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Effect of Presence of Outliers in the Estimate of the Parameters of a Johnson S_B Distribution

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Abstract- An outlier is an observation that deviates in some sense from the rest of the observations. Outliers have strong influence on estimates of the parameters of a model that is being fitted to the data. In this paper, the effect of presence of outlying observation was studied in the estimates of the parameters of a Johnson S_B distribution with slippage alternative of location parameters. The largest observation of the data was shifted towards right or the smallest observation was shifted to the left, the estimates of the parameters would be affected. Some examples are also given for highlighting the result.

Keywords: Johnson S_B distribution, Transformation function, Outlier, Slippage alternative, Maximum likelihood estimation, Least square estimation.

I. INTRODUCTION

Johnson (1949) derived a family of distributions using the method of translation and was called as Johnson family of distributions, which provides flexibility of covering wide varieties of distributional shapes.

The estimation of the parameters of Johnson S_B distribution is done by George and Ramachandran (2011), Parresol (2003), Slifker and Shapiro (1980), Kudus *et al*. (2011) *etc.* They have used methods like maximum likelihood-least square method, method of moments, percentile method and linear or non-linear regression methods. They have also used some computational techniques like Newton-Raphson method *etc*. George and Ramachandran (2011) has compared their method with two commonly used procedures and established that their procedure to be superior.

An outlier is defined as an observation that deviates from the rest of the data set in some sense. Here, we study the effect of outlier in the estimates of the parameters of Johnson S_B distribution. The outliers are introduced in the estimates of the parameters with a shift in largest as well as smallest observation.

For introducing an outlying observation, a slippage alternative was considered. Hence, in the beginning an observation from the same distribution with a slipped location parameter was introduced and its effect was studied. Note that if any such observation is introduced anywhere in the middle of the data, it may not show up and will have a negligent effect on the estimates. Hence, the shifted observation was introduced in the smallest or the largest observation. Thus if it was assumed that the largest observation of the data was shifted towards right or the smallest observation was shifted to the left, the estimates of the parameters would be affected.

II. THE JOHNSON S_B DISTRIBUTION AND ITS ESTIMATES

For a continuous random variable x whose distribution function is not known, Johnson proposed three normalizing transformations having a general form:

$$z = \gamma + \delta f\left(\frac{x-\xi}{\lambda}\right), \tag{2.1}$$

where $f(.)$ is the transformation function, z is a standard normal variable, γ and δ are shape parameters, λ a scale parameter and ξ a location parameter. Without loss of generality, it is assumed that $\delta > 0$ and $\lambda > 0$.

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If X follows a Johnson S_B distribution with location parameter ξ, scale parameter λ, and two shape parameters γ and δ, then the density and distribution function X of are as follows.

$$f(x;\xi,\lambda,\gamma,\delta)=\frac{\delta}{\sqrt{2\pi}}\frac{\lambda}{(\lambda-(x-\xi))(x-\xi)}\ exp\left[-\frac{1}{2}\left\{\gamma+\delta\ln\left(\frac{x-\xi}{\lambda-(x-\xi)}\right)\right\}^2\right], \quad (2.2)$$
$$\xi\le x\le\xi+\lambda, \delta>0, -\infty<\gamma<\infty, \lambda>0, -\infty<\xi<\infty;$$

and $F(x)=\Phi\left(\gamma+\delta\ln\left(\frac{y}{1-y}\right)\right)$, (2.3)

where $y=\left(\frac{x-\xi}{\lambda}\right)$ and $\Phi(z)$ is the cumulative distribution function of a standard normal distribution up to the point z.

The null hypothesis (H_0) states that there is no outlier in the data. Under this assumption, the estimates of the parameters of this distribution as given by George & Ramachandran (2011) using maximum likelihood least square method are as follows.

$$\hat{\gamma}=-\frac{\delta\sum_{i=1}^{n-1}g\left(\frac{x_i-\xi}{\lambda}\right)}{n-1}=-\delta\bar{g} \text{ and} \quad (2.4)$$

$$\hat{\delta}^2=\frac{n-1}{\sum_{i=1}^{n-1}\left[g\left(\frac{x_i-\xi}{\lambda}\right)\right]^2-\frac{1}{n-1}\left[\sum_{i=1}^{n-1}g\left(\frac{x_i-\xi}{\lambda}\right)\right]^2}=\frac{1}{var(g)}, \quad (2.5)$$

where, $g\left(\frac{x-\xi}{\lambda}\right)=\log\left(\frac{x-\xi}{\lambda-(x-\xi)}\right)$, $\bar{g}$ is the mean and $var(g)$ is the variance of the values of g defined here.

Since, the partial derivatives of the log-likelihood function of the Johnson S_B distribution with respect to ξ and λ are not simple, the least square method was used to estimate the parameters ξ and λ. From (2.1) for fixed values of γ and δ, the general form of the Johnson translation system equation $x=\xi+\lambda f^{-1}\left(\frac{z-\gamma}{\delta}\right)$, was considered as a linear equation with parameters ξ and λ.

Here, $z=\frac{x-\xi}{\lambda}$ is a standard normal variate. Hence, the quantiles of x and the corresponding quantiles of z can be considered as paired observations. When there were 100 or more x values, the percentiles 1 through 99 were considered, while for k number of data points of x, where k is less than 100, $k-1$ quantiles of x were considered. These $k-1$ quantiles of x and the corresponding $k-1$ quantiles of z were considered as paired observations. The estimates of λ and ξ were as follows;

$$\hat{\lambda}=\frac{(n-1)\sum_{i=1}^{n-1}x_i f^{-1}\left(\frac{z_i-\gamma}{\delta}\right)-\sum_{i=1}^{n-1}f^{-1}\left(\frac{z_i-\gamma}{\delta}\right)\sum_{i=1}^{n-1}x_i}{(n-1)\sum_{i=1}^{n-1}\left[f^{-1}\left(\frac{z_i-\gamma}{\delta}\right)\right]^2-\left[\sum_{i=1}^{n-1}f^{-1}\left(\frac{z_i-\gamma}{\delta}\right)\right]^2} \quad (2.6)$$

and $\hat{\xi}=\bar{x}-\lambda^{*}mean\left[f^{-1}\left(\frac{z-\gamma}{\delta}\right)\right]$, (2.7)

where $\bar{x}$ is the mean of x –quantiles and $\bar{z}$ is the mean of z –quantiles. These involve only $n-1$ observations. Starting with some initial values of ξ and λ, these values were obtained by using either percentile method or quantile method. Using the values of ξ and λ, the estimates of γ and δ were obtained using (2.4) and (2.5) respectively. After obtaining the estimates of γ and δ, (2.6) and (2.7) were used to revise the ξ and λ estimates. Now these steps were repeated, each time using the most recent estimates, tracking the Residual Sum of Squares (RSS) simultaneously. After a few steps, the estimates with minimum RSS value was selected and the values of ξ and λ that resulted in obtaining the minimum value of RSS is the required estimates of the location and scale parameters.

III. WHEN THE LARGEST OBSERVATION IS FROM A DISTRIBUTION WITH A SHIFTED LOCATION PARAMETER

Let $x_1,x_2,\cdots,x_n$ be n observations from a Johnson S_B distribution with location parameter ξ, scale parameter λ, and two shape parameters γ and δ and $x_{(1)},x_{(2)},\cdots,x_{(n)}$ be the corresponding order statistics of the n observations. If a shift in the largest observation $x_{(n)}$is made by an amount 'a' in the positive direction, then the effect of shift on the estimates of the parameters are as follows.

From (2.6) and (2.7), we have

$$\hat{\lambda}=\frac{(n-1)\sum_{i=1}^{n-1}x_i f^{-1}\left(\frac{z_i-\gamma}{\delta}\right)-\sum_{i=1}^{n-1}f^{-1}\left(\frac{z_i-\gamma}{\delta}\right)\sum_{i=1}^{n-1}x_i}{(n-1)\sum_{i=1}^{n-1}\left[f^{-1}\left(\frac{z_i-\gamma}{\delta}\right)\right]^2-\left[\sum_{i=1}^{n-1}f^{-1}\left(\frac{z_i-\gamma}{\delta}\right)\right]^2}, \text{ and}$$

$$\hat{\xi}=\bar{x}-\lambda^{*}mean\left[f^{-1}\left(\frac{z-\gamma}{\delta}\right)\right].$$

Since, $\bar{x}$ and $\bar{z}$ is the mean of x –quantiles and z –quantiles respectively, this only depends upon $n-1$ observations, not on n^{th} or the largest observation. Hence, a shift in largest observation $x_{(n)}$ does not affect the estimates of location parameter ξ and scale parameter λ.

If a shift in the second largest observation $x_{(n\text{-}1)}$ is made by an amount 'a' on the right side, then the effect of shift on the estimates of the parameters are as follows. According to the procedure initial values of ξ and λ were taken from percentile method.

Effect of shift on estimate of the location parameter (ξ)

From (2.7), the estimate of location parameter ξ can be written as

$$\hat{\xi}=\bar{x}-\frac{\lambda_p}{n-1}\left[\sum_{i=1}^{n-1}\left(\frac{x_{(i)}-\xi_p}{\lambda_p}\right)\right]$$

$$=\frac{1}{n-1}\sum_{i=1}^{n-1}x_{(i)}-\frac{1}{n-1}\left[\sum_{i=1}^{n-1}\left(x_{(i)}-\xi_p\right)\right] \qquad (3.1)$$

Shifting the second largest observation $x_{(n-1)}$ towards right by an amount 'a', then the expression (3.1) becomes,

$$\hat{\xi}^{*}=\frac{1}{n-1}\left\{\sum_{i=1}^{n-2}x_{(i)}+x_{(n-1)}+a\right\}-\frac{1}{n-1}\left\{\sum_{i=1}^{n-2}x_{(i)}+x_{(n-1)}+a\right\}+\frac{1}{n-1}(n-1)\xi_p$$

$$=\xi_p.$$

It can be seen from above that the shifted effect gets cancelled out in expression of the location parameter $\hat{\xi}$, so it has no effect of shift in any observation.

Effect on estimate of the scale parameter (λ)

The expression for the estimate of the scale parameter λ given by (2.6) can be rewritten as,

$$\hat{\lambda}=\frac{(n-1)\sum_{i=1}^{n-1}x_{(i)}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}-\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\sum_{i=1}^{n-1}x_{(i)}}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}$$

$$=\frac{\frac{(n-1)}{\lambda_p}\left\{\sum_{i=1}^{n-1}x_{(i)}^2-\xi_p\sum_{i=1}^{n-1}x_{(i)}\right\}-\frac{1}{\lambda_p}\left\{\sum_{i=1}^{n-1}x_{(i)}-(n-1)\xi_p\right\}\sum_{i=1}^{n-1}x_{(i)}}{\frac{(n-1)}{\lambda_p^{2}}\left[\sum_{i=1}^{n-1}x_{(i)}^2+(n-1)\xi_p^{2}-2\xi_p\sum_{i=1}^{n-1}x_{(i)}\right]-\frac{1}{\lambda_p^{2}}\left[\left\{\sum_{i=1}^{n-1}x_{(i)}\right\}^2+(n-1)^2\xi_p^{2}-2\xi_p(n-1)\sum_{i=1}^{n-1}x_{(i)}\right]}. \qquad (3.2)$$

Let the second largest observation $x_{(n-1)}$ be shifted by an amount 'a' in (3.2), then expression becomes $\hat{\lambda}^{*}$ as given below.

$$\hat{\lambda}^{*}=\frac{\frac{(n-1)}{\lambda_p}\left[\sum_{i=1}^{n-2}x_{(i)}^2+\left\{x_{(n-1)}+a\right\}^2-\xi_p\left\{\sum_{i=1}^{n-2}x_{(i)}+x_{(n-1)}+a\right\}\right]-\frac{1}{\lambda_p}\left[\sum_{i=1}^{n-2}x_{(i)}+x_{(n-1)}+a-(n-1)\xi_p\right]\left[\sum_{i=1}^{n-2}x_{(i)}+x_{(n-2)}+a\right]}{\frac{(n-1)}{\lambda_p^{2}}\left[\sum_{i=1}^{n-2}x_{(i)}^2+\left\{x_{(n-1)}+a\right\}^2+(n-1)\xi_p^{2}-2\xi_p\left\{\sum_{i=1}^{n-2}x_{(i)}+x_{(n-1)}+a\right\}\right]-\frac{1}{\lambda_p^{2}}\left[\left\{\sum_{i=1}^{n-2}x_{(i)}+x_{(n-1)}+a\right\}^2+(n-1)^2\xi_p^{2}-2\xi_p(n-1)\left\{\sum_{i=1}^{n-2}x_{(i)}+x_{(n-1)}+a\right\}\right]}$$

$$=\frac{(n-1)\sum_{i=1}^{n-1}x_{(i)}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}+\frac{(n-1)}{\lambda_p}\left\{a^2+2ax_{(n-1)}-\xi_p a\right\}-\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\sum_{i=1}^{n-1}x_{(i)}-\frac{1}{\lambda_p}a\sum_{i=1}^{n-1}x_{(i)}-a\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}-\frac{a^2}{\lambda_p}}{(n-1)\sum_{i=1}^{n}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2+\frac{(n-1)}{\lambda_p^{2}}\left\{a^2+2ax_{(n-1)}-2a\xi_p\right\}-\left[\sum_{i=1}^{n}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2-\frac{1}{\lambda_p^{2}}\left\{a^2+2a\sum_{i=1}^{n-1}x_{(i)}-2a(n-1)\xi_p\right\}}$$

$$=\frac{(n-1)\sum_{i=1}^{n-1}x_{(i)}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}-\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\sum_{i=1}^{n-1}x_{(i)}+\frac{1}{\lambda_p}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2+\frac{1}{\lambda_p^{2}}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}$$

$$=\frac{\hat{\lambda}\left[(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2\right]+\frac{1}{\lambda_p}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2+\frac{1}{\lambda_p^{2}}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}.$$

Or
$$\hat{\lambda}^{*}=\frac{\hat{\lambda}+\dfrac{\frac{1}{\lambda_p}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left(\frac{x_{(i)}-\xi_p}{\lambda_p}\right)^2-\left(\sum_{i=1}^{n-1}\left(\frac{x_{(i)}-\xi_p}{\lambda_p}\right)\right)^2}}{1+\dfrac{\frac{1}{\lambda_p^{2}}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}}.$$

Or
$$\left(\hat{\lambda}^{*}-\hat{\lambda}\right)+\left(\hat{\lambda}^{*}-\hat{\lambda}\right)\left[\frac{\frac{1}{\lambda_p^{2}}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}\right]$$

$$=\frac{\frac{1}{\lambda_p}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}-\hat{\lambda}\left[\frac{\frac{1}{\lambda_p^{2}}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}\right].$$

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$$\text{Or } \left(\hat{\lambda}^* - \hat{\lambda}\right) = \frac{\dfrac{\frac{1}{\lambda_p}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}-\hat{\lambda}\left[\dfrac{\frac{1}{\lambda_p^2}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}\right]}{1+\left\{\dfrac{\frac{1}{\lambda_p^2}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}\right\}}.$$

After solving the above expression, we get

$$\hat{\lambda}^* - \hat{\lambda} = \frac{\frac{1}{\lambda_p}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]-\frac{\hat{\lambda}}{\lambda_p^2}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2+\frac{1}{\lambda_p^2}\left[(n-2)a^2+2a\left\{(n-1)x_{(n-1)}-\sum_{i=1}^{n-1}x_{(i)}\right\}\right]}.$$

Percentage change in the scale parameter λ due to a shift in the location parameter by an amount a is given as

$$\lambda_{LP} = \frac{|\hat{\lambda}^*-\hat{\lambda}|}{\hat{\lambda}} \times 100 . \qquad (3.3)$$

Effect on estimate of the shape parameter (γ)

The expression for the estimate of the shape parameter γ given by (2.4) can be written as;

$$\hat{\gamma} = -\frac{\hat{\delta}}{(n-1)}\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}. \qquad (3.4)$$

The second largest observation $x_{(n-1)}$ was shifted by amount 'a' towards right and the (3.4) becomes,

$$\hat{\gamma}^* = -\frac{\hat{\delta}}{(n-1)}\left[\sum_{i=1}^{n-2}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}+\log\left\{\frac{x_{(n-1)}+a-\xi_p}{\lambda_p-(x_{(n-1)}+a-\xi_p)}\right\}\right]. \qquad (3.5)$$

Consider the term

$$\log\left\{\frac{x_{(n-1)}+a-\xi_p}{\lambda_p-(x_{(n-1)}+a-\xi_p)}\right\} = \log\left\{x_{(n-1)}+a-\xi_p\right\}-\log\left\{\lambda_p-\left(x_{(n-1)}-\xi_p\right)-a\right\}$$

$$=\log\left\{x_{(n-1)}-\xi_p\right\}+log\left\{1+\frac{a}{x_{(n-1)}-\xi_p}\right\}-\log\left\{\lambda_p-\left(x_{(n-1)}-\xi_p\right)\right\}-\log\left[1-\frac{a}{\{\lambda_p-(x_{(n-1)}-\xi_p)\}}\right]$$

$$= \log\left\{\frac{x_{(n-1)}-\xi_p}{\lambda_p-(x_{(n-1)}-\xi_p)}\right\}+\log\left[\frac{\{x_{(n-1)}+a-\xi_p\}\{\lambda_p-(x_{(n-1)}-\xi_p)\}}{\{x_{(n-1)}-\xi_p\}\{\lambda_p-(x_{(n-1)}-\xi_p)-a\}}\right]. \qquad (3.6)$$

Using (3.6) in (3.5) gives

$$\hat{\gamma}^* = -\frac{\hat{\delta}}{(n-1)}\left[\sum_{i=1}^{n-2}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}+\log\left\{\frac{x_{(n-1)}-\xi_p}{\lambda_p-(x_{(n-1)}-\xi_p)}\right\}+\log\left[\frac{\{x_{(n-1)}+a-\xi_p\}\{\lambda_p-(x_{(n-1)}-\xi_p)\}}{\{x_{(n-1)}-\xi_p\}\{\lambda_p-(x_{(n-1)}-\xi_p)-a\}}\right]\right].$$

$$\text{Or } \hat{\gamma}^* = -\frac{\hat{\delta}}{(n-1)}\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}-\frac{\hat{\delta}}{n}\left[\log\left\{\frac{(x_{(n-1)}+a-\xi_p)\left(\lambda_p-(x_{(n-1)}-\xi_p)\right)}{(x_{(n-1)}-\xi_p)(\lambda_p-(x_{(n-1)}-\xi_p)-a)}\right\}\right].$$

$$\text{Or } \hat{\gamma}^*-\hat{\gamma} = -\frac{\hat{\delta}}{(n-1)}\left[\log\left\{\frac{(x_{(n-1)}+a-\xi_p)\left(\lambda_p-(x_{(n-1)}-\xi_p)\right)}{(x_{(n-1)}-\xi_p)(\lambda_p-(x_{(n-1)}-\xi_p)-a)}\right\}\right].$$

Percentage change in the shape parameter γdue to a shift in the location parameter by an amount a is given as

$$\gamma_{LP} = \frac{|\hat{\gamma}^*-\hat{\gamma}|}{\hat{\gamma}} \times 100. \qquad (3.7)$$

Effect on estimate of the shape parameter (δ)

Using (2.5), the estimate of δ can be written as

$$\hat{\delta}^2 = \frac{(n-1)}{\sum_{i=1}^{n-1}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2-\frac{1}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2}. \qquad (3.8)$$

Considering the first term of denominator of (3.8),

$$\sum_{i=1}^{n-1}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2 = \sum_{i=1}^{n-2}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2+\left[\log\left\{\frac{x_{(n-1)}-\xi_p}{\lambda_p-(x_{(n-1)}-\xi_p)}\right\}\right]^2$$

$$= \sum_{i=1}^{n-2}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2+\left[\log\left\{\frac{x_{(n-1)}+a-\xi_p}{\lambda_p-(x_{(n-1)}+a-\xi_p)}\right\}\right]^2. \qquad (3.9)$$

After shifting the second largest observation $x_{(n-1)}$ by an amount 'a' in (3.9) and using (3.6), this reduces to

$$\sum_{i=1}^{n-1}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2 = \sum_{i=1}^{n-1}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2+\left[\log\left\{\frac{(x_{(n-1)}+a-\xi_p)\left(\lambda_p-(x_{(n-1)}-\xi_p)\right)}{(x_{(n-1)}-\xi_p)(\lambda_p-(x_{(n-1)}-\xi_p)-a)}\right\}\right]^2$$

$$+2\log\left\{\frac{x_{(n-1)}-\xi_p}{\lambda_p-(x_{(n-1)}-\xi_p)}\right\}\log\left\{\frac{(x_{(n-1)}+a-\xi_p)\left(\lambda_p-(x_{(n-1)}-\xi_p)\right)}{(x_{(n-1)}-\xi_p)(\lambda_p-(x_{(n-1)}-\xi_p)-a)}\right\}. \qquad (3.10)$$

Considering the second term of denominator of (3.8),

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$$\frac{1}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-\left(x_{(i)}-\xi_p\right)}\right\}\right]^2=\frac{1}{(n-1)}\left[\sum_{i=1}^{n-2}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-\left(x_{(i)}-\xi_p\right)}\right\}+\log\left\{\frac{x_{(n-1)}-\xi_p}{\lambda_p-\left(x_{(n-1)}-\xi_p\right)}\right\}\right]^2$$

$$=\frac{1}{(n-1)}\left[\sum_{i=1}^{n-2}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-\left(x_{(i)}-\xi_p\right)}\right\}+\log\left\{\frac{x_{(n-1)}+a-\xi_p}{\lambda_p-\left(x_{(n-1)}+a-\xi_p\right)}\right\}\right]^2. \qquad (3.11)$$

Again shifting the second largest observation $x_{(n-1)}$ by 'a' amount in (3.11) and using (3.6), this becomes

$$\frac{1}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-\left(x_{(i)}-\xi_p\right)}\right\}\right]^2$$

$$=\frac{1}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-\left(x_{(i)}-\xi_p\right)}\right\}\right]^2+\frac{1}{(n-1)}\left[\log\left\{\frac{\left(x_{(n-1)}+a-\xi_p\right)\left(\lambda_p-\left(x_{(n-1)}-\xi_p\right)\right)}{\left(x_{(n-1)}-\xi_p\right)\left(\lambda_p-\left(x_{(n-1)}-\xi_p\right)-a\right)}\right\}\right]^2$$

$$+\frac{2}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-\left(x_{(i)}-\xi_p\right)}\right\}\right]\left[\log\left\{\frac{\left(x_{(n-1)}+a-\xi_p\right)\left(\lambda_p-\left(x_{(n-1)}-\xi_p\right)\right)}{\left(x_{(n-1)}-\xi_p\right)\left(\lambda_p-\left(x_{(n-1)}-\xi_p\right)-a\right)}\right\}\right]. \qquad (3.12)$$

Using (3.10) and (3.12) in (3.8), then expression becomes ${\hat{\delta}^*}^2$ as given below,

$${\hat{\delta}^*}^2=\frac{(n-1)}{\sum_{i=1}^{n-1}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-\left(x_{(i)}-\xi_p\right)}\right\}\right]^2+A^2+2A\log\left\{\frac{x_{(n-1)}-\xi_p}{\lambda_p-\left(x_{(n-1)}-\xi_p\right)}\right\}-\frac{1}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-\left(x_{(i)}-\xi_p\right)}\right\}\right]^2-\frac{A^2}{(n-1)}-\frac{2A}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-\left(x_{(i)}-\xi_p\right)}\right\}\right]}$$

where, $A=\log\left\{\frac{\left(x_{(n-1)}+a-\xi_p\right)\left(\lambda_p-\left(x_{(n-1)}-\xi_p\right)\right)}{\left(x_{(n-1)}-\xi_p\right)\left(\lambda_p-\left(x_{(n-1)}-\xi_p\right)-a\right)}\right\}$.

Or $${\hat{\delta}^*}^2=\frac{(n-1)}{\frac{(n-1)}{\hat{\delta}^2}+\left[1-\frac{1}{(n-1)}\right][A]^2+2A\left[\log\left\{\frac{x_{(n-1)}-\xi_p}{\lambda_p-\left(x_{(n-1)}-\xi_p\right)}\right\}-\frac{1}{(n-1)}\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-\left(x_{(i)}-\xi_p\right)}\right\}\right]}. \qquad \{\text{from (3.8)}\}$$

Or $$\hat{\delta}^*=\sqrt{\frac{(n-1)}{\frac{(n-1)}{\hat{\delta}^2}+\left[1-\frac{1}{(n-1)}\right][A]^2+2A\left[\log\left\{\frac{x_{(n-1)}-\xi_p}{\lambda_p-\left(x_{(n-1)}-\xi_p\right)}\right\}-\frac{1}{(n-1)}\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-\left(x_{(i)}-\xi_p\right)}\right\}\right]}}.$$

Or $$\hat{\delta}^*-\hat{\delta}=\sqrt{\frac{(n-1)}{\frac{(n-1)}{\hat{\delta}^2}+\left[1-\frac{1}{(n-1)}\right][A]^2+2A\left[\log\left\{\frac{x_{(n-1)}-\xi_p}{\lambda_p-\left(x_{(n-1)}-\xi_p\right)}\right\}-\frac{1}{(n-1)}\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-\left(x_{(i)}-\xi_p\right)}\right\}\right]}}-\hat{\delta}.$$

Percentage change in the shape parameter δ due to a shift in the location parameter by an amount a is given as

$$\delta_{LP}=\frac{|\hat{\delta}^*-\hat{\delta}|}{\hat{\delta}}\times 100. \qquad (3.13)$$

IV. WHEN THE SMALLEST OBSERVATION IS FROM A DISTRIBUTION WITH A SHIFTED LOCATION PARAMETER

Now when the smallest observation $x_{(1)}$ was shifted towards the negative direction by an amount 'b', then the effects of shift in the estimates of the parameters are as follows.

Effect on estimate of the location parameter (ξ)

From (3.1), when the smallest observation $x_{(1)}$ was shifted towards left by an amount 'b', then its effect on the location parameter is as follows.

$$\hat{\xi}^*=\frac{1}{n-1}\left\{x_{(1)}-b+\sum_{i=2}^{n-1}x_{(i)}\right\}-\frac{1}{n-1}\left\{x_{(1)}-b+\sum_{i=2}^{n-1}x_{(i)}\right\}+\frac{1}{n-1}(n-1)\xi_p$$
$$=\xi_p.$$

Since the shifted effect gets cancelled out in the above expression, shifting of observation has no effect on the estimate of the location parameter.

Effect on estimate of the scale parameter (λ)

Using (3.2), it can be written as

$$\hat{\lambda}=\frac{\frac{(n-1)}{\lambda_p}\left\{x_{(1)}^2+\sum_{i=2}^{n-1}x_{(i)}^2-\xi_p x_{(1)}-\xi_p\sum_{i=2}^{n-1}x_{(i)}\right\}-\frac{1}{\lambda_p}\left\{x_{(1)}+\sum_{i=2}^{n-1}x_{(i)}-(n-1)\xi_p\right\}\left\{x_{(1)}+\sum_{i=2}^{n-1}x_{(i)}\right\}}{\frac{(n-1)}{\lambda_p^2}\left\{x_{(1)}^2+\sum_{i=2}^{n-1}x_{(i)}^2+(n-1)\xi_p^2-2\xi_p x_{(1)}-2\xi_p\sum_{i=2}^{n-1}x_{(i)}\right\}-\frac{1}{\lambda_p^2}\left\{\left(x_{(1)}+\sum_{i=2}^{n-1}x_{(i)}\right)^2+(n-1)^2\xi_p^2-2(n-1)\xi_p\left(x_{(1)}+\sum_{i=2}^{n-1}x_{(i)}\right)\right\}} \qquad (4.1)$$

Shifting the smallest observation $x_{(1)}$ towards left by an amount 'b' in (4.1), then expression becomes $\hat{\lambda}^*$ as given below

$$\hat{\lambda}^*=\frac{\frac{(n-1)}{\lambda_p}\left[\left\{x_{(1)}-b\right\}^2+\sum_{i=2}^{n-1}x_{(i)}^2-\xi_p\left\{x_{(1)}-b\right\}-\xi_p\sum_{i=2}^{n-1}x_{(i)}\right]-\frac{1}{\lambda_p}\left[\left\{x_{(1)}-b\right\}+\sum_{i=2}^{n-1}x_{(i)}-(n-1)\xi_p\right]\left[\left\{x_{(1)}-b\right\}+\sum_{i=2}^{n-1}x_{(i)}\right]}{\frac{(n-1)}{\lambda_p^2}\left[\left\{x_{(1)}-b\right\}^2+\sum_{i=2}^{n-1}x_{(i)}^2+(n-1)\xi_p^2-2\xi_p\left\{x_{(1)}-b\right\}-2\xi_p\sum_{i=2}^{n-1}x_{(i)}\right]-\frac{1}{\lambda_p^2}\left[\left\{x_{(1)}-b+\sum_{i=2}^{n-1}x_{(i)}\right\}^2+(n-1)^2\xi_p^2-2(n-1)\xi_p\left\{x_{(1)}-b+\sum_{i=2}^{n-1}x_{(i)}\right\}\right]}$$

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$$= \frac{(n-1)\sum_{i=1}^{n-1} x_{(i)}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}+\frac{(n-1)}{\lambda_p}\{b^2-2bx_{(1)}+b\xi_p\}-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\sum_{i=1}^{n-1} x_{(i)}-\frac{b}{\lambda_p}\sum_{i=1}^{n-1} x_{(i)}-b\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}+\frac{b^2}{\lambda_p}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2+\frac{(n-1)}{\lambda_p^2}\{b^2-2bx_{(1)}+2b\xi_p\}-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2-\frac{1}{\lambda_p^2}\{b^2-2b\sum_{i=1}^{n-1} x_{(i)}+2b(n-1)\xi_p\}}$$

$$= \frac{(n-1)\sum_{i=1}^{n-1} x_{(i)}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}-\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\sum_{i=1}^{n-1} x_{(i)}+\left[(n-2)\frac{b^2}{\lambda_p}+\frac{2b}{\lambda_p}\{\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)}\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2+\left[(n-2)\frac{b^2}{\lambda_p^2}+\frac{2b}{\lambda_p^2}\{\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)}\}\right]}$$

$$= \frac{\hat{\lambda}\left[(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2\right]+\left[(n-2)\frac{b^2}{\lambda_p}+\frac{2b}{\lambda_p}\{\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)}\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2+\left[(n-2)\frac{b^2}{\lambda_p^2}+\frac{2b}{\lambda_p^2}\{\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)}\}\right]}.$$

Or $$\hat{\lambda}^* = \frac{\hat{\lambda}+\frac{\frac{1}{\lambda_p}\left[(n-2)b^2+2b\left\{\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}}{1+\frac{\frac{1}{\lambda_p^2}\left[(n-2)b^2+2b\left\{\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)}\right\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}}.$$

Or $$\hat{\lambda}^*+\hat{\lambda}^*\left[\frac{\frac{1}{\lambda_p^2}\{(n-2)b^2+2b(\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)})\}}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}\right] = \hat{\lambda}+\left[\frac{\frac{1}{\lambda_p}\{(n-2)b^2+2b(\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)})\}}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}\right].$$

Or $$\left(\hat{\lambda}^*-\hat{\lambda}\right)+\left(\hat{\lambda}^*-\hat{\lambda}\right)\left[\frac{\frac{1}{\lambda_p^2}\{(n-2)b^2+2b(\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)})\}}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}\right]$$

$$= \frac{\frac{1}{\lambda_p}\left[(n-2)b^2+2b\{\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)}\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}-\hat{\lambda}\left[\frac{\frac{1}{\lambda_p^2}\{(n-2)b^2+2b(\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)})\}}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}\right].$$

Or $$\left(\hat{\lambda}^*-\hat{\lambda}\right)\left[1+\left\{\frac{\frac{1}{\lambda_p^2}\{(n-2)b^2+2b(\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)})\}}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}\right\}\right]$$

$$= \frac{\frac{1}{\lambda_p}\left[(n-2)b^2+2b\{\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)}\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}-\hat{\lambda}\left[\frac{\frac{1}{\lambda_p^2}\{(n-2)b^2+2b(\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)})\}}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}\right].$$

Or $$\left(\hat{\lambda}^*-\hat{\lambda}\right) = \frac{\frac{\frac{1}{\lambda_p}\left[(n-2)b^2+2b\{\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)}\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}-\hat{\lambda}\left[\frac{\frac{1}{\lambda_p^2}\{(n-2)b^2+2b(\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)})\}}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}\right]}{1+\left\{\frac{\frac{1}{\lambda_p^2}\{(n-2)b^2+2b(\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)})\}}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2}\right\}}.$$

Simplification of the above equation gives

$$\left(\hat{\lambda}^*-\hat{\lambda}\right) = \frac{\frac{1}{\lambda_p}\left[(n-2)b^2+2b\{\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)}\}\right]-\frac{\hat{\lambda}}{\lambda_p^2}\left[(n-2)b^2+2b\{\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)}\}\right]}{(n-1)\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}^2-\left[\sum_{i=1}^{n-1}\left\{\frac{x_{(i)}-\xi_p}{\lambda_p}\right\}\right]^2+\frac{1}{\lambda_p^2}\left[(n-2)b^2+2b\{\sum_{i=1}^{n-1} x_{(i)}-(n-1)x_{(1)}\}\right]}.$$

Percentage change in the scale parameter λ due to a shift in the location parameter by an amount b is given as

$$\lambda_{SP} = \frac{|\hat{\lambda}^*-\hat{\lambda}|}{\hat{\lambda}}\times 100\ . \quad (4.2)$$

<u>*Effect on estimate of the shape parameter* (γ)</u>

From (3.4),

$$\hat{\gamma} = -\frac{\hat{\delta}}{(n-1)}\left[\log\left\{\frac{x_{(1)}-\xi_p}{\lambda_p-(x_{(1)}-\xi_p)}\right\}+\sum_{i=2}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]. \quad (4.3)$$

Smallest observation $x_{(1)}$ was getting shifted by an amount 'b' in (4.3), we get

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$$\hat{\gamma}^* = -\frac{\hat{\delta}}{n}\left[\log\left\{\frac{x_{(1)}-b-\xi_p}{\lambda_p-(x_{(1)}-b-\xi_p)}\right\} + \sum_{i=2}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]$$

$$= -\frac{\hat{\delta}}{(n-1)}\left[\log\left\{\frac{x_{(1)}-\xi_p-b}{\lambda_p-(x_{(1)}-\xi_p)+b}\right\} + \sum_{i=2}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]$$

$$= -\frac{\hat{\delta}}{(n-1)}\left[\log\{x_{(1)}-\xi_p-b\} - \log\{\lambda_p-\left(x_{(1)}-\xi_p\right)+b\} + \sum_{i=2}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]$$

$$= -\frac{\hat{\delta}}{(n-1)}\left[\log\left\{\frac{(x_{(1)}-\xi_p-b)\left(\lambda_p-(x_{(1)}-\xi_p)\right)}{(x_{(1)}-\xi_p)(\lambda_p-(x_{(1)}-\xi_p)+b)}\right\} + \sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]. \quad \{\text{from } (3.6)\}$$

Or $\hat{\gamma}^* = -\frac{\hat{\delta}}{(n-1)}\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\} - \frac{\hat{\delta}}{n}\log\left\{\frac{(x_{(1)}-\xi_p-b)\left(\lambda_p-(x_{(1)}-\xi_p)\right)}{(x_{(1)}-\xi_p)(\lambda_p-(x_{(1)}-\xi_p)+b)}\right\}$.

Or $\hat{\gamma}^* - \hat{\gamma} = -\frac{\hat{\delta}}{(n-1)}\log\left\{\frac{(x_{(1)}-\xi_p-b)\left(\lambda_p-(x_{(1)}-\xi_p)\right)}{(x_{(1)}-\xi_p)(\lambda_p-(x_{(1)}-\xi_p)+b)}\right\}$.

Percentage change in the shape parameter γ due to a shift in the location parameter by an amount b is given as

$$\gamma_{SP} = \frac{|\hat{\gamma}^*-\hat{\gamma}|}{\hat{\gamma}} \times 100. \qquad (4.4)$$

Effect on estimate of the shape parameter (δ)

The first term of denominator of (3.8) can be written as

$$\sum_{i=1}^{n-1}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2 = \left[\log\left\{\frac{x_{(1)}-\xi_p}{\lambda_p-(x_{(1)}-\xi_p)}\right\}\right]^2 + \sum_{i=2}^{n-1}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2. \qquad (4.5)$$

In (4.5) the smallest observation $x_{(1)}$ was shifted by an amount 'b' towards left, then the above equation becomes

$$\sum_{i=1}^{n-1}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2 = \left[\log\left\{\frac{x_{(1)}-b-\xi_p}{\lambda_p-(x_{(1)}-b-\xi_p)}\right\}\right]^2 + \sum_{i=2}^{n-1}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2$$

$$= \left[\log\left\{\frac{x_{(1)}-\xi_p}{\lambda_p-(x_{(1)}-\xi_p)}\right\} + \log\left\{\frac{(x_{(1)}-\xi_p-b)\{\lambda_p-(x_{(1)}-\xi_p)\}}{(x_{(1)}-\xi_p)(\lambda_p-(x_{(1)}-\xi_p)+b)}\right\}\right]^2 + \sum_{i=2}^{n-1}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2$$

$$= \sum_{i=1}^{n-1}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2 + \left[\log\left\{\frac{(x_{(1)}-\xi_p-b)\{\lambda_p-(x_{(1)}-\xi_p)\}}{(x_{(1)}-\xi_p)(\lambda_p-(x_{(1)}-\xi_p)+b)}\right\}\right]^2$$

$$+ 2\log\left\{\frac{x_{(1)}-b-\xi_p}{\lambda_p-(x_{(1)}-b-\xi_p)}\right\}\log\left\{\frac{(x_{(1)}-\xi_p-b)\{\lambda_p-(x_{(1)}-\xi_p)\}}{(x_{(1)}-\xi_p)(\lambda_p-(x_{(1)}-\xi_p)+b)}\right\}. \qquad (4.6)$$

Considering the second term of denominator of (3.8),

$$\frac{1}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2 = \frac{1}{(n-1)}\left[\log\left\{\frac{x_{(1)}-\xi_p}{\lambda_p-(x_{(1)}-\xi_p)}\right\} + \sum_{i=2}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2. (4.7)$$

On shifting the smallest observation $x_{(1)}$ by an amount 'b' and using equation (3.6), the R.H.S of (4.7) becomes

$$\frac{1}{(n-1)}\left[\log\left\{\frac{x_{(1)}-\xi_p}{\lambda_p-(x_{(1)}-\xi_p)}\right\} + \log\left\{\frac{(x_{(1)}-\xi_p-b)\{\lambda_p-(x_{(1)}-\xi_p)\}}{(x_{(1)}-\xi_p)(\lambda_p-(x_{(1)}-\xi_p)+b)}\right\} + \sum_{i=2}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2$$

$$= \frac{1}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\} + \log\left\{\frac{(x_{(1)}-\xi_p-b)\{\lambda_p-(x_{(1)}-\xi_p)\}}{(x_{(1)}-\xi_p)(\lambda_p-(x_{(1)}-\xi_p)+b)}\right\}\right]^2$$

$$= \frac{1}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2 + \frac{1}{(n-1)}\left[\log\left\{\frac{(x_{(1)}-\xi_p-b)\{\lambda_p-(x_{(1)}-\xi_p)\}}{(x_{(1)}-\xi_p)(\lambda_p-(x_{(1)}-\xi_p)+b)}\right\}\right]^2$$

$$+ \frac{2}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]\log\left\{\frac{(x_{(1)}-\xi_p-b)\{\lambda_p-(x_{(1)}-\xi_p)\}}{(x_{(1)}-\xi_p)(\lambda_p-(x_{(1)}-\xi_p)+b)}\right\}. \qquad (4.8)$$

Using (4.6) and (4.8) in (3.8), then the expression of $\hat{\delta}^{*2}$ is as given below

$$\hat{\delta}^{*2} = \frac{(n-1)}{\sum_{i=1}^{n-1}\left[\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2 + B^2 + 2B\log\left\{\frac{x_{(1)}-b-\xi_p}{\lambda_p-(x_{(1)}-b-\xi_p)}\right\} - \frac{1}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]^2 - \frac{B^2}{(n-1)} - \frac{2B}{(n-1)}\left[\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right]},$$

where, $B = \log\left\{\frac{(x_{(1)}-\xi_p-b)\{\lambda_p-(x_{(1)}-\xi_p)\}}{(x_{(1)}-\xi_p)(\lambda_p-(x_{(1)}-\xi_p)+b)}\right\}$.

Or $\hat{\delta}^{*2} = \frac{(n-1)}{\frac{(n-1)}{\hat{\delta}^2} + 2B\left[\log\left\{\frac{x_{(1)}-\xi_p}{\lambda_p-(x_{(1)}-\xi_p)}\right\} - \frac{1}{(n-1)}\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right] + \left[1-\frac{1}{(n-1)}\right][B]^2}$.

Or $\hat{\delta}^* = \sqrt{\frac{(n-1)}{\frac{(n-1)}{\hat{\delta}^2} + 2B\left[\log\left\{\frac{x_{(1)}-\xi_p}{\lambda_p-(x_{(1)}-\xi_p)}\right\} - \frac{1}{(n-1)}\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_p}{\lambda_p-(x_{(i)}-\xi_p)}\right\}\right] + \left[1-\frac{1}{(n-1)}\right][B]^2}}$.

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Or $\hat{\delta}^* - \hat{\delta} = \sqrt{\frac{(n-1)}{\frac{(n-1)}{\hat{\delta}^2}+2B\left[\log\left\{\frac{x_{(1)}-\xi_P}{\lambda_P-\left(x_{(1)}-\xi_P\right)}\right\}-\frac{1}{(n-1)}\sum_{i=1}^{n-1}\log\left\{\frac{x_{(i)}-\xi_P}{\lambda_P-\left(x_{(i)}-\xi_P\right)}\right\}\right]+\left[1-\frac{1}{(n-1)}\right][B]^2}} - \hat{\delta}$.

Percentage change in the shape parameter δ due to a shift in the location parameter by an amount b is given by

$$\delta_{SP} = \frac{|\hat{\delta}^* - \hat{\delta}|}{\hat{\delta}} \times 100. \qquad (4.9)$$

V. AN EXAMPLE

Random samples of size n = 1000, 2000, 2500, 3000, 4000 from Johnson S_B distribution with value 1, 1 as the two shape parameters, 10, 10 as the scale and the location parameters respectively were generated using R software. Using these data sets, the percentage variations in scale and shape parameters were calculated.

Percentage variation of the estimates of the scale parameter λ and both the shape parameters δ and γ when the second largest observation is from a distribution with a shifted location parameter obtained from equations (3.3), (3.7) and (3.13) the effect of shift in the estimates of scale and shape parameters are shown in tables 5.1 and 5.2 respectively.

Table 5.1. Percentage variation in the scale parameter λ when the shift in second largest observation is made by an amount '*a*' with a shifted location parameter

Sample size *n*	% absolute variation of λ	
	a=10000	*a*=50000
1000	35.42695	35.428202
2000	36.89483	36.89772
2500	37.35046	37.35287
3000	37.57519	37.57892
4000	37.58212	37.58704

It can be seen from this table that the percentage variation of λ is almost a constant, *i.e.* variations in the sample size as well as the shift '*a'* does not have much effect on the percentage variation of λ**.**

Table 5.2. Percentage variation in the shape parameters δ and γ when the shift in second largest observation is made by an amount '*a*' with a shifted location parameter

Sample size *n*	% absolute variation of δ		% absolute variation of γ	
	a=3	*a*=4	*a*=3	*a*=4
1000	0.270114	0.1192171	0.8194109	0.3758388
2000	0.116984	0.0519911	0.1162732	0.2517942
2500	0.060378	0.0276644	0.0916069	0.04336292
3000	0.048966	0.0225438	0.0817969	0.0388729
4000	0.037140	0.0171025	0.0583097	0.02771801

It can be noticed from table 5.2 that there is an extremely small variation in both the shape parameters due to any shift in location parameter. Also, it can be seen that the percentage variation decreases in the shape parameters as the sample sizes increase.

Percentage variation of the estimates of the scale parameter λ and both the shape parameters δ and γ when the smallest observation is from a distribution with a shifted location parameter obtained from equations (4.3), (4.4) and (4.9) the effect of shift in the estimates of scale and shape parameters are shown in tables 5.3 and 5.4 respectively.

Table 5.3. Percentage of variation in the scale parameter λ when the shift in smallest observation is made by an amount '*b*' with a shifted location parameter

Sample Size *n*	% variation in λ	
	b=10000	*b*=50000

1000	35.42741	35.42804
2000	36.89607	36.89777
2500	37.35162	37.35292
3000	37.57689	37.57899
4000	37.58427	37.58712

This table shows that the percentage variation of λ is almost a constant, *i.e.* variations in the sample size as well as the shift '*b*' does not have much effect on the percentage variation of λ.

Table 5.4. Percentage variation in both the shape parameters δ and γ when the shift in smallest observation is made by an amount '*b*' with a shifted location parameter

Sample Size *n*	% variation of δ		% variation of γ	
	b=3	*b*=4	*b*=3	*b*=4
1000	0.018047	0.092697	0.6529123	0.3551242
2000	0.019352	0.0855136	0.1519352	0.08094967
2500	0.016882	0.1131722	0.0595022	0.03137204
3000	0.017629	0.0353589	0.0557616	0.02938813
4000	0.013218	0.0265765	0.0381041	0.02004436

It can be seen from table 5.4 that there is an extremely small variation in both the shape parameters due to any shift in location parameter. Also, it can be seen that the percentage variation decreases in the shape parameters as the sample sizes increase.

VI. CONCLUSION

From the above study it can be concluded that, even a large variations in sample size does not have much effect in the variation of the estimates of any of the parameters of Johnson S_B distribution, while any shift in the parameters certainly lead to variation in the estimate of the parameters. It was also observed that the percentage variation of the estimate of λ is almost a constant for any variation in any of the parameters.

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