International Journal of Scientific Research in Mathematical and Statistical Sciences **Research Paper**
Vol.6, Issue.1, pp.167-172, February (2019) **E-ISSN:** 2348-4519
DOI: https://doi.org/10.26438/ijsrmss/v6i1.167172

Relation between L_p-Riemann Derivative, Approximate Riemann Derivative and Riemann Derivative

T. K. Garai

Department of Mathematics, Bolpur College, Bolpur-731204, Birbhum, West Bengal,

Corresponding Author: tg70841@gmail.com, Tel.: +919434543368

Available online at: www.isroset.org

Received: 09/Feb/2019, Accepted: 22/Feb/2019, Online: 28/Feb/2019

***Abstract*—**As the Definition of L_p-derivative is such that which contains only the absolute value of the function and therefore it is not possible to define the L_p-derivates from the definition of L_p-derivative. So to remove this difficulty S.N. Mukhopadyay and S.Ray uses a special technique to define them in their paper [2]. In this article we define the Lp-Riemann Derivative using the same technique as it is used to define the Lp-derivates in [2] and relation between approximate Riemann Derivative and Riemann derivative are studied.

***Keywords*—** *Riemann derivative, Approximate Riemann derivative L_p-Riemann derivative Holder's inequality*

I. Introduction

Let $f: R \to R$ be a function. The Riemann derivative of a function f at x of order is r denoted by $RD_{(r)} f(x)$ and defined as $RD_{(r)} f(x) = {}_t\underrightarrow{\lim}_0 \dfrac{\Delta^r(f,x,t)}{t^r}$,where,

$$\Delta^r(f,x,t) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(x+it), r = 1,2,3,...$$

The upper right and lower right Riemann derivative of f at x of order r are denoted by $\overline{RD}^+_{(r)} f(x)$ and $\underline{RD}^+_{(r)} f(x)$ and are defined as,

$$\overline{RD}^+_{(r)} f(x) = {}_t\underrightarrow{\lim}_{0+} \sup \frac{\Delta^r(f,x,t)}{t^r} \quad \text{and} \quad \underline{RD}^+_{(r)} f(x) = {}_t\underrightarrow{\lim}_{0+} \inf \frac{\Delta^r(f,x,t)}{t^r}$$

Similarly the upper left and lower left Riemann derivatives $\overline{RD}^-_{(r)} f(x)$ and $\underline{RD}^-_{(r)} f(x)$ are defined. The approximate Riemann derivatives $\overline{RD}^+_{(r),a} f(x)$, $\underline{RD}^+_{(r),a} f(x)$, $\overline{RD}^-_{(r),a} f(x)$, $\underline{RD}^-_{(r),a} f(x)$ are defined by taking approximate limit instead of ordinary limits in above definitions.

In this article we shall use the following notations: For any function $A: R \to R$, its positive and negative parts are defined as, $[A]_+ = \max[A,0], [A]_- = \max[-A,0]$ respectively. Clearly,

(1) $A = [A]_+ - [A]_-$

(2) $|A| = [A]_+ + [A]_-$

If $A: R \to R$ and $B: R \to R$ then

(3) $[A+B]_+ \leq [A]_+ + [B]_+$ and $[A+B]_- \leq [A]_- + [B]_+$

and if $A \leq B$ then

(4) $[A]_+ \leq [B]_+$ and $[B]_- \leq [A]_-$

In this paper there are four section in which Section-I deals with the introduction of the total work and some preliminary ideas. In Section-II we define the Riemann derivative in a new manner which helps us to find the relation with L_p-Riemann derivative. In Section-III we define the L_p-Riemann derivative using the technique as used in [2]. In Section-IV we established the relation between L_p-Riemann derivative, approximate Riemann derivative and Riemann derivative. In Section-V we conclude about its future aspects.

II. THE RIEMANN DERIVATIVE

Lemma 2.1. *Let $\psi(x,t)$ be a function of $x,t \in R, t \neq 0$ then the right hand upper limit of ψ at x as $t \to 0_+$ is given by* $\psi^+ = \inf S$ *where,* $\psi^+(x) = \limsup_{t \to 0+} \psi(x,t)$ *and*

$$S = \{a : a \in R, [\psi(x,t) - a]_+ = o(1), as, t \to 0_+\}$$

(It is proved in [2], for definiteness we give the proof here.)

Proof. Let x be fixed. Suppose $\psi^+(x) = \infty$. We show that S is empty. If possible let $a \in S$. Then

$$\lim_{t \to 0+} [\psi(x,t) - a]_+ = 0$$

Since $\psi(x,t) - a \leq [\psi(x,t) - a]_+$, $\limsup_{t \to 0+} (\psi(x,t) - a) \leq 0$ and so $\limsup_{t \to 0+} (\psi(x,t) \leq a$ which is a contradiction, since $\psi^+(x) = \infty$. So, S is empty. Next, suppose $\psi^+(x)$ is finite and $\psi^+(x) < M$ .Then there is $\delta > 0$ such that $\psi(x,t) < M$ for $0 < t < \delta$. So $[\psi(x,t) - M]_+ = 0$ for $0 < t < \delta$ and hence $M \in S$. This shows that every $a > \psi^+(x)$ is a member of S. Again let $m < \psi^+(x)$. Then there is a sequence $\{t_n\}$ such that $t_n > 0$ for all n and $t_n \to 0$ as $n \to \infty$ and $\psi(x,t_n) > m + \varepsilon$ for all n where $m < m + \varepsilon < \psi^+(x)$ .Hence $[\psi(x,t_n) - m]_+ > \varepsilon$ for all n and so $m \in S$. This shows that if $a < \psi^+(x)$ then $a \in S$. Therefore $\psi^+(x) = \inf S$ .Finally suppose $\psi^+(x) = -\infty$. Then $\lim_{t \to 0+} \psi(x,t) = -\infty$. Let $a \in R$. Then there is $\delta > 0$ such that $\psi(x,t) < a$ for $0 < t < \delta$ .Hence $a \in R$ .Thus every member of R is a member of S and hence $\inf S = -\infty$.

Corollary 2.2. *Let $f : R \to R$ and $x \in R$ be fixed. Then the r-th order right hand upper Riemann derivative of f at x,* $\overline{RD}^+_{(r)} f(x)$ *is given by,*

$$\overline{RD}^+_{(r)} f(x) = \lim_{t \to 0+} \sup \frac{\Delta^r(f,x,t)}{t^r}$$

$$= \inf \{a : a \in R; [\Delta^r(f,x,t) - t^r a]_+ = o(t^r) as, t \to 0_+\}$$

Proof. Putting $\psi(x,t) = \dfrac{\Delta^r(f,x,t)}{t^r}$ *Lemma-2.1* we get,

$$\overline{RD}^+_{(r)} f(x) = \inf \left\{ a : a \in R; \left[\frac{\Delta^r(f,x,t)}{t^r} - a \right]_+ = o(1) as, t \to 0_+ \right\}$$

$$= \inf \{a : a \in R; [\Delta^r(f,x,t) - t^r a]_+ = o(t^r) as, t \to 0_+\}$$

III. THE L_p-RIEMANN DERIVATIVE

The following theorem can be proved using the same technique as used in Theorem-3.1 of [2].

Theorem 3.1: *Let $f : R \to R$ and $x \in R$ be fixed. Let $f \in L_p, 1 \le p < \infty$, in some neighbourhood of x and r in a fixed positive integer. If*

$$U_+(f) = \left\{ a : a \in R; \left(\frac{1}{h} \int_0^h ([\Delta^r(f, x, t) - at^r]_+)^p \, dt \right)^{\frac{1}{p}} = o(h^r) as, h \to 0_+ \right\} \qquad (5)$$

and

$$U_-(f) = \left\{ a : a \in R; \left(\frac{1}{h} \int_0^h ([\Delta^r(f, x, t) - at^r]_-)^p \, dt \right)^{\frac{1}{p}} = o(h^r) as, h \to 0_+ \right\} \qquad (6)$$

then

$$\inf U_+(f) \ge \sup U_-(f) \qquad (7)$$

Moreover,if

$$\inf U_+(f) = \sup U_-(f) = \mu \text{ say, } \mu \text{ is finite} \qquad (8)$$

then

$$\left(\frac{1}{h} \int_0^h (| \Delta^r(f, x, t) - \mu t^r |)^p \, dt \right)^{\frac{1}{p}} = o(h^r), as, h \to 0_+ \qquad (9)$$

and conversely, if (9) holds for some μ then (8) holds.

Now the Theorem 3.1 helps us to define upper and lower L_p-Riemann derivatives.

Definition 3.2. Let $f : R \to R$ and $x \in R$ be fixed. Let $f \in L_p, 1 \le p < \infty$, in some neighbourhood of x and r in a fixed positive integer. The right upper and right lower L_p-Riemann derivative of f at x of order r are denoted by $\overline{RD}^+_{(r),p} f(x)$ and $\underline{RD}^+_{(r),p} f(x)$ respectively and are defined as,

$$\overline{RD}^+_{(r),p} f(x) = \inf \left\{ a : a \in R; \left(\frac{1}{h} \int_0^h ([\Delta^r(f, x, t) - at^r]_+)^p \, dt \right)^{\frac{1}{p}} = o(h^r) as, h \to 0_+ \right\}$$

And

$$\underline{RD}^+_{(r),p} f(x) = \sup \left\{ a : a \in R; \left(\frac{1}{h} \int_0^h ([\Delta^r(f, x, t) - at^r]_-)^p \, dt \right)^{\frac{1}{p}} = o(h^r) as, h \to 0_+ \right\}$$

Similarly the left upper and left lower L_p-Riemann derivative of f at x of order r can be defined and are denoted by $\overline{RD}^-_{(r),p} f(x)$ and $\underline{RD}^-_{(r),p} f(x)$ respectively. Both sided upper and lower derivatives are

$$\overline{RD}_{(r),p} f(x) = \max[\overline{RD}^+_{(r),p} f(x), \overline{RD}^-_{(r),p} f(x)]$$

and

$$\underline{RD}_{(r),p} f(x) = \min[\underline{RD}^+_{(r),p} f(x), \underline{RD}^-_{(r),p} f(x)]$$

If $\overline{RD}_{(r),p} f(x) = \underline{RD}_{(r),p} f(x)$, the common value is the Lp-Riemann derivative of f at x of order r and is denoted by $RD_{(r),p} f(x)$.

IV. RELATION BETWEEN Lp-RIEMANN DERIVATIVE, APPROXIMATE RIEMANN DERIVATIVE AND RIEMANN DERIVATIVE

Theorem 4.1. If $f \in L_p$ then,

$$\underline{RD}^{+}_{(r),p} f(x) \le \underline{RD}^{+}_{(r),a} f(x) \le \overline{RD}^{+}_{(r),a} f(x) \le \overline{RD}^{+}_{(r),p} f(x)$$

With similar relations for left derivatives.

Proof. Let $\overline{RD}^{+}_{(r),a} f(x) = \alpha$ and $\overline{RD}^{+}_{(r),p} f(x) = \beta$. If possible let $\alpha > \beta$ then there exists γ such that $\alpha > \gamma > \beta$. Then by definition of α the set

$$E = \{t : t > 0; (\Delta^r(f,x,t) - \gamma t^r) > 0\}$$

has positive upper density in the right of $t = 0$. Hence there exists $\delta > 0$ and a sequence $\{h_n\}$ such that $h_n \to 0_+$ as $n \to \infty$ and

$$\frac{\mu(E \cap [0,h_n]}{h_n} > \delta \text{ for all } n$$

Hence

$$\mu(E \cap [0,h_n] > \delta h_n \text{ for all } n$$

Also by the definition of for all β there is $\sigma \in R, \beta \le \sigma < \gamma$ such that

$$\left(\frac{1}{h}\int_0^h ([\Delta^r(f,x,t) - \sigma t^r]_+)^p \, dt\right)^{\frac{1}{p}} = o(h^r) as, h \to 0_+$$

So,

$$\left(\frac{1}{h_n}\int_0^{h_n} ([\Delta^r(f,x,t) - \sigma t^r]_+)^p \, dt\right)^{\frac{1}{p}} = o({h_n}^r) as, n \to \infty \qquad (10)$$

Now for a fixed n we have by (4),

$$\int_0^{h_n} ([\Delta^r(f,x,t) - \sigma t^r]_+)^p \, dt \ge \int_0^{h_n} ([\Delta^r(f,x,t) - \gamma t^r]_+)^p \, dt \ge \int_{E\cap[0,h_n]} ([\Delta^r(f,x,t) - \gamma t^r]_+)^p \, dt = C(say)$$

Then C > 0 . For, if C = 0 then by the property of Lebesgue integral the integrand of the last expression would vanish a.e. on $E \cap [0,h_n]$ which is a contradiction since E has positive upper density in the right of the point t = 0. Therefore

$$\frac{1}{h_n^r}\left(\frac{1}{h_n}\int_0^{h_n} ([\Delta^r(f,x,t) - \sigma t^r]_+)^p \, dt\right)^{\frac{1}{p}} \ge \frac{1}{h_n^r} C^{\frac{1}{p}} \left(\frac{1}{h_n}\right)^{\frac{1}{p}} = \frac{1}{h_n^{r+\frac{1}{p}}} C^{\frac{1}{p}} \to \infty \text{ as } n \to \infty$$

Which contradict (10). Therefore last inequality of theorem is proved. Similarly the first inequality can be proved.

Theorem 4.2. If $f \in L_p$ then,

$$\underline{RD}^{+}_{(r)} f(x) \le \underline{RD}^{+}_{(r),p} f(x) \le \overline{RD}^{+}_{(r),p} f(x) \le \overline{RD}^{+}_{(r)} f(x)$$

With similar relations for left derivatives.

Proof. Let

$$E_{+}(f) = \left\{ a : a \in R; \left(\frac{1}{h} \int_0^h ([\Delta^r(f,x,t) - at^r]_{+})^p \, dt \right)^{\frac{1}{p}} = o(h^r) as, h \to 0_{+} \right\}$$

and

$$F_{+}(f) = \left\{ a : a \in R; \left[\Delta^r(f,x,t) - t^r a\right]_{+} = o(t^r) as, t \to 0_{+} \right\}$$

Let $a \in F_{+}(f)$. Let $\varepsilon > 0$ be arbitrary. Then since $a \in F_{+}(f)$ there is $\delta > 0$ such that $\frac{1}{t^r}[\Delta^r(f,x,t) - t^r a]_{+} < \varepsilon$ for $0 < t < \delta$ and so $[\Delta^r(f,x,t) - t^r a]_{+} < \varepsilon t^r$ for $0 < t < \delta$. Hence

$$(\frac{1}{h} \int_0^h ([\Delta^r(f,x,t) - at^r]_{+})^p \, dt)^{\frac{1}{p}} < \varepsilon \frac{h^r}{(rp+1)^{\frac{1}{p}}} \text{ for } 0 < h < \delta$$

Since $\varepsilon > 0$ is arbitrary

$$(\frac{1}{h} \int_0^h ([\Delta^r(f,x,t) - at^r]_{+})^p \, dt)^{\frac{1}{p}} = o(h^r) \text{ as } h \to 0_{+}$$

Therefore $a \in E_{+}(f)$. So $F_{+}(f) \subset E_{+}(f)$. Hence from definition of $\overline{RD}^{+}_{(r),p} f(x)$ and from Corollary2.2

$$\overline{RD}^{+}_{(r),p} f(x) = \inf E_{+}(f) \le \inf F_{+}(f) = \overline{RD}^{+}_{(r)} f(x)$$

This proves the last inequality, the proof of the first inequality is similar.

Theorem 4.3. If $f \in L_p$ and $1 \le q < p < \infty$ then,

$$\underline{RD}^{+}_{(r),p} f(x) \le \underline{RD}^{+}_{(r),q} f(x) \le \overline{RD}^{+}_{(r),q} f(x) \le \overline{RD}^{+}_{(r),p} f(x)$$

Proof. Since $f \in L_p$, $(\Delta^r(f,x,t) - t^r a) \in L_p$ and so $[(\Delta^r(f,x,t) - t^r a]_{+} \in L_p$. Hence $([\Delta^r(f,x,t) - t^r a]_{+})^q \in L_{\frac{p}{q}}$. Since $1 \in L_{\frac{p}{p-q}}$, by Holder's inequality we get,

$$\int_0^h ([\Delta^r(f,x,t) - at^r]_{+})^q \, dt \le (\int_0^h ([\Delta^r(f,x,t) - at^r]_{+})^p \, dt)^{\frac{p}{q}} h^{\frac{p-q}{p}}$$

Hence

$$(\frac{1}{h} \int_0^h ([\Delta^r(f,x,t) - at^r]_{+})^q \, dt)^{\frac{1}{q}} \le (\frac{1}{h} \int_0^h ([\Delta^r(f,x,t) - at^r]_{+})^p \, dt)^{\frac{1}{p}}$$

which shows that

$$\left\{ a : a \in R; \left(\frac{1}{h} \int_0^h ([\Delta^r(f,x,t) - at^r]_{+})^p \, dt \right)^{\frac{1}{p}} = o(h^r) as, h \to 0_{+} \right\}$$

is a subset of

$$\left\{ a : a \in R; \left(\frac{1}{h} \int_0^h ([\Delta^r(f,x,t) - at^r]_+)^q \, dt \right)^{\frac{1}{q}} = o(h^r) as, h \to 0_+ \right\} 8<$$

Therefore from definition $\overline{RD}^{+}_{(r),q} f(x) \le \overline{RD}^{+}_{(r),p} f(x)$, this complete the proof of the last inequality of (11). The proof of the first inequality of (11) is same.

Theorem 4.4. If $f \in L_p$ and $1 \le q < p < \infty$ then,

$$\underline{RD}^{+}_{(r)} f(x) \le \underline{RD}^{+}_{(r),p} f(x) \le \underline{RD}^{+}_{(r),q} f(x) \le \underline{RD}^{+}_{(r),a} f(x)$$
$$\le \overline{RD}^{+}_{(r),a} f(x) \le \overline{RD}^{+}_{(r),q} f(x) \le \overline{RD}^{+}_{(r),p} f(x) \le \overline{RD}^{+}_{(r)} f(x)$$

Proof. The theorem is a combination of the Theorem-4.1, Theorem-4.2 and Theorem-4.3

V. CONCLUSION AND FUTURE SCOPE

In Definition 3.2 the L_p-Riemann derivative is defined in a way so that the absolute value of the function can be removed. Theorem 4.1 shows that if a function possesses L_p-Riemann derivate then it possesses the approximate Riemann derivate. Theorem 4.2 shows that Lp-Riemann derivative is a generalization of Riemann derivative.

ACKNOWLEDGMENT

The author wish to express his sincere gratitude to Dr.S.Ray, Associate Professor of the department of Mathematics , Siksha Bhavana, Visva-Bharati, for his kind help and suggestions in preparation of this paper.

REFERENCES

[1] S. N. Mukhopadhyay, Higher order derivatives, Chapman and Hall/CRC, Monographs and surveys in Pure and Applied Methematics, 144 (2012).

[2] S.N.Mukhopadhyay and S.Ray, "Relation between Lp-derivates and Peano, approximate Peano and Borel derivates of higher order". Real Analysis Exchange, 41(1)(2015/2016) 1-22.

[3] J.M.Ash, Generalisations of Riemman derivetive, Trans. American Math. Soc., 126 (1967), 181-199.

AUTHORS PROFILE

T.K.Garai is an Assistant Professor in Mathematics, Department of Mathematics, Bolpur College, Bolpur, West Bengal. He is pursuing Ph.D degree AT Visva-Bharati University, West Bengal. He has published seven research paper in reputed national and international journal. He is working as assistant professor since 2005.