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K-Forcing Number of Some Graphs and Their Splitting Graphs

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Abstract: Amos et al. instigated the idea of k-forcing number of a graph. For a graph $G = (V, E)$ and a subset $F$of G, the vertices in F are called initially colored black vertices and the vertices in $V(G) - F$ are called initially non colored black vertices or white vertices. Then the set F is a k-forcing set of a graph G if all vertices in G will become colored black after applying the subsequent color changing rule. Color changing rule: If a black colored vertex is adjacent to atmost k-white vertices, then the white vertices change to be colored black. The cardinality of a smallest k-forcing set is known as the k-forcing number of the graph G and is represented as $Z_k(G)$. This work is intended to investigate the k-forcing number of the splitting graph of a graph in which $k \geq 2$.

Keywords: Zero Forcing Number, K-forcing Number and Splitting Graph.

I. INTRODUCTION

In this paper, we consider only simple graphs with vertex set $V(G)$ and edge set $E(G)$. The splitting graph $S(G)$ of a graph G is the graph obtained from G by taking a vertex u' corresponding to each vertex $u \in G$ and join u' to all vertices in $N(u)$, the open neighborhood of u. This graph was introduced by E. Sampathkumar et al. in [1]. In [2] the authors studied about the zero forcing number of the splitting graph of a graph.

For a simple graph G and a positive integer $k > 0$, the k-forcing number of G, denoted by $Z_k(G)$ is the minimum number of vertices that are needed to be initially colored black so that all vertices after a finite number of steps become colored black during the subsequent color changing rule.

Color changing rule : If a black colored vertex has at most k non-colored neighbors, then each of its non-colored neighbors becomes colored as black. When $k = 1$, this definition is same as that of the zero forcing number, denoted by $Z(G)$ (See [3]).

The zero forcing number can be applied in quantum physics and logic circuits (See [4], [5] and [6]). The k-forcing number of a graph was introduced by D Amos, Y Caro, R Davila and R Pepper in [7]. In this paper, we consider the case when $2 \leq k \leq \Delta$, where Δ is the maximum degree of the graph G, also we initiate the study of the k-forcing number of the splitting graph $S(G)$ of a graph G. When the color changing rule is given to an arbitrary vertex v to change the color of w, we say v forces w and we represent it as $v \rightarrow w$. The following definitions are necessary for the further development of this article. We recall them from [7] and [8].

- Cartesian Product: The Cartesian product $G_1 \square G_2$ of the graphs G_1 and G_2 is the graph with vertex set equal to the Cartesian product of $V(G_1) \times V(G_2)$ and two vertices (g, g') and (h, h') are adjacent in $G_1 \square G_2$ if and only if either $g = h$ and $g' \sim h'$ in G_2 or $g' = h'$ and $g \sim h$ in G_1, where the symbol $\sim$ represents the adjacency between two vertices.
- Three vertices u, v and w in a graph G are said to be 3-consecutive if uv and vw are edges in G (See [8]).
- The square graph of a graph G is represented as $G^{(2)}$ and is the graph with vertex set is same as that of the vertex set of G that is $V(G)$ and two vertices are adjacent in $G^{(2)}$ if their distance in G is either 1 or 2.
- The line graph of a graph G is denoted by $L(G)$ and is the graph obtained by taking the edges of G as vertices of $L(G)$, with two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are. These graphs were defined by Whitney [9]. For more definitions on graphs, we refer to [10].

II. Results and Discussion

In this section, we obtain some simple graphs for which $1 \leq Z_2(G) \leq 4$. We start with paths and cylces. The following propositions are easy to observe.

Proposition 1. *Let G be a connected graph. If $\Delta(G) \leq 2$, then $Z_2(G) = 1$.*

Proposition 2. *If G is a connected graph with minimum degree $\delta \geq 3$, then $Z_2(G) \geq 2$.*

Next we consider one more class of graph for which $Z_2(G) = 1$. The corona product of graphs were defined as follows: Let G_1 and G_2 be two graphs. Then the *corona product* ($G = G_1 \circ G_2$) of G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and by joining each vertex of the i-th copy of G_2 to the i-th vertex of G_1, where $1 \leq i \leq |V(G)|$ (See[11]).

Proposition 3. *Let G be the graph $C_n \circ K_1$. Then $Z_2(G) = 1$.*

Proposition 4. *Let G be the square graph of the path P_n on $n \geq 3$ vertices. Then $Z_2(G) = 1$.*

Proposition 5. *Let G be the graph $C_n \circ K_1$. Let $G_1, G_2, \dots, G_m, m \geq n$ be the m- copies of the graph G and let G^* be the graph obtained by identifying a pendant vertex of G_1 with a pendent vertx of G_2, a pendant vertex of G_2 with a pendant vertex of G_3 etc a pendant vertex of G_{n-1} with a pendant vertex of G_n. Then $Z_2(G^*) = 1$.*

Now we consider more classes of graphs for which $Z_2(G) = 1$.

Proposition 6. *Let G be the graph $C_n \square K_2$, where $n \geq 2$ and let $v_1, v_2, \dots, v_n$ be the vertices of the cycle C_n in $C_n \square K_2$ and $v'_1, v'_2, \dots, v'_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ in $C_n \square K_2$. Let H be the graph obtained by sub dividing the edges $v_1v'_1, v_2v'_2, \dots, v_nv'_n$ in $C_n \square K_2$ exactly once. Then $Z_2(H) = 1$.*

Proposition 7. *Let G be the graph $P_n \square P_m$, where P_m and P_n are the paths on $n \geq 2$ and $m \geq 2$ vertices. Then $Z_2(G) = 1$.*

It is an open problem to charcterize graphs and splitting graphs for which $Z_2(S(G)) = 1,2,3$ and 4. Now we consider some splitting graphs.

Proposition 8. *Let G be a path on $n \geq 3$ vertices and $2 \leq k \leq \Delta$ be a positive integer. Then the k-forcing number of $S(G)$ is 1.*

Proof. Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of G and $\{v'_1, v'_2, \dots, v'_n\}$ be the corresponding neighbors of the vertices $\{v_1, v_2, \dots, v_n\}$ in $S(G)$. Consider the vertex $Z = \{v'_2\}$ and color $\{v'_2\}$ black and the remaining vertices as white. Clearly, the vertex v'_2 is adjacent to only two vertices v_1 and v_3, therefore the vertex v'_2 2-forces v_1 and v_3 to black. Now the vertex v_2 2-forces v'_1 and v'_3 to black and the process continues till we get a derived coloring. So $Z = \{v'_2\}$ forms a 2-forcing set. Hence $Z_2[S(G)] = 1$. It can be easily verify that if $k > 2$, then $Z_k[S(G)] = 1$.

Next we consider the cycle graph C_n

Proposition 9. *Let G be a cycle graph on n-vertices ($n \geq 4$). Then the 2-forcing number of the splitting graph of G is 2, that is $Z_2[S(G)] = 2$. If $4 \geq k \geq 3$, then $Z_k[S(G)] =$* 1.

Proof. Let $A = \{u_1, u_2, \dots, u_n\}$ be the vertices of the cycle graph in $S(G)$ and $B = \{u'_1, u'_2, \dots, u'_n\}$ be the corresponding vertices of the cycle graph in $S(G)$. Note that the degree of each vertex in A is 4 and that of B is 2 in $S(G)$. Assume that there exist a zero forcing set with cardinality one. Let $v \in S(G)$ be the black vertex.

Case 1. Assume $v \in B$. Clearly, the vertex v is black and all other vertices in $S(G)$ are white. Now the vertex v can 2-forces two more vertices as black not all the vertices , because each vertex of B is adjacent to two vertices in A having degree 4, a contradiction.

Case 2. Assume $v \in A$. It can be easily verify that the colour change rule is not possible since all vertices in A have degree 4.

Therefore, $$2 \leq Z_2[S(G)] \qquad \textbf{(1)}$$

To prove the reverse part we consider the following case.

Let $Z = \{u_1, u'_1\}$. Color the vertices $u_1$and u'_1 as black and all other vertices as white. Clearly, u'_1 2-forces u_n and u_2 to black. Now u_1 2-forces u'_2, u'_n as black. Now consider the black vertex u_2. u_2 is adjacent to u_1, u'_1, u_3, u'_3 and the vertices u_1, u'_1 are already colored black. So, u_2 2-forces u_3 and u'_3 to black, and the process continues untill we get the derived colouring. So, $Z = \{u_1, u'_1\}$ forms a 2-forcing set. Therefore,

$$Z_2[S(G)] \leq 2 \qquad \textbf{(2)}$$

Hence from **(1)** and **(2)** the result follows.

Corollary 10. *If G is the cycle graph C_3 (triangle), then* $Z_2[S(G)] = 1$.

Proof. Let u_1, u_2 and u_3 be the vertices of the cycle C_3 in $S(G)$ and u'_1, u'_2 and u'_3 be the corresponding vertices of

u_1, u_2 and u_3 in $S(G)$. Color the vertex u'_1 as black. Clearly, the vertex u'_1 2-forces the vertices u_2 and u_3, again the vertices u_2 and u_3 2-forces all other vertices as black. Therefore, $Z_2[S(G)] = 1$.

The Ladder graph $G = P_n \square P_2$ is the graph obtained by taking the Cartesian product of P_n with P_2.

Proposition 11. *Let G be the ladder graph. Then the 2-forcing number of the splitting graph of G is 2. That is $Z_2[S(G)] = 2$.*

Proof. Let $A = \{u_1, u_2, \ldots, u_n\}$ and $B = \{v_1, v_2, \ldots, v_n\}$ be the vertices of the copies of the paths P_n in $S(G)$. Then $C = \{u'_1, u'_2, \ldots, u'_n\}$ and $D = \{v'_1, v'_2, \ldots, v'_n\}$ be the corresponding vertices of the copies of the paths P_n in $S(G)$.

It can be easily verify that, it is not possible to get a zero frocing set of cardinality one. Therefore,

$$Z_2[S(G)] \geq 2 \qquad \textbf{(3)}$$

Let us generate a derived coloring by taking 2 black vertices and all other vertices colored white. Let $Z_2 = \{u_1, u'_1\}$ be a set of black vertices in $S(G)$. Since u'_1 is adjacent to v_1 and u_2, therefore, u'_1 2-forces v_1 and u_2 to black. Now u_1 is adjacent to the vertices u_2,u'_2, v_1,v'_1. The vertices v_1 and u_2 are already colored black, therefore the vertex u_1 2-forces v'_1, u'_2 to black and the process continues. So

$$Z_2[S(G)] \geq 2 \qquad \textbf{(4)}$$

From the above two inequalities, we have $Z_2[S(G)] = 2$.

Prism graph or the circular ladder graph is the graph obtained by taking the Cartesian product of the cycle C_n with the complete graph K_2. This graph was defined by Hladink et al. in [12]. The circular ladder graph can be denoted as $G = C_n \square K_2$.

Proposition 12. *Let G be a Prism graph on $n \geq 3$ vertices. Then $Z_2(G) = 2$.*

The 3-regular cube Q_3 is the graph $C_4 \square K_2$ and by using the above Proposition, for the 3- regular cube $Z_2(G) = 2$.

We recall the following Proposition from [7] to prove the next result.

Proposition 13. [7] *For any connected graph G with minimum degree δ, $Z_2(G) \geq \delta - 1$.*

Proposition 14. *Let G be the wheel graph on $n \geq 5$ vertices. Then $Z_2(G) = 2$.*

Proof. We have from Proposition 13, $2 \leq Z_2(G)$. Consider two adjacent vertices on the outer cycle C_{n-1} of the wheel graph G and color it as black. Clearly, these two vertices forms a 2-forcing set of the graph G and hence the result follows.

Next Proposition deals with a particular class of graphs for which $Z_2[S(G)] = 3$.

Proposition 15. *Let G be the wheel graph (The graph obtained by connecting a single vertex to all vertices of the cycle graph C_{n-1}). Then $Z_2[S(G)] = 3$.*

Proof. It can be easily observe that in $S(G)$, with 2 black vertices we can 2-force maximum of two more vertices to black not all. Therefore,

$$3 \leq Z_2[S(G)] \qquad \textbf{(5)}$$

Now let $v_1, v_2, \ldots, v_{n-1}$ be the vertices of the cycle C_{n-1} and v_n be the universal vertex in the splitting graph of the wheel graph. Let $v'_1, v'_2, \ldots, v'_{n-1}$ be the corresponding vertices of the cycle C_{n-1} in $S(G)$ and v'_n be the vertex corresponds to the universal vertex v_n in $S(G)$. Consider the set $Z_2 = \{v'_1, v'_2, v_n\}$ in $S(G)$. Clearly the vertex $v'_1 \rightarrow \{v_2, v_{n-1}\}$ and $v'_2 \rightarrow \{v_3, v_1\}$. Now in $N(v_1)$ we have exactly two white neighbors v'_{n-1} and v'_n. Therefore, $v_1 \rightarrow \{v'_{n-1}, v'_n\}$. The remaining white vertices forms a splitting graph of the cycle graph and hence we can force all the vertices as black. Hence

$$Z_2[S(G)] \geq 3. \qquad \textbf{(6)}$$

Thus the result follows from **(5)** and **(6)**.

Next proposition deals with another classes of graphs G for which $Z_2(G) = 3$.

Proposition 16. *Let G be the cycle graph C_n and let G' be the square graph of the cycle graph of C_n. Then $Z_2(G') = 3$.*

Proof. By construction the graph G' is a 4- regular graph. So by using Proposition 13, $Z_2(G') \geq 4 - 1 = 3$. In order to get the reverse inequality, it is convenient to consider the vertices of G' as $v_1, v_2, \ldots, v_{n-1}, v_n$. Now color the vertices v_1, v_2 and v_3 as black. Clearly, the vertex v_1 2-forces v_n and v_{n-1} as bkack. Now $v_2 \rightarrow v_4$, $v_3 \rightarrow v_5$ and so on. Hence the set $\{v_1, v_2, v_3\}$ forms a 2-forcing set. Therefore, $Z_2(G') = 3$.

Proposition 17. *Let G be the prism graph $C_n \square K_2$ of order $n \geq 3$. Then $Z_2(S(G)) = 4$.*

Proof. It can be easily verify that with two black vertices, we can force a maximum of two more vertices to black. Therefore, color changing rule is not possible with two black vertices. This implices $Z_2(S(G)) \neq 2$. Now assume that $Z_2(S(G)) = 3$. Since the vertices in $S(G)$ are of degree 3 or 6, we have the following cases. Suppose that u, v and w are three black vertices in $S(G)$.

Case 1. Suppose u, v and w are mutually non adjacent. Now $|N(u)| = 3$ or 6, $|N(v)| = 3$ or 6 and $|N(w)| = 3$ or 6.
Since $N(u)$, $N(v)$ and $N(w)$ contain 3 or 6 white vertces, color changing rule is not applicable. This implies that there exist at least two adjacent black vertices in the zero forcing set. Therefore, $deg(u) = deg(v) = deg(w) = 3$ is not possible since vertices with degre three are independent.

Case 2. Suppose that $deg(u) = deg(v) = deg(w) = 6$. The number of white vertices adjacent to u, v or w is atleast 4. Therefore, these vertices will never force any of the the other vertices, a contradiction to our assumption that $Z_2(S(G)) = 3$.

Case 3. Suppose that $deg(u) = deg(v) = 6$ and $deg(w) = 3$. In this case, at least two vertices must be adjacent by case 1. Assume that $u \sim v$ and $deg(w) = 3$. We observed that to force any other vertex, the vertices u, v and w must form a path of length 2. That is they must form 3- consecutive vertices. Since the graph is triangle free, therefore in this case, we can force maximum of two more vertices to black, a contradiction.

Case 4. $deg(u) = deg(v) = deg(w) = 3$. Clearly u, v and w are mutually non adjacent and $N(u), N(v)$ and $N(w)$ contains 3 white vertices. Therefore, color chaning rule is not applicable, this contradicts the fact that $Z_2(S(G)) = 3$.

Case 5. $deg(u) = deg(v) = 3$ and $deg(w) = 6$. Since $deg(u) = deg(v) = 3$. This implies that u is not adjacent to v. We have two sub cases

Subcase 5.1. Assume that u is adjacent to w and v is not adjacent to w. The the vertex u will force two white vertices of degree 6 and these two vertices will have atleast 4 white vertices adjacent to it. Therefore, further forcing is not possible, since the vertices forced by the vertex u are not adjacent to the black vertex v , a contradiction.

Subcase 5. 2. Assume that $w \sim u$ and $w \sim v$. In this case u and v together can force a maximum of 4 more vetices to black. By considering the above cases, it can be concluded that

$$Z_2[S(G)] \geq 4 \quad \textbf{(7)}$$

To prove the reverse inequality let us consider the graph depicted in Figure 1.

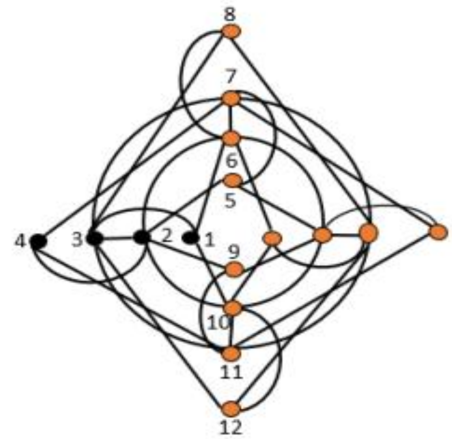

Figure 1: The splitting graph of the prism graph $C_4 \square K_2$.

In figure 1, the orange vertices represents white vertices. Consider the set of black vertices {1,2,3,4}. The vertex 1 forces the vertices 6 and 10 to black. Now consider the vertex 2. Since 6 and 10 are black and they are in the open neighborhood of 2. Therefore the vertex 2 again forces the vertices 5 and 9. Since there are two white neighbors in the open neighborhood of the vertex 4. Therefore, vertex 4 forces the vertices 7 and 11. Now if we consider the vertex 3 it can forces the vertices 8 and 12. We can continue the process with the vertex set {5,6,7,8} or {9,10,11,12} to force the remaining vertices to black. The same argument is true for splitting graph of the prism graph of order $n \geq 5$. Hence

$$Z_2[S(G)] \leq 4 \quad \textbf{(8)}$$

Therefore from **(7)** and **(8)** the result follows.

The star graph $K_{1,n}$ is a tree on n vertices with one vertex having degree n and all other vertices have degree one.

III. GRAPHS FOR WHICH $Z_2(G) > 4$

In this section, we consider some more graph classes for which $Z_2(G) > 4$. We start with the splitting graph of a star graph.

Proposition 18. *Let G be the star graph $K_{1,n}$ on $n + 1$ vertices, $n \geq 3$. Then* $Z_2[S(G)] = 2n - 4$.

Proof. Assume that we have a zero forcing set consisting of $2n - 5$ vertices. Then the number of white vertices in $S(G)$ is 7 , that is $\{2n + 2 - (2n - 5) = 7\}$. We divide the vertex set of $S(G)$ into four sets $A = \{u'\}$, $B = \{u_1, u_2, \dots, u_n\}$, $C = \{u\}$ and $D = \{u'_1, u'_2, \dots, u'_n\}$ as depicted in Figure 2

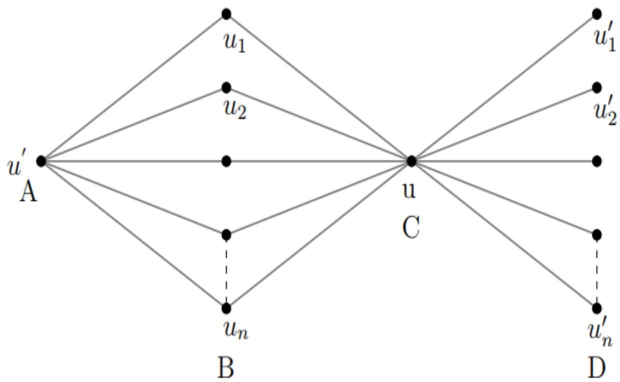

Figure 2: Splitting graph of the graph $K_{1,n}$

Case 1. Assume that the vertices in A and C are white. Consider the remaining 5 white vertices in $S(G)$. Then we have the following subcases.

Subcase 1.1. Assume that either 3 of them will be in B and 2 of them will be in D or vice versa. It can be easily seen that the color changing rule is not applicable in this case because three vertices will remain as white either in B or in D.

Subcase 1.2. Assume that 4 of them will be in B and 1 will be in D or vice versa. In this case, colour changing rule is not possible since four vertices will remain as white either in B or in D.

Subcase 1.3. Assume that there exist no white vertices in the set B and all 5 white vertices are in the set D or vice versa. Here also we cannot apply the colour changing rule.

Case 2. Assume that the vertices in A and C are black. The remaining 7 white vertices can be distributed in the sets B and D are as follows .

Table 1

B	7	6	5	4
D	0	1	2	3

From the above partition, we can observe that in each case, at least 3 white vertices will be either in B or in D. There fore, colour changing rule is not possible.

Case 3. Assume that the vertx $u' \in A$ or the vertex $u \in C$ is black. Consider the following distribution of the 7 white vertices among the sets A, B, C and D.

Table 2

B	6	0	1	5	4	3	2
D	0	6	5	1	2	3	4
A or C	1	1	1	1	1	1	1

From the above partition, it can be seen that at least 3 white vertices will be in B or in D. Therefore, the colour changing rule is not possible in this case. From the above cases, we conclude that

$$Z_2[S(G)] \geq 2n - 4 \quad \textbf{(9)}$$

Let $A = \{u'\}$, $B = \{u_1, u_2, \dots, u_{n-1}, u_n\}$, $C = \{u\}$, $D = \{u'_1, u'_2, \dots, u'_{n-1}, u'_n\}$. Consider the 6 white vertices $\{u, u', u_n, u_{n-1}, u'_n, u'_{n-1}\}$. Consider one black vertex, say u_1 in B. Clearly $u_1 \to u$ and $u_1 \to u'$ to black. Consider the vertex u', clearly the vertex u', 2-forces the vertices u_{n-1} andu_n as black. Again the vertex u 2-forces u'_n and u'_{n-1} to black. Therefore, the set $\{u_1, u_2, \dots, u_{n-2}\} \cup \{u'_1, u'_2, \dots, u'_{n-2}\}$ forms a zero forcing set Z for the graph $S(G)$. There fore, we get a derived colouring of $S(G)$ with allvertices coloured black. This implies $Z_2[S(G)] \leq n - 2 + n - 2 = 2n - 4$. Therefore,

$$Z_2[S(G)] \leq 2n - 4 \quad \textbf{(10)}$$

Hence from (**9**) and (**10**) the result follows.

Remark 19. *If G is the star graph $K_{1,2}$ then* $Z_2[S(G)] = 1$.

Let F_p^k be the graph obtained by taking k copies of the cycle graph $C_p, p \geq 4, k \geq p$ by joining the k copies of the cycle graph C_p with a common vertex v_p of each cycle C_p. For example, the graph F_3^5 is the graph obtained by taking the cycle graph C_3 five times by joning each vetex v_3 of each cycle C_3. The graph F_4^5 is obtained by joinig 5 copies of C_4 with a common vertex v_4 and is dipicted in Figure 3.

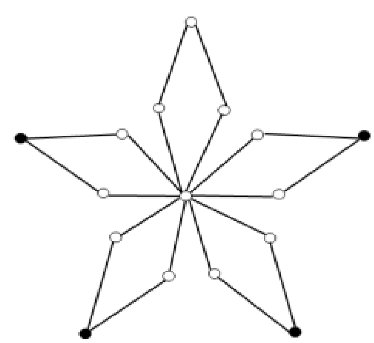

Figure 3: The graph F_3^5.

Proposition 20. *Let G be the graph F_p^k, $p \geq 4$. Then* $Z_2(G) = k - 1$.

Proof. Denote the cycles $C_1, C_2, \dots, C_k$ in F_p^k as follows.

$$C_1 = v_1^1, v_2^1, \dots, v_p, v_1^1$$

$$C_2 = v_1^2, v_2^2, \dots, v_p, v_1^2$$

$$\vdots \quad \vdots \quad \vdots$$

$$C_k = v_1^k, v_2^k, \dots, v_p, v_1^k$$

Consider the set of black vertices $Z = \{v_1^1, v_1^2, \dots, v_1^{k-1}\}$. We can see that $N(v_1^1)$ contains only two white vertices v_p and v_2^1. Therefore, v_1^1 2-forces v_p and v_2^1 to black. v_2^1 2-forces v_3^1 to black and so on . Similarly, we can observe that $N(v_1^2)$ contains the vertices v_p and v_2^2 of which v_p is already black. Now, $v_1^2 \to v_2^2$ and $v_2^2 \to v_3^2$ and so on. In a similar way $v_1^{k-1} \to v_2^{k-1}$ and $v_2^{k-1} \to v_3^{k-1}$ and so on. Now consider the cycle C_k. The vertex set of the cycle C_k is $\{v_1^k, v_2^k, \dots, v_p\}$. Now the black vertex v_p is adacent to two white vertices v_1^k and v_n^k. So the black vertex v_p 2 forces v_1^k and v_n^k to black. Now $v_1^k \to v_2^k$, $v_2^k \to v_3^k$ and so on. Therefore the set Z forms a 2- forcing set for F_p^k. The cardinality of the set Z is $k - 1$ and one can easily observe that with $k - 2$ black vertices it is not possible to form a 2-forcing set. Therefore, $Z_2(G) = k - 1$.

IV. Bounds on $Z_2(G)$

In this section we consider some bounds on $Z_2[G]$.

Proposition 21. *For any connected graph G of order $n \geq 3$, $Z_2[S(G)] \leq 2\, Z_2[G]$.*

Proof. Assume that $Z_2 = \{u_1, u_2, \dots, u_m\}, 1 \leq m \leq n$ be a 2-forcing set of G. Now consider the set

$Z_2' = \{u_1, u_2, \dots, u_m\} \cup \{u'_1, u'_2, \dots, u'_m\} \in V[S(G)]$, where $\{u'_1, u'_2, \dots, u'_m\}$ be the copies of the vertices of $\{u_1, u_2, \dots, u_m\}$ in $V[S(G)]$. Color all the vertices in Z_2' as black.

We prove that the set Z_2' will form a 2-forcing set for $S(G)$. Assume that G is colored with black and white vertices and the vertices in Z_2 are black. Consider the vertices in G which has exactly two white neighbors in G. Let it be $u_1, u_2, \dots, u_l, l \leq m$ and $u'_1, u'_2, \dots, u'_l$ be the vertices corresponds to $u_1, u_2, \dots, u_l$ in $S(G)$. We see that in $S(G)$, $N(u'_1), N(u'_2), \dots, N(u'_l)$, each one contains exactly two white vertices. Let it be $v_1, v_2, \dots, v_l$. Now clearly $u'_1 \to v_1$, $u'_2 \to v_2$, ..., $u'_l \to v_l$. Now again consider the set $\{u_1, u_2, \dots, u_l\}$ in $S(G)$. At this point of time we can see that $u_1 \to v'_1$, $u_2 \to v'_2$,..., $u_l \to v'_l$. Consider the white vertices which are adjacent to $v_1, v_2, \dots, v_l$ in G. Let it be $w_1, w_2, \dots, w_l$. Clearly $v'_1 \to w_1 v_1 \to w'_1$ and so on. Therefore, the set Z_2' forms a 2- forcing set for $S(G)$.

We consider the following result from [7], [2] and [13] to prove a relationship between $Z(G)$, $Z_2(G)$ and $Z(S(G))$.

Proposition 22[2]. *Let G be a connected graph of order $n \geq 3$. Then $Z(S(G)) \leq 2Z(G)$.*

Proposition 23[7]. *Let $G = (V, E)$ be a connected graph. Then $Z(G) \geq Z_2(G)$.*

Next we prove a relationship between $Z(G)$, $Z_2(G)$ and $Z(S(G))$.

Proposition 24. *Let G be a connected graph of order $n \geq 3$. Then $Z_2(G) + Z(S(G)) \leq 3Z(G)$, and the bound is sharp if G is a path on 3 vertices, that is if $G \equiv P_3$.*

Proof. We have from Proposition 22 and Propositon 23 ,

$$Z_2(G) + Z(S(G)) \leq Z(G) + 2Z(G) = 3Z(G).$$

V. CONCLUSION

In this article we deals with the characterization of graphs G for which $1 \leq Z_2(G) \leq 4$. Also we determined the k-forcing number of some classes of splitting graphs. Infact there are many graph classes for which $1 \leq Z_2(G) \leq 4$. Therefore, it is open to characterize $S(G)$ and G for which $Z_2(S(G)) = 1$, $Z_2(G) = 1$, $Z_2(S(G)) = 2$, $Z_2(G) = 2$, $Z_2(S(G)) = 3$, $Z_2(G) = 3$, $Z_2(S(G)) = 4$ and $Z_2(G) = 4$. It is also open to characterize $Z_2(G) + Z(S(G)) = 3Z(G)$.

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