|q_6(x,y,z,t)\right|$

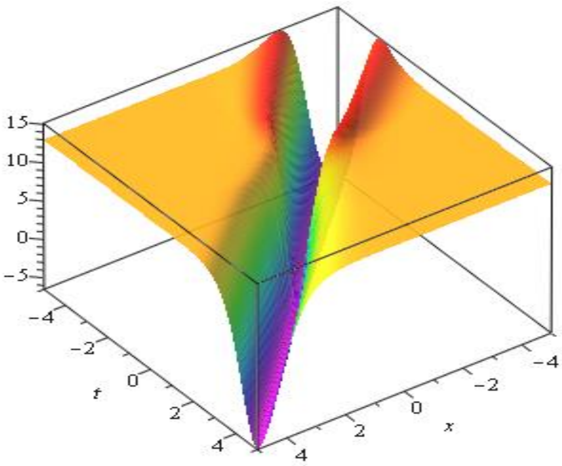


d. 2D conspiracy of $|q_6(x,y,z,t)|$

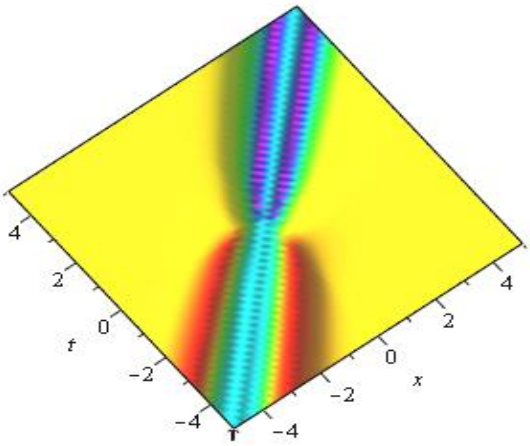
Figure 4: Some conspiracy of the solution $q_6(x,y,z,t)$ at $\beta = 1.$



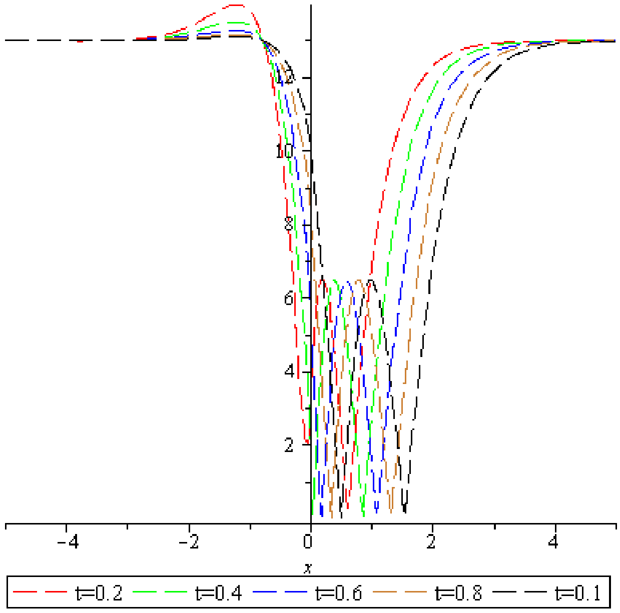
a. 3D conspiracy of $|q_6(x,y,z,t)|$



b. 3D conspiracy of $\mathrm{Re}(q_6(x,y,z,t))$

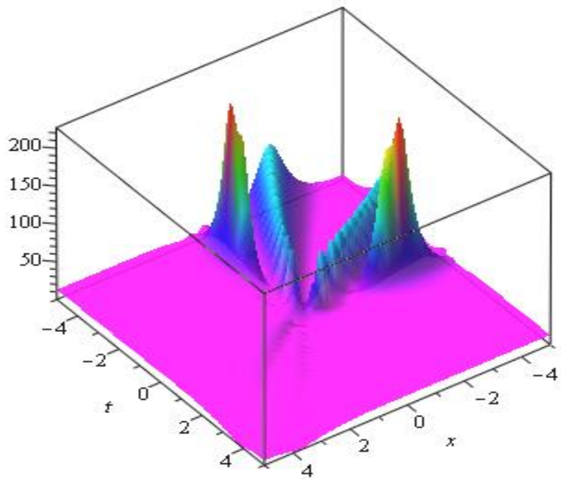


c. curve plot of $|q_6(x,y,z,t)|$

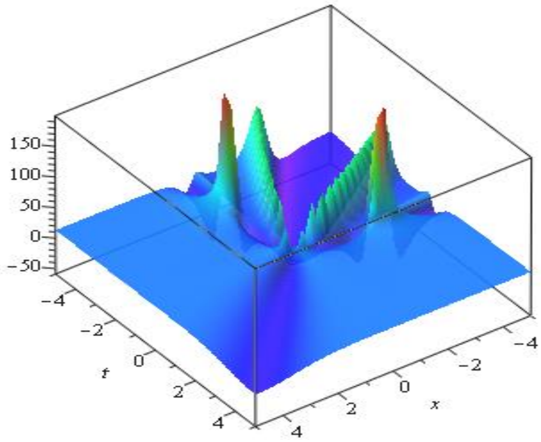


d. 2D conspiracy of $|q_6(x,y,z,t)|$

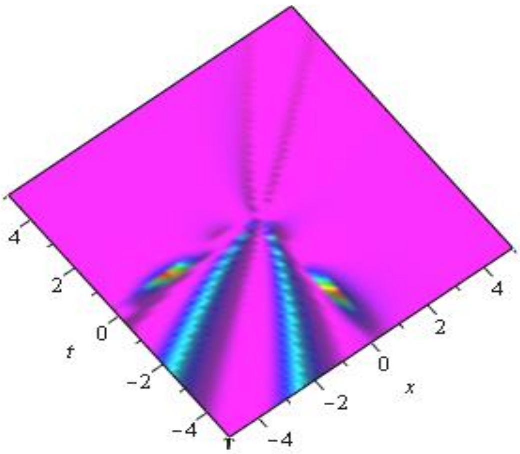
Figure 5: Some conspiracy of the solution $q_6(x,y,z,t)$ at $\beta = 0.8.$



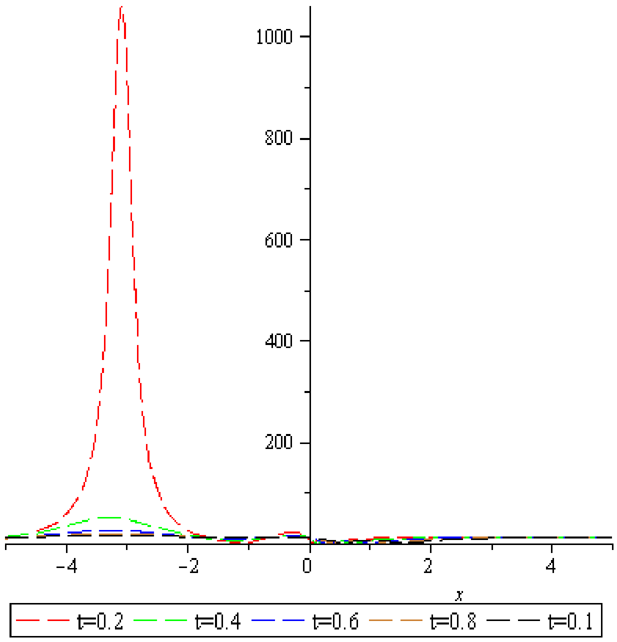
a. 3D conspiracy of $|q_6(x,y,z,t)|$



b. 3D conspiracy of $\mathrm{Re}\left(q_6(x,y,z,t)\right)$

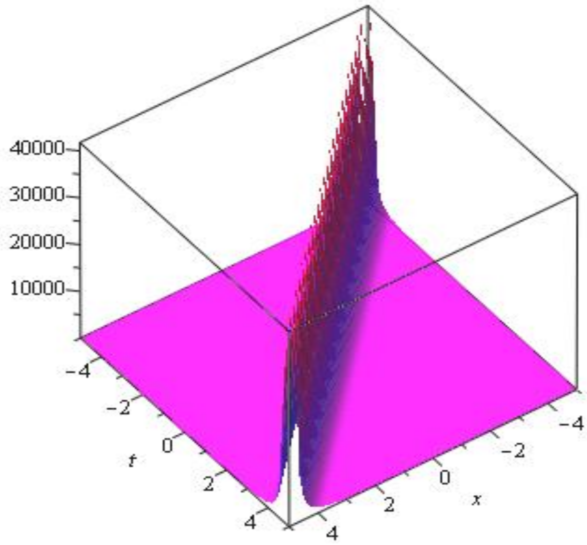


c. curve plot of $\left|q_6(x,y,z,t)\right|$

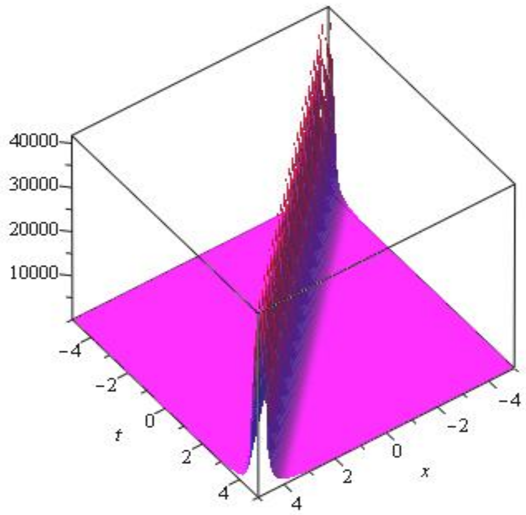


d. 2D conspiracy of $\left|q_6(x,y,z,t)\right|$

Figure 6: Some conspiracy of the solution $q_6(x,y,z,t)$ at $\beta = 0.4$.



a. 3D conspiracy of $\left|q_9(x,y,z,t)\right|$



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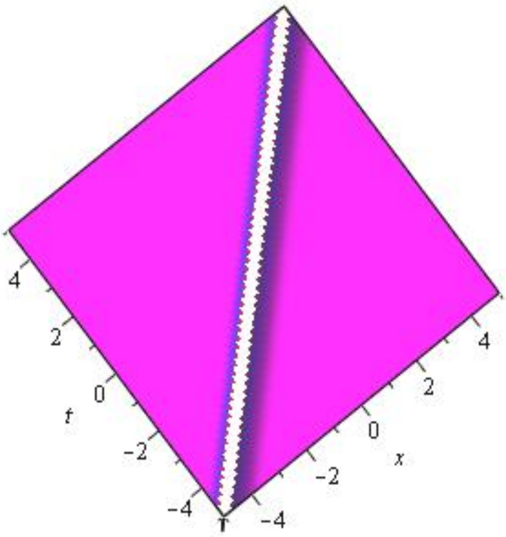
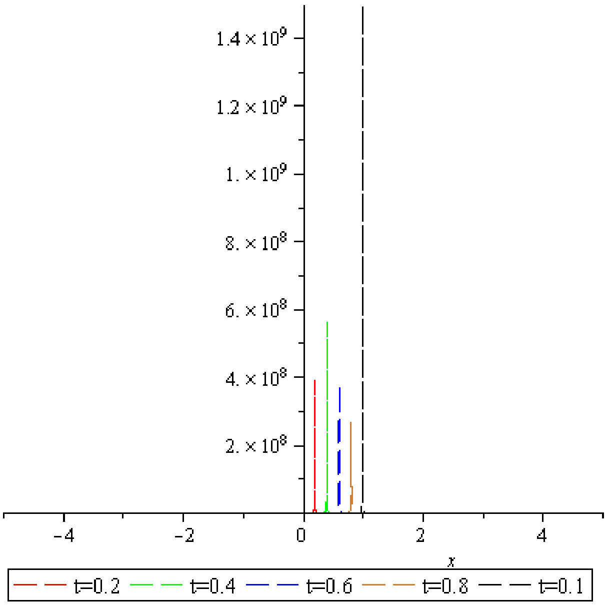
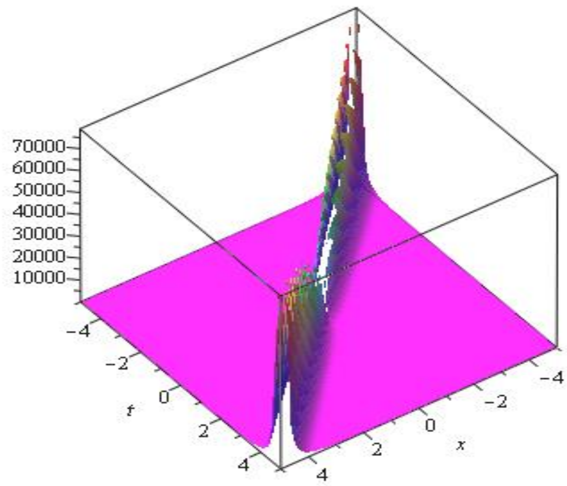


c. curve plot of $\left|q_9(x,y,z,t)\right|$

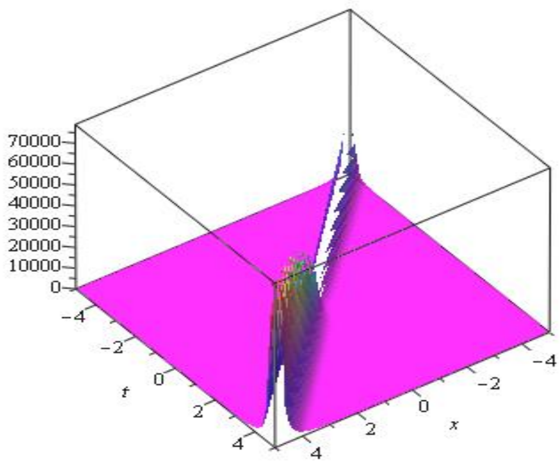


d. 2D conspiracy of $|q_9(x,y,z,t)|$

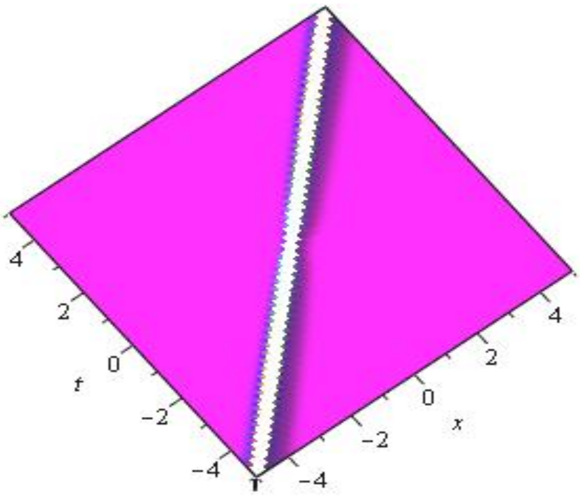
Figure 7: Some conspiracy of the solution $q_9(x,y,z,t)$ at $\beta=1$.



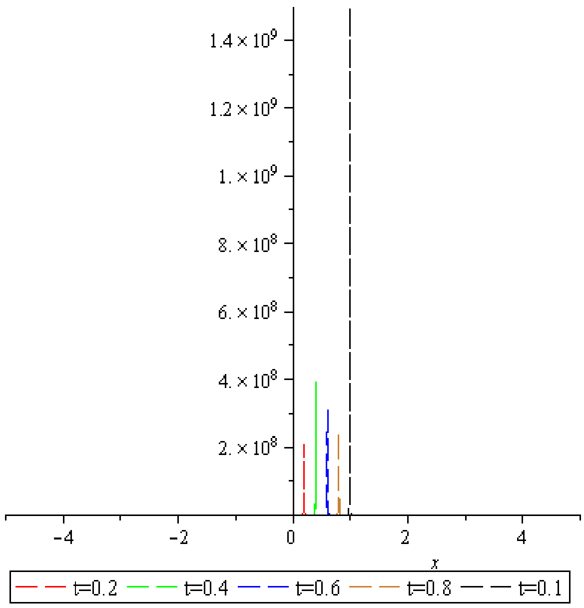
a. 3D conspiracy of $|q_9(x,y,z,t)|$



b. 3D conspiracy of $\mathrm{Re}(q_9(x,y,z,t))$

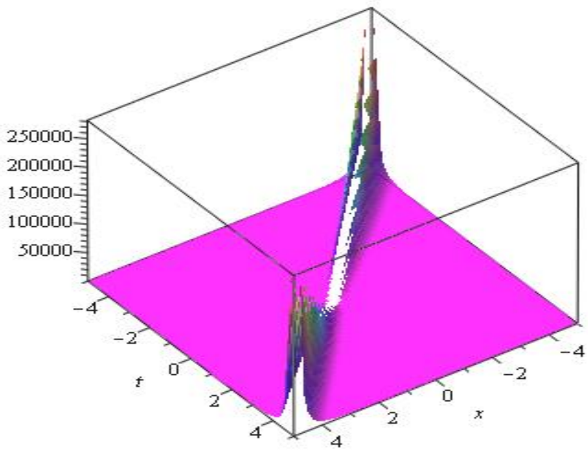


c. curve plot of $|q_9(x,y,z,t)|$



d. 2D conspiracy of $|q_9(x,y,z,t)|$

Figure 8: Some conspiracy of the solution $q_9(x,y,z,t)$ at $\beta=0.8$.



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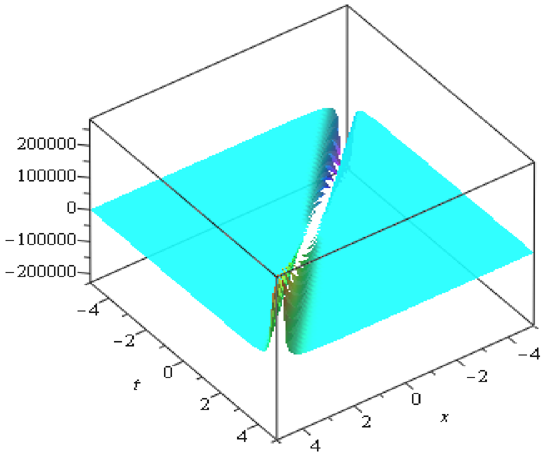
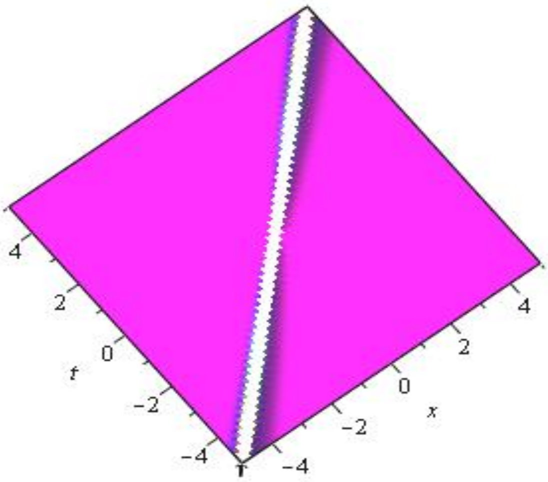
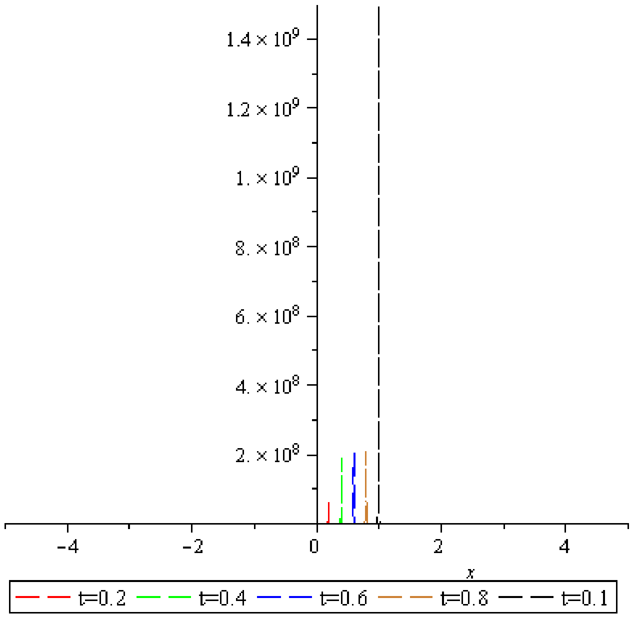


b. 3D conspiracy of $\mathrm{Re}\left(q_9(x,y,z,t)\right)$



c. curve plot of $\left|q_9(x,y,z,t)\right|$



d. 2D conspiracy of $\left|q_9(x,y,z,t)\right|$

Figure 9: Some conspiracy of the solution $q_9(x,y,z,t)$ at $\beta = 0.4.$

VI. CONCLUSION

This article investigated the exact desolate (lonely) wave solutions for the fractional (divided) (3+1) generalized nonlinear wave identify with gas bubbles using the well-known ansatz techniques and the sine-cosine method. Liquids with gas bubbles mainly arise in manifold or crowded applications like science, engineering and physics. The methods yielded different periodic and hyperbolic desolate (lonely) wave solutions. Moreover, solving the (3+1)-dimensional (aspect) generalized fractional nonlinear (discriminate) wave equation with gas bubbles is equivalent to solving many physical models, such as the (2+1)-dimensional (aspect) Kadomtsev-Petviashvil model with negative dispersion, the (3+1)-dimensional (aspect) Kadomtsev-Petviashvili model, the (3+1)-dimensional (aspect) nonlinear waves with bubble liquid mixture, and other special cases of the considered model. Finally, we conspired both 2D and 3D as well as the contour plots to understand the physical application of the considered model using maple.

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