

Optimal Decision Model under Mixture Inventory System

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Available online at: www.isroset.org

Accepted 25/Aug/2018, Online 30/Aug/2018

Abstract- In this paper, the behavior of an inventory model in a production system with Weibull demand and time dependent deterioration is depicted. Holding cost is changed per unit time. Mathematical model of both crisp and fuzzy models have been developed to determine the optimal cycle time and optimal inventory cost. Fuzzy set theory is primarily concerned with imprecision and uncertainty. It provides the decision maker as a mixture inventory system which is used in modeling real-world problems as compared to the classical deterministic and probabilistic mathematical tools. The demand, deterioration rate, holding cost, unit cost and shortage cost are taken mixture and subsequently as pentagonal fuzzy numbers. Both graded mean integration and signed distance method are used to defuzzify the total cost function. Numerical illustrations are provided to illustrate the applications of the model with useful graphs and tables to validate the developed model. Sensitivity analysis is carried out to analyze the variability in the optimal solution with respect to change in various parameters.

Keywords- EOQ Model, Weibull Demand, Pentagonal Fuzzy Number, Graded Mean Integration, Signed Distance.

AMS Classification No: 90B05

I. INTRODUCTION

Deterioration is a trending research area in inventory model. It is a challenging issue to develop inventory model for deteriorating items such as food stuffs, vegetables, food grains, medicines, fashion cosmetics and electronics items etc. Hariga [3] developed a model under general continuous time dependent demand and three replenishment policies. Jaggi, Aggarwal and Goel [5] studied replenishment model of deteriorating items with function of inflation induced demand. The model with three-parameter Weibull as well as parabolic deterioration and various demands and holding cost are proposed by Sahoo and Tripathy [15] & [16].

One of the weaknesses of recent model which is vastly used in business world is the unrealistic assumption of the different parameters. Fuzzy logic can deal with information arising from cognition and computational perception, that is, imprecise, vague, uncertain, partially true, or without sharp boundaries. Fuzzy inventory models are more realistic than the conventional inventory models. The uncertainties are due to fuzziness, and such cases are expanded in the fuzzy set theory which was demonstrated by Zadeh [20] and Park [12]. Chang *et al.* [2] and Hsieh [4] developed model with fuzzy back order quantity and

fuzzy production inventory. Kao and Hsu [7] proposed lot size reorder point inventory with fuzzy demands. Yao *et al.* [18] and Jaggi *et al.* [6] developed deteriorating model with some parameters as triangular fuzzy number. The analysis of deteriorating inventory model began with Yao and Lee [19] who developed fuzzy inventory with or without backorder with trapezoidal fuzzy number. Since then, many related research with various parameters was found in Behera *et al.* [1] Mohanty and Tripathy [9], Sahoo *et al.* [14] and Tripathy *et al.* [17]. Panda *et al.* [11] studied on Pentagonal fuzzy number and its corresponding matrices. Nagar and Surana [10] proposed fuzzy deteriorating inventory model with fluctuating demand and using inventory parameters as pentagonal fuzzy numbers, while, Mondal and Mandal [8] discussed properties and applications of pentagonal fuzzy number.

This model is discussed for items which deteriorate from the beginning of time cycle where demand and deterioration are considered as Weibull and time proportional respectively. Holding cost is also changed per unit of time. Further the demand, deterioration rate and all costs are taken as mixture and subsequently pentagonal fuzzy numbers. Both graded mean integration and signed distance method are used to defuzzify the total cost function. Mathematical model has been developed for determining the optimal cycle time and optimal total inventory fuzzy cost. Numerical illustrations

are given to validate the developed model. The useful graphs and tables with sensitivity analysis are carried out to analyze the variability in the optimal solution with respect to change in various system parameters.

II. PRELIMINARIES

Definition 1: (By Pu and Liu¹³) A fuzzy set \tilde{a} defined on real number set R is called a fuzzy point if its membership function is

$$\mu_{\tilde{a}}(x) = \begin{cases} 0, & \text{if } x < a \\ 1, & \text{if } x = a \\ 0, & \text{if } x > a \end{cases}$$

where the point a is called the support of fuzzy set \tilde{a} .

Definition 2: A fuzzy set $[a_{1\alpha}, a_{2\alpha}]$ where $0 \leq \alpha \leq 1$ and $a_1 < a_2$ defined on R is called a level of a fuzzy interval if its membership function is

$$\mu_{[a_{1\alpha}, a_{2\alpha}]}(x) = \begin{cases} \alpha, & a_1 \leq x \leq a_2 \\ 0, & \text{otherwise} \end{cases}$$

Definition 3: A triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$, $a_1 < a_2 < a_3$ defined on R with membership function $\mu_{\tilde{A}}$ as:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & , \text{if } x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & \text{if } x \in [a_1, a_2] \\ \frac{a_3-x}{a_3-a_2}, & \text{if } x \in [a_2, a_3] \\ 0 & , \text{if } x > a_3 \end{cases}$$

The α -cut of $\tilde{A} = (a_1, a_2, a_3)$, $0 \leq \alpha \leq 1$ is

$$A(\alpha) = [A_L(\alpha), A_R(\alpha)]$$

where $A_L(\alpha) = a_1 + (a_2 - a_1)\alpha$

and $A_R(\alpha) = a_3 - (a_3 - a_2)\alpha$ are the left and right end points of $A(\alpha)$.

Definition 4: A trapezoidal fuzzy number is depicted by $\tilde{A} = (a_1, a_2, a_3, a_4)$, $a_1 < a_2 < a_3 < a_4$ with membership function $\mu_{\tilde{A}}$ is defined on R as:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & , \text{if } x < a_1 \\ L(x) = \frac{x-a_1}{a_2-a_1}, & \text{if } x \in [a_1, a_2] \\ 1 & , \text{if } x \in [a_2, a_3] \\ R(x) = \frac{a_4-x}{a_4-a_3}, & \text{if } x \in [a_3, a_4] \\ 0 & , \text{if } x > a_4 \end{cases}$$

The α -cut of $\tilde{A} = (a_1, a_2, a_3, a_4)$ $0 < \alpha < 1$ is

$$A(\alpha) = [A_L(\alpha), A_R(\alpha)],$$

where $A_L(\alpha) = a_1 + (a_2 - a_1)\alpha$

and $A_R(\alpha) = a_4 - (a_4 - a_3)\alpha$ are the left and right end points of $A(\alpha)$.

Definition 5: A pentagonal fuzzy number is depicted by $\tilde{A} = (a_1, a_2, a_3, a_4, a_5)$ $a_1 < a_2 < a_3 < a_4 < a_5$ is obtained on R with membership function $\mu_{\tilde{A}}$ as:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & , \text{if } x < a_1 \\ L_1(x) = \frac{x-a_1}{a_2-a_1}, & \text{if } x \in [a_1, a_2] \\ L_2(x) = \frac{x-a_2}{a_3-a_2}, & \text{if } x \in [a_2, a_3] \\ 1 & , \text{if } x = a_3 \\ R_1(x) = \frac{a_4-x}{a_4-a_3}, & \text{if } x \in [a_3, a_4] \\ R_2(x) = \frac{a_5-x}{a_5-a_4}, & \text{if } x \in [a_4, a_5] \\ 0 & , \text{if } x > a_5 \end{cases}$$

The α -cut of $\tilde{A} = (a_1, a_2, a_3, a_4, a_5)$, for $0 < \alpha < 1$ is

$A(\alpha) = [A_L(\alpha), A_R(\alpha)]$, where $A_L(\alpha) = \frac{A_{L_1}(\alpha) + A_{L_2}(\alpha)}{2}$

and $A_R(\alpha) = \frac{A_{R_1}(\alpha) + A_{R_2}(\alpha)}{2}$ are the left and right end points of $A(\alpha)$ such that

$$A_{L_1}(\alpha) = a_1 + (a_2 - a_1)\alpha ;$$

$$A_{L_2}(\alpha) = a_2 + (a_3 - a_2)\alpha$$

$$\text{and } A_{R_1}(\alpha) = a_4 - (a_4 - a_3)\alpha ;$$

$$A_{R_2}(\alpha) = a_5 - (a_5 - a_4)\alpha .$$

Figure 1 portrays a pentagonal inventory during production cycle

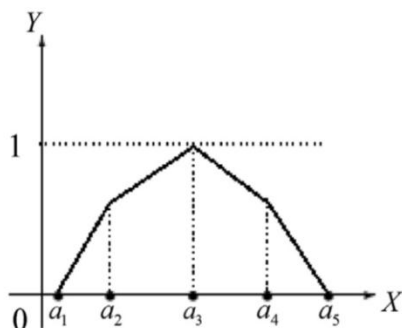


Figure 1. Pentagonal fuzzy number

Definition 6: If $\tilde{A} = (a_1, a_2, a_3, a_4, a_5)$ be a pentagonal fuzzy number, then the graded mean integration method of \tilde{A} is defined as:

$$G_M(\tilde{A}) = \frac{\int_0^{w_A} \alpha [A_L(\alpha) + A_R(\alpha)] d\alpha}{\int_0^1 \alpha d\alpha}$$

with $0 < \alpha \leq w_A$ and $0 < w_A \leq 1$

$$= \frac{\frac{1}{2} \int_0^1 \alpha \left[\frac{a_1 + a_2 + (a_3 - a_1)\alpha}{2} + \frac{a_4 + a_5 - (a_5 - a_4)\alpha}{2} \right] d\alpha}{\int_0^1 \alpha d\alpha}$$

$$= \frac{1}{12} (a_1 + 3a_2 + 4a_3 + 3a_4 + a_5)$$

Definition 7: The signed distance method of \tilde{A} with pentagonal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4, a_5)$ is evaluated as:

$$S_D(\tilde{A}) = \frac{1}{2} \int_0^{w_A} [A_L(\alpha) + A_R(\alpha)] d\alpha$$

with $0 < \alpha \leq w_A$ and $0 < w_A \leq 1$

$$= \frac{1}{2} \int_0^1 \left[\frac{a_1 + a_2 + (a_3 - a_1)\alpha}{2} + \frac{a_4 + a_5 - (a_5 - a_4)\alpha}{2} \right] d\alpha$$

$$= \frac{1}{8} (a_1 + 2a_2 + 2a_3 + 2a_4 + a_5)$$

III. NOTATIONS AND ASSUMPTIONS

Notations

- (i) $d(t)$ is the demand rate at any time.
- (ii) θ is the deterioration rate per unit time.
- (iii) A, c, h and S are the ordering cost, unit cost, holding cost and shortage cost respectively per unit time.

- (iv) t_2 is the length of the order cycle.
- (v) Q is the ordering quantity per unit.
- (vi) $C(t_1, t_2)$ is the total inventory cost per unit time.
- (vii) \tilde{d} is the fuzzy demand per unit time.
- (viii) $\tilde{\theta}$ is the fuzzy deterioration rate per unit time.
- (ix) \tilde{c}, \tilde{h} and \tilde{S} are the fuzzy unit cost, fuzzy holding cost and fuzzy shortage cost respectively per unit time.
- (x) $\tilde{C}(t_1, t_2)$ is the total fuzzy inventory cost per unit time.
- (xi) $C_{GM}(t_1, t_2)$ is the defuzzified value of $\tilde{C}(t_1, t_2)$ by applying graded mean integration method.
- (xii) $C_{SD}(t_1, t_2)$ is the defuzzified value of $\tilde{C}(t_1, t_2)$ by applying signed distance method.

Assumptions

- (xiii) Demand $d(t) = \alpha\beta t^{\beta-1}$ is Weibull distribution function where $\alpha > 0$ is a scale parameter and $\beta > 0$ is a shape parameter.
- (xiv) $\theta(t) = \theta t, 0 < \theta < 1$ is the deterioration rate per unit time.
- (xv) $h(t) = ht$ is the time proportional holding cost per unit time.
- (xvi) The replenishment rate is instantaneous and lead-time is zero.
- (xvii) Shortages are allowed and fully backlogged.

IV. MATHEMATICAL MODEL

Let $I(t)$ be the inventory level at any time t with initial inventory Q . The inventory level gradually diminishes during the period $[0, t_1]$ due to demand and deterioration and exhausted at time t_1 . Finally, a shortage is occurred during the interval of time period $[t_1, t_2]$, which are fully backlogged. Since the inventory is depleted by the combined effect of demand and deterioration the inventory level $I(t)$ at time t is governed by the following differential equations.

Crisp Model

$$\frac{dI(t)}{dt} + \theta I(t) = -d(t), 0 \leq t \leq t_1 \tag{1}$$

with $I(0) = Q$ and $I(t_1) = 0$.

And $\frac{dI(t)}{dt} = -d(t), t_1 \leq t \leq t_2$ (2)

with $I(t_1) = 0$

As $d(t) = \alpha\beta t^{\beta-1}$ and $\theta(t) = \theta t$ then the equations (1) and (2) becomes

$\frac{dI(t)}{dt} + \theta t I(t) = -\alpha\beta t^{\beta-1}, 0 \leq t \leq t_1$ (3)

with $I(0) = Q$ and $I(t_1) = 0$.

And, $\frac{dI(t)}{dt} = -\alpha\beta t^{\beta-1}, t_1 \leq t \leq t_2$ (4)

with $I(t_1) = 0$.

Solution of equation (3) using condition $I(0) = Q$ is calculated as

$I(t) = Qe^{-\frac{\theta t^2}{2}} - (\alpha t^\beta + \frac{\alpha\beta\theta}{2(\beta+2)} t^{\beta+2})e^{-\frac{\theta t^2}{2}}, 0 \leq t \leq t_1$ (5)

As $I(t_1) = 0 \Rightarrow Q = \alpha t_1^\beta + \frac{\alpha\beta\theta}{2(\beta+2)} t_1^{\beta+2}$ (6)

Thus

$I(t) = [(\alpha t_1^\beta + \frac{\alpha\beta\theta}{2(\beta+2)} t_1^{\beta+2}) - (\alpha t^\beta + \frac{\alpha\beta\theta}{2(\beta+2)} t^{\beta+2})]e^{-\frac{\theta t^2}{2}}$
 $= [\alpha(t_1^\beta - t^\beta) + \frac{\alpha\beta\theta}{2(\beta+2)}(t_1^{\beta+2} - t^{\beta+2})](1 - \frac{\theta t^2}{2})$
 neglecting higher power of θ
 $= \left[\alpha(t_1^\beta - t^\beta) - \frac{\alpha\theta}{2}(t_1^\beta t^2 - t^{\beta+2}) + \frac{\alpha\beta\theta}{2(\beta+2)}(t_1^{\beta+2} - t^{\beta+2}) \right], 0 \leq t \leq t_1$ (7)

Solution of equation (4) will be

$I(t) = \alpha(t_1^\beta - t^\beta), t_1 \leq t \leq t_2$ (8)

Inventory is available in the system during the time period $(0, t_1)$. The cost for holding in stock inventory is computed for time period $(0, t_1)$ only.

Total no. of holding cost unit I_{HC} during period $[0, t_2]$ is given by

$I_{HC} = \int_0^{t_1} h(t)I(t)dt$
 $= h \int_0^{t_1} t \left[\alpha(t_1^\beta - t^\beta) - \frac{\alpha\theta}{2}(t_1^\beta t^2 - t^{\beta+2}) + \frac{\alpha\beta\theta}{2(\beta+2)}(t_1^{\beta+2} - t^{\beta+2}) \right] dt$
 $= h \left[\frac{\alpha\beta}{2(\beta+2)} t_1^{\beta+2} + \frac{\alpha\beta\theta}{8(\beta+4)} t_1^{\beta+4} \right]$ (9)

Total no. of deteriorated units I_D during period $[0, t_2]$ is determined as

$I_D = c[Q - \text{Total Demand}]$
 $= c \left[Q - \int_0^{t_1} d(t)dt \right]$
 $= \frac{\alpha\beta\theta}{2(\beta+2)} t_1^{\beta+2}$ (10)

Total no. of units of shortage I_S during period $[0, t_2]$ is illustrated as

$I_S = S \int_{t_1}^{t_2} -I(t)dt$
 $= -S(\alpha t_1^\beta t_2 - \frac{\alpha}{\beta+1} t_2^{\beta+1} - \frac{\alpha\beta}{\beta+1} t_1^{\beta+1})$ (11)

Total cost of the system per unit time is calculated as

$C(t_1, t_2) = \frac{1}{t_2} [A + I_{HC} + I_D + I_S]$
 $\Rightarrow C(t_1, t_2) = \frac{1}{t_2} \left[A + \frac{\alpha\beta h}{2(\beta+2)} t_1^{\beta+2} + \frac{\alpha\beta\theta h}{8(\beta+4)} t_1^{\beta+4} + \frac{\alpha\beta\theta c}{2(\beta+2)} t_1^{\beta+2} - S \left(\alpha t_1^\beta t_2 - \frac{\alpha}{\beta+1} t_2^{\beta+1} - \frac{\alpha\beta}{\beta+1} t_1^{\beta+1} \right) \right]$ (12)

Fuzzy Model

It is not easy to define all the system parameters precisely due to uncertainty in the environment. Accordingly it is assumed that some of these parameters namely $\tilde{\alpha}, \tilde{\beta}, \tilde{c}, \tilde{S}, \tilde{\theta}, \tilde{h}$ may change within some limits. Let $\tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, $\tilde{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$, $\tilde{h} = (h_1, h_2, h_3, h_4, h_5)$, $\tilde{c} = (c_1, c_2, c_3, c_4, c_5)$, $\tilde{S} = (S_1, S_2, S_3, S_4, S_5)$, $\tilde{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, are pentagonal fuzzy numbers.

Total cost of the system per unit time in fuzzy sense is governed by

$\tilde{C}(t_1, t_2) = \frac{1}{t_2} \left[A + \frac{\tilde{\alpha}\tilde{\beta}\tilde{h}}{2(\tilde{\beta}+2)} t_1^{\tilde{\beta}+2} + \frac{\tilde{\alpha}\tilde{\beta}\tilde{\theta}\tilde{h}}{8(\tilde{\beta}+4)} t_1^{\tilde{\beta}+4} + \frac{\tilde{\alpha}\tilde{\beta}\tilde{\theta}\tilde{c}}{2(\tilde{\beta}+2)} t_1^{\tilde{\beta}+2} - \tilde{S} \left(\tilde{\alpha} t_1^{\tilde{\beta}} t_2 - \frac{\tilde{\alpha}}{\tilde{\beta}+1} t_2^{\tilde{\beta}+1} - \frac{\tilde{\alpha}\tilde{\beta}}{\tilde{\beta}+1} t_1^{\tilde{\beta}+1} \right) \right]$ (13)

The fuzzy total cost $\tilde{C}(t_1, t_2)$ is defuzzified by graded mean integration as well as signed distance method

(i) **By graded mean integration method**, the total cost is given as

$$C_{GM}(t_1, t_2) = \frac{1}{12} (C_{GM_1}(t_1, t_2) + 3C_{GM_2}(t_1, t_2) + 4K_{GM_3}(t_1, t_2) + 3C_{GM_4}(t_1, t_2) + K_{GM_5}(t_1, t_2))$$

$$= \frac{1}{12t_2} \left[\begin{aligned} & 12A + \left\{ \frac{\alpha_1\beta_1h_1}{2(\beta_1+2)} t_1^{\beta_1+2} + \frac{\alpha_1\beta_1\theta_1h_1}{8(\beta_1+4)} t_1^{\beta_1+4} + \frac{\alpha_1\beta_1\theta_1c_1}{2(\beta_1+2)} t_1^{\beta_1+2} - S_1(\alpha_1 t_1^{\beta_1} t_2 - \frac{\alpha_1}{\beta_1+1} t_2^{\beta_1+1} - \frac{\alpha_1\beta_1}{\beta_1+1} t_1^{\beta_1+1}) \right\} \\ & + 3 \left\{ \frac{\alpha_2\beta_2h_2}{2(\beta_2+2)} t_1^{\beta_2+2} + \frac{\alpha_2\beta_2\theta_2h_2}{8(\beta_2+4)} t_1^{\beta_2+4} + \frac{\alpha_2\beta_2\theta_2c_2}{2(\beta_2+2)} t_1^{\beta_2+2} - S_2(\alpha_2 t_1^{\beta_2} t_2 - \frac{\alpha_2}{\beta_2+1} t_2^{\beta_2+1} - \frac{\alpha_2\beta_2}{\beta_2+1} t_1^{\beta_2+1}) \right\} \\ & + 4 \left\{ \frac{\alpha_3\beta_3h_3}{2(\beta_3+2)} t_1^{\beta_3+2} + \frac{\alpha_3\beta_3\theta_3h_3}{8(\beta_3+4)} t_1^{\beta_3+4} + \frac{\alpha_3\beta_3\theta_3c_3}{2(\beta_3+2)} t_1^{\beta_3+2} - S_3(\alpha_3 t_1^{\beta_3} t_2 - \frac{\alpha_3}{\beta_3+1} t_2^{\beta_3+1} - \frac{\alpha_3\beta_3}{\beta_3+1} t_1^{\beta_3+1}) \right\} \\ & + 3 \left\{ \frac{\alpha_4\beta_4h_4}{2(\beta_4+2)} t_1^{\beta_4+2} + \frac{\alpha_4\beta_4\theta_4h_4}{8(\beta_4+4)} t_1^{\beta_4+4} + \frac{\alpha_4\beta_4\theta_4c_4}{2(\beta_4+2)} t_1^{\beta_4+2} - S_4(\alpha_4 t_1^{\beta_4} t_2 - \frac{\alpha_4}{\beta_4+1} t_2^{\beta_4+1} - \frac{\alpha_4\beta_4}{\beta_4+1} t_1^{\beta_4+1}) \right\} \\ & + \left\{ \frac{\alpha_5\beta_5h_5}{2(\beta_5+2)} t_1^{\beta_5+2} + \frac{\alpha_5\beta_5\theta_5h_5}{8(\beta_5+4)} t_1^{\beta_5+4} + \frac{\alpha_5\beta_5\theta_5c_5}{2(\beta_5+2)} t_1^{\beta_5+2} - S_5(\alpha_5 t_1^{\beta_5} t_2 - \frac{\alpha_5}{\beta_5+1} t_2^{\beta_5+1} - \frac{\alpha_5\beta_5}{\beta_5+1} t_1^{\beta_5+1}) \right\} \end{aligned} \right] \tag{14}$$

The optimal values of t_1 and t_2 can be obtained so as to minimize the total fuzzy cost $C_{GM}(t_1, t_2)$ by solving the following equations

$$\frac{\partial C_{GM}(t_1, t_2)}{\partial t_1} = 0 \text{ and } \frac{\partial C_{GM}(t_1, t_2)}{\partial t_2} = 0 \tag{15}$$

Equation (15) becomes

$$\frac{1}{12t_2} \left[\begin{aligned} & \left\{ \frac{\alpha_1\beta_1h_1}{2} t_1^{\beta_1+1} + \frac{\alpha_1\beta_1\theta_1h_1}{8} t_1^{\beta_1+3} + \frac{\alpha_1\beta_1\theta_1c_1}{2} t_1^{\beta_1+1} - S_1\alpha_1\beta_1(t_1^{\beta_1-1}t_2 - t_1^{\beta_1}) \right. \\ & + 3 \left\{ \frac{\alpha_2\beta_2h_2}{2} t_1^{\beta_2+1} + \frac{\alpha_2\beta_2\theta_2h_2}{8} t_1^{\beta_2+3} + \frac{\alpha_2\beta_2\theta_2c_2}{2} t_1^{\beta_2+1} - S_2\alpha_2\beta_2(t_1^{\beta_2-1}t_2 - t_1^{\beta_2}) \right\} \\ & + 4 \left\{ \frac{\alpha_3\beta_3h_3}{2} t_1^{\beta_3+1} + \frac{\alpha_3\beta_3\theta_3h_3}{8} t_1^{\beta_3+3} + \frac{\alpha_3\beta_3\theta_3c_3}{2} t_1^{\beta_3+1} - S_3\alpha_3\beta_3(t_1^{\beta_3-1}t_2 - t_1^{\beta_3}) \right\} \\ & + 3 \left\{ \frac{\alpha_4\beta_4h_4}{2} t_1^{\beta_4+1} + \frac{\alpha_4\beta_4\theta_4h_4}{8} t_1^{\beta_4+3} + \frac{\alpha_4\beta_4\theta_4c_4}{2} t_1^{\beta_4+1} - S_4\alpha_4\beta_4(t_1^{\beta_4-1}t_2 - t_1^{\beta_4}) \right\} \\ & \left. + \frac{\alpha_5\beta_5h_5}{2} t_1^{\beta_5+1} + \frac{\alpha_5\beta_5\theta_5h_5}{8} t_1^{\beta_5+3} + \frac{\alpha_5\beta_5\theta_5c_5}{2} t_1^{\beta_5+1} - S_5\alpha_5\beta_5(t_1^{\beta_5-1}t_2 - t_1^{\beta_5}) \right\} = 0 \tag{16} \end{aligned} \right]$$

&

$$\frac{1}{12t_2} \left[\begin{aligned} & S_1\alpha_1(t_1^{\beta_1} - t_2^{\beta_1}) + 3S_2\alpha_2(t_1^{\beta_2} - t_2^{\beta_2}) + 4S_3\alpha_3(t_1^{\beta_3} - t_2^{\beta_3}) \\ & + 3S_4\alpha_4(t_1^{\beta_4} - t_2^{\beta_4}) + S_5\alpha_5(t_1^{\beta_5} - t_2^{\beta_5}) \end{aligned} \right]$$

$$+ \frac{1}{12t_2^2} \left[\begin{aligned} & 12A + \left\{ \frac{\alpha_1\beta_1h_1}{2(\beta_1+2)} t_1^{\beta_1+2} + \frac{\alpha_1\beta_1\theta_1h_1}{8(\beta_1+4)} t_1^{\beta_1+4} + \frac{\alpha_1\beta_1\theta_1c_1}{2(\beta_1+2)} t_1^{\beta_1+2} - S_1(\alpha_1 t_1^{\beta_1} t_2 - \frac{\alpha_1}{\beta_1+1} t_2^{\beta_1+1} - \frac{\alpha_1\beta_1}{\beta_1+1} t_1^{\beta_1+1}) \right\} \\ & + 3 \left\{ \frac{\alpha_2\beta_2h_2}{2(\beta_2+2)} t_1^{\beta_2+2} + \frac{\alpha_2\beta_2\theta_2h_2}{8(\beta_2+4)} t_1^{\beta_2+4} + \frac{\alpha_2\beta_2\theta_2c_2}{2(\beta_2+2)} t_1^{\beta_2+2} - S_2(\alpha_2 t_1^{\beta_2} t_2 - \frac{\alpha_2}{\beta_2+1} t_2^{\beta_2+1} - \frac{\alpha_2\beta_2}{\beta_2+1} t_1^{\beta_2+1}) \right\} \\ & + 4 \left\{ \frac{\alpha_3\beta_3h_3}{2(\beta_3+2)} t_1^{\beta_3+2} + \frac{\alpha_3\beta_3\theta_3h_3}{8(\beta_3+4)} t_1^{\beta_3+4} + \frac{\alpha_3\beta_3\theta_3c_3}{2(\beta_3+2)} t_1^{\beta_3+2} - S_3(\alpha_3 t_1^{\beta_3} t_2 - \frac{\alpha_3}{\beta_3+1} t_2^{\beta_3+1} - \frac{\alpha_3\beta_3}{\beta_3+1} t_1^{\beta_3+1}) \right\} \\ & + 3 \left\{ \frac{\alpha_4\beta_4h_4}{2(\beta_4+2)} t_1^{\beta_4+2} + \frac{\alpha_4\beta_4\theta_4h_4}{8(\beta_4+4)} t_1^{\beta_4+4} + \frac{\alpha_4\beta_4\theta_4c_4}{2(\beta_4+2)} t_1^{\beta_4+2} - S_4(\alpha_4 t_1^{\beta_4} t_2 - \frac{\alpha_4}{\beta_4+1} t_2^{\beta_4+1} - \frac{\alpha_4\beta_4}{\beta_4+1} t_1^{\beta_4+1}) \right\} \\ & + \left\{ \frac{\alpha_5\beta_5h_5}{2(\beta_5+2)} t_1^{\beta_5+2} + \frac{\alpha_5\beta_5\theta_5h_5}{8(\beta_5+4)} t_1^{\beta_5+4} + \frac{\alpha_5\beta_5\theta_5c_5}{2(\beta_5+2)} t_1^{\beta_5+2} - S_5(\alpha_5 t_1^{\beta_5} t_2 - \frac{\alpha_5}{\beta_5+1} t_2^{\beta_5+1} - \frac{\alpha_5\beta_5}{\beta_5+1} t_1^{\beta_5+1}) \right\} \end{aligned} \right] = 0 \tag{17}$$

Further, for the convexity of total fuzzy cost function $C_{GM}(t_1, t_2)$, the following conditions must be satisfied

$$\frac{\partial^2 C_{GM}(t_1, t_2)}{\partial t_1^2} > 0, \frac{\partial^2 C_{GM}(t_1, t_2)}{\partial t_2^2} > 0 \tag{18}$$

$$\& \left(\frac{\partial^2 C_{GM}(t_1, t_2)}{\partial t_1^2} \right) \left(\frac{\partial^2 C_{GM}(t_1, t_2)}{\partial t_2^2} \right) - \left(\frac{\partial^2 C_{GM}(t_1, t_2)}{\partial t_1 \partial t_2} \right) > 0 \tag{19}$$

It is difficult to prove the convexity mathematically, since it is complicated to determine the second derivatives of the total fuzzy cost function $C_{GM}(t_1, t_2)$. It has constrained to show the convexity of total fuzzy cost in graph (Figure 2).

(ii) **By signed distance method**, the total fuzzy cost is illustrated as

$$C_{SD}(t_1, t_2) = \frac{1}{8} (C_{GM_1}(t_1, t_2) + 2C_{GM_2}(t_1, t_2) + 2K_{GM_3}(t_1, t_2) + 2C_{GM_4}(t_1, t_2) + K_{GM_5}(t_1, t_2))$$

$$= \frac{1}{8t_2} \left[\begin{aligned} & 8A + \left\{ \frac{\alpha_1 \beta_1 h_1}{2(\beta_1+2)} t_1^{\beta_1+2} + \frac{\alpha_1 \beta_1 \theta_1 h_1}{8(\beta_1+4)} t_1^{\beta_1+4} + \frac{\alpha_1 \beta_1 \theta_1 c_1}{2(\beta_1+2)} t_1^{\beta_1+2} - S_1 (\alpha_1 t_1^{\beta_1} t_2 - \frac{\alpha_1}{\beta_1+1} t_2^{\beta_1+1} - \frac{\alpha_1 \beta_1}{\beta_1+1} t_1^{\beta_1+1}) \right\} \\ & + 2 \left\{ \frac{\alpha_2 \beta_2 h_2}{2(\beta_2+2)} t_1^{\beta_2+2} + \frac{\alpha_2 \beta_2 \theta_2 h_2}{8(\beta_2+4)} t_1^{\beta_2+4} + \frac{\alpha_2 \beta_2 \theta_2 c_2}{2(\beta_2+2)} t_1^{\beta_2+2} - S_2 (\alpha_2 t_1^{\beta_2} t_2 - \frac{\alpha_2}{\beta_2+1} t_2^{\beta_2+1} - \frac{\alpha_2 \beta_2}{\beta_2+1} t_1^{\beta_2+1}) \right\} \\ & + 2 \left\{ \frac{\alpha_3 \beta_3 h_3}{2(\beta_3+2)} t_1^{\beta_3+2} + \frac{\alpha_3 \beta_3 \theta_3 h_3}{8(\beta_3+4)} t_1^{\beta_3+4} + \frac{\alpha_3 \beta_3 \theta_3 c_3}{2(\beta_3+2)} t_1^{\beta_3+2} - S_3 (\alpha_3 t_1^{\beta_3} t_2 - \frac{\alpha_3}{\beta_3+1} t_2^{\beta_3+1} - \frac{\alpha_3 \beta_3}{\beta_3+1} t_1^{\beta_3+1}) \right\} \\ & + 2 \left\{ \frac{\alpha_4 \beta_4 h_4}{2(\beta_4+2)} t_1^{\beta_4+2} + \frac{\alpha_4 \beta_4 \theta_4 h_4}{8(\beta_4+4)} t_1^{\beta_4+4} + \frac{\alpha_4 \beta_4 \theta_4 c_4}{2(\beta_4+2)} t_1^{\beta_4+2} - S_4 (\alpha_4 t_1^{\beta_4} t_2 - \frac{\alpha_4}{\beta_4+1} t_2^{\beta_4+1} - \frac{\alpha_4 \beta_4}{\beta_4+1} t_1^{\beta_4+1}) \right\} \\ & + \left\{ \frac{\alpha_5 \beta_5 h_5}{2(\beta_5+2)} t_1^{\beta_5+2} + \frac{\alpha_5 \beta_5 \theta_5 h_5}{8(\beta_5+4)} t_1^{\beta_5+4} + \frac{\alpha_5 \beta_5 \theta_5 c_5}{2(\beta_5+2)} t_1^{\beta_5+2} - S_5 (\alpha_5 t_1^{\beta_5} t_2 - \frac{\alpha_5}{\beta_5+1} t_2^{\beta_5+1} - \frac{\alpha_5 \beta_5}{\beta_5+1} t_1^{\beta_5+1}) \right\} \end{aligned} \right] \tag{20}$$

The optimal values of t_1 and t_2 can be obtained by solving the following equations to minimize the total fuzzy cost $C_{SD}(t_1, t_2)$.

$$\frac{\partial C_{SD}(t_1, t_2)}{\partial t_1} = 0 \text{ and } \frac{\partial C_{SD}(t_1, t_2)}{\partial t_2} = 0 \tag{21}$$

Equation (6.21) becomes

$$\frac{1}{8t_2} \left[\begin{aligned} & \frac{\alpha_1 \beta_1 h_1}{2} t_1^{\beta_1+1} + \frac{\alpha_1 \beta_1 \theta_1 h_1}{8} t_1^{\beta_1+3} + \frac{\alpha_1 \beta_1 \theta_1 c_1}{2} t_1^{\beta_1+1} - S_1 \alpha_1 \beta_1 (t_1^{\beta_1-1} t_2 - t_1^{\beta_1}) \\ & + 2 \left\{ \frac{\alpha_2 \beta_2 h_2}{2} t_1^{\beta_2+1} + \frac{\alpha_2 \beta_2 \theta_2 h_2}{8} t_1^{\beta_2+3} + \frac{\alpha_2 \beta_2 \theta_2 c_2}{2} t_1^{\beta_2+1} - S_2 \alpha_2 \beta_2 (t_1^{\beta_2-1} t_2 - t_1^{\beta_2}) \right\} \\ & + 2 \left\{ \frac{\alpha_3 \beta_3 h_3}{2} t_1^{\beta_3+1} + \frac{\alpha_3 \beta_3 \theta_3 h_3}{8} t_1^{\beta_3+3} + \frac{\alpha_3 \beta_3 \theta_3 c_3}{2} t_1^{\beta_3+1} - S_3 \alpha_3 \beta_3 (t_1^{\beta_3-1} t_2 - t_1^{\beta_3}) \right\} \\ & + 2 \left\{ \frac{\alpha_4 \beta_4 h_4}{2} t_1^{\beta_4+1} + \frac{\alpha_4 \beta_4 \theta_4 h_4}{8} t_1^{\beta_4+3} + \frac{\alpha_4 \beta_4 \theta_4 c_4}{2} t_1^{\beta_4+1} - S_4 \alpha_4 \beta_4 (t_1^{\beta_4-1} t_2 - t_1^{\beta_4}) \right\} \\ & + \frac{\alpha_5 \beta_5 h_5}{2} t_1^{\beta_5+1} + \frac{\alpha_5 \beta_5 \theta_5 h_5}{8} t_1^{\beta_5+3} + \frac{\alpha_5 \beta_5 \theta_5 c_5}{2} t_1^{\beta_5+1} - S_5 \alpha_5 \beta_5 (t_1^{\beta_5-1} t_2 - t_1^{\beta_5}) \end{aligned} \right] = 0 \tag{22}$$

$$\& \left[\begin{aligned} & S_1 \alpha_1 (t_1^{\beta_1} - t_2^{\beta_1}) + 2S_2 \alpha_2 (t_1^{\beta_2} - t_2^{\beta_2}) + 2S_3 \alpha_3 (t_1^{\beta_3} - t_2^{\beta_3}) \\ & + 2S_4 \alpha_4 (t_1^{\beta_4} - t_2^{\beta_4}) + S_5 \alpha_5 (t_1^{\beta_5} - t_2^{\beta_5}) \end{aligned} \right]$$

$$\left. + \frac{1}{8t_2^2} \left[\begin{aligned} & 8A + \left\{ \frac{\alpha_1\beta_1h_1}{2(\beta_1+2)} t_1^{\beta_1+2} + \frac{\alpha_1\beta_1\theta_1h_1}{8(\beta_1+4)} t_1^{\beta_1+4} + \frac{\alpha_1\beta_1\theta_1c_1}{2(\beta_1+2)} t_1^{\beta_1+2} - S_1 \left(\alpha_1 t_1^{\beta_1} t_2 - \frac{\alpha_1}{\beta_1+1} t_2^{\beta_1+1} - \frac{\alpha_1\beta_1}{\beta_1+1} t_1^{\beta_1+1} \right) \right\} \right. \\ & + 2 \left\{ \frac{\alpha_2\beta_2h_2}{2(\beta_2+2)} t_1^{\beta_2+2} + \frac{\alpha_2\beta_2\theta_2h_2}{8(\beta_2+4)} t_1^{\beta_2+4} + \frac{\alpha_2\beta_2\theta_2c_2}{2(\beta_2+2)} t_1^{\beta_2+2} - S_2 \left(\alpha_2 t_1^{\beta_2} t_2 - \frac{\alpha_2}{\beta_2+1} t_2^{\beta_2+1} - \frac{\alpha_2\beta_2}{\beta_2+1} t_1^{\beta_2+1} \right) \right\} \\ & + 2 \left\{ \frac{\alpha_3\beta_3h_3}{2(\beta_3+2)} t_1^{\beta_3+2} + \frac{\alpha_3\beta_3\theta_3h_3}{8(\beta_3+4)} t_1^{\beta_3+4} + \frac{\alpha_3\beta_3\theta_3c_3}{2(\beta_3+2)} t_1^{\beta_3+2} - S_3 \left(\alpha_3 t_1^{\beta_3} t_2 - \frac{\alpha_3}{\beta_3+1} t_2^{\beta_3+1} - \frac{\alpha_3\beta_3}{\beta_3+1} t_1^{\beta_3+1} \right) \right\} \\ & + 2 \left\{ \frac{\alpha_4\beta_4h_4}{2(\beta_4+2)} t_1^{\beta_4+2} + \frac{\alpha_4\beta_4\theta_4h_4}{8(\beta_4+4)} t_1^{\beta_4+4} + \frac{\alpha_4\beta_4\theta_4c_4}{2(\beta_4+2)} t_1^{\beta_4+2} - S_4 \left(\alpha_4 t_1^{\beta_4} t_2 - \frac{\alpha_4}{\beta_4+1} t_2^{\beta_4+1} - \frac{\alpha_4\beta_4}{\beta_4+1} t_1^{\beta_4+1} \right) \right\} \\ & \left. + \left\{ \frac{\alpha_5\beta_5h_5}{2(\beta_5+2)} t_1^{\beta_5+2} + \frac{\alpha_5\beta_5\theta_5h_5}{8(\beta_5+4)} t_1^{\beta_5+4} + \frac{\alpha_5\beta_5\theta_5c_5}{2(\beta_5+2)} t_1^{\beta_5+2} - S_5 \left(\alpha_5 t_1^{\beta_5} t_2 - \frac{\alpha_5}{\beta_5+1} t_2^{\beta_5+1} - \frac{\alpha_5\beta_5}{\beta_5+1} t_1^{\beta_5+1} \right) \right\} \right] = 0 \tag{23}
 \end{aligned}$$

Further, for the convexity of the total fuzzy cost function $C_{SD}(t_1, t_2)$, the following conditions must be satisfied

$$\frac{\partial^2 C_{SD}(t_1, t_2)}{\partial t_1^2} > 0, \quad \frac{\partial^2 C_{SD}(t_1, t_2)}{\partial t_2^2} > 0 \tag{24}$$

&

$$\left(\frac{\partial^2 C_{SD}(t_1, t_2)}{\partial t_1^2} \right) \left(\frac{\partial^2 C_{SD}(t_1, t_2)}{\partial t_2^2} \right) - \left(\frac{\partial^2 C_{SD}(t_1, t_2)}{\partial t_1 \partial t_2} \right)^2 > 0 \tag{25}$$

It is difficult to prove the convexity mathematically as it is complicated to determine the second derivatives of the total fuzzy cost function $C_{SD}(t_1, t_2)$. Hence, the convexity of the total fuzzy cost is shown graphically (Figure 3).

V. NUMERICAL ILLUSTRATIONS

Crisp Model:

Illustration-1: To illustrate the solution process, setting $A = 100$, $\alpha = 50$, $\beta = 0.25$, $\theta = 0.25$, $C = 10$, $h = 2.5$ and $S = 7.5$ with proper units and using Mathematica-5.1, the optimal replenishment time is determined as $t_1 = 1.29018$, $t_2 = 1.8739$ and minimum total cost is found as $C(t_1, t_2) = 78.5062$.

Illustration-2: To understand the effect of solution process, assuming $A = 200$, $\alpha = 100$, $\beta = 0.5$, $\theta = 0.5$, $C = 20$, $h = 5$ and $S = 15$ with proper units and using Mathematica-5.1, the optimal cycle time is evaluated as $t_1 = 0.807497$, $t_2 = 1.14238$. and optimum total cost is calculated as $C(t_1, t_2) = 308.77$.

Illustration-3: Based on the computational process, considering $A = 300$, $\alpha = 150$, $\beta = 0.75$, $\theta = 0.75$, $C = 30$, $h = 7.5$ and $S = 22.5$ with proper units and using Mathematica-5.1, the time period is determined as $t_1 = 0.623543$, $t_2 = 0.887471$ and cost value is found as $C(t_1, t_2) = 663.909$.

Fuzzy Model:

By Graded Mean Integration Method (GMIM):

Illustration-4: To illustrate the solution process, setting $A = 100$, $\tilde{\alpha} = (30, 40, 50, 60, 70)$, $\tilde{\beta} = \begin{pmatrix} 0.05, 0.15, 0.25 \\ 0.35, 0.45 \end{pmatrix}$, $\tilde{\theta} = \begin{pmatrix} 0.05, 0.15, 0.25 \\ 0.35, 0.45 \end{pmatrix}$, $\tilde{C} = \begin{pmatrix} 5, 7.5, 10 \\ 12.5, 15 \end{pmatrix}$, $\tilde{h} = \begin{pmatrix} 0.5, 1.5, 2.5 \\ 3.5, 4.5 \end{pmatrix}$ and $\tilde{S} = (5.5, 6.5, 7.5, 8.5, 9.5)$ with proper units and using Mathematica-5.1, the optimal replenishment time is obtained as $t_1 = 1.53738$, $t_2 = 2.65927$ and minimum fuzzy cost calculated as $C_{GM}(t_1, t_2) = 76.9063$.

Illustration-5: To understand the effect of solution process, assuming $A = 200$, $\tilde{\alpha} = \begin{pmatrix} 60, 80, 100 \\ 120, 120 \end{pmatrix}$, $\tilde{\beta} = \begin{pmatrix} 0.1, 0.3, 0.5 \\ 0.7, 0.9 \end{pmatrix}$, $\tilde{\theta} = \begin{pmatrix} 0.1, 0.3, 0.5 \\ 0.7, 0.9 \end{pmatrix}$, $\tilde{C} = (10, 15, 20, 25, 30)$, $\tilde{h} = (1, 3, 5, 7, 9)$ and $\tilde{S} = (11, 13, 15, 17, 19)$ with proper units and using Mathematica-5.1, the optimal cycle time is found as

$t_1 = 0.725419, t_2 = 1.07932$ and optimal fuzzy cost is determined as $C_{GM}(t_1, t_2) = 327.657$.

Illustration-6: Based on the computational process, considering $A = 300$, $\tilde{\alpha} = \begin{pmatrix} 90, 120, 150, \\ 180, 210 \end{pmatrix}$, $\tilde{\beta} = \begin{pmatrix} 0.15, 0.45, 0.75, \\ 1.05, 1.35 \end{pmatrix}$, $\tilde{\theta} = \begin{pmatrix} 0.15, 0.45, 0.75, \\ 1.05, 1.35 \end{pmatrix}$, $\tilde{C} = \begin{pmatrix} 15, 22.5, 30, \\ 37.5, 45 \end{pmatrix}$, $\tilde{h} = \begin{pmatrix} 1.5, 4.5, 7.5, \\ 10.5, 13.5 \end{pmatrix}$ and $\tilde{S} = (16.5, 19.5, 22.5, 25.5, 28.5)$ with proper units and using Mathematica-5.1, the time period is obtained as $t_1 = 0.505067, t_2 = 0.724772$ and cost is calculated as $C_{GM}(t_1, t_2) = 674.746$.

By **Signed Distance Method (SDM):**

Illustration-7: To illustrate the solution process, setting $A = 100$, $\tilde{\alpha} = (30, 40, 50, 60, 70)$, $\tilde{\beta} = \begin{pmatrix} 0.05, 0.15, 0.25, \\ 0.35, 0.45 \end{pmatrix}$, $\tilde{\theta} = \begin{pmatrix} 0.05, 0.15, 0.25, \\ 0.35, 0.45 \end{pmatrix}$, $\tilde{C} = \begin{pmatrix} 5, 7.5, 10, \\ 12.5, 15 \end{pmatrix}$, $\tilde{h} = \begin{pmatrix} 0.5, 1.5, 2.5, \\ 3.5, 4.5 \end{pmatrix}$ and $\tilde{S} = (5.5, 6.5, 7.5, 8.5, 9.5)$ with proper units and using Mathematica-5.1, the optimal replenishment time is evaluated as $t_1 = 1.35926, t_2 = 2.26523$ and minimum total fuzzy cost is determined as $C_{SD}(t_1, t_2) = 80.5096$.

Illustration-8: To understand the effect of solution process, assuming $A = 200$, $\tilde{\alpha} = \begin{pmatrix} 60, 80, 100, \\ 120, 120 \end{pmatrix}$, $\tilde{\beta} = \begin{pmatrix} 0.1, 0.3, 0.5, \\ 0.7, 0.9 \end{pmatrix}$, $\tilde{\theta} = \begin{pmatrix} 0.1, 0.3, 0.5, \\ 0.7, 0.9 \end{pmatrix}$, $\tilde{C} = (10, 15, 20, 25, 30)$, $\tilde{h} = (1, 3, 5, 7, 9)$ and $\tilde{S} = (11, 13, 15, 17, 19)$ with proper units and using Mathematica-5.1, the optimum time is determined as $t_1 = 0.659012, t_2 = 0.965084$ and optimum fuzzy cost is found as $C_{SD}(t_1, t_2) = 337.583$.

Illustration-9: Based on the computational process, considering $A = 300$, $\tilde{\alpha} = \begin{pmatrix} 90, 120, 150, \\ 180, 210 \end{pmatrix}$,

$\tilde{\beta} = \begin{pmatrix} 0.15, 0.45, 0.75, \\ 1.05, 1.35 \end{pmatrix}$, $\tilde{\theta} = \begin{pmatrix} 0.15, 0.45, 0.75, \\ 1.05, 1.35 \end{pmatrix}$, $\tilde{C} = \begin{pmatrix} 15, 22.5, 30, \\ 37.5, 45 \end{pmatrix}$, $\tilde{h} = \begin{pmatrix} 1.5, 4.5, 7.5, \\ 10.5, 13.5 \end{pmatrix}$ and $\tilde{S} = (16.5, 19.5, 22.5, 25.5, 28.5)$ with proper units and using Mathematica-5.1, the time period is found as $t_1 = 0.505067, t_2 = 0.724772$ and total fuzzy cost is evaluated as $C_{SD}(t_1, t_2) = 686.035$. Taking various parameters as pentagonal fuzzy numbers, the performance of two methods can be compared with optimal time period and optimal cost

Table 1: The optimum time period and optimum cost when $\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}, \tilde{h}, \tilde{c}, \tilde{S}$ are pentagonal fuzzy numbers

Observed Method	t_1	t_2	Optimal Fuzzy cost
GMIM	0.725419	1.07932	327.657
SDM	0.659012	0.965084	337.583

Table 2: The optimum time period and optimum cost when $\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}, \tilde{h}, \tilde{c}$ are pentagonal fuzzy numbers

Observed Method	t_1	t_2	Optimal Fuzzy cost
GMIM	0.723845	1.10348	323.284
SDM	0.650257	0.976168	332.456

Table 3: The optimum time period and optimum cost when $\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}, \tilde{h}$ are pentagonal fuzzy numbers

Observed Method	t_1	t_2	Optimal Fuzzy cost
GMIM	0.763865	1.13657	312.609
SDM	0.693558	1.01294	319.566

Table 4: The optimum time period and optimum cost when $\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}$ are pentagonal fuzzy numbers

Observed Method	t_1	t_2	Optimal Fuzzy cost
GMIM	0.787699	1.15731	307.268
SDM	0.718747	1.03489	313.084

Table 5: The optimum time period and optimum cost when $\tilde{\alpha}, \tilde{\beta}$ are pentagonal fuzzy numbers

Observed Method	t_1	t_2	Optimal Fuzzy cost
GMIM	0.839459	1.20215	295.997
SDM	0.776242	1.08508	299.037

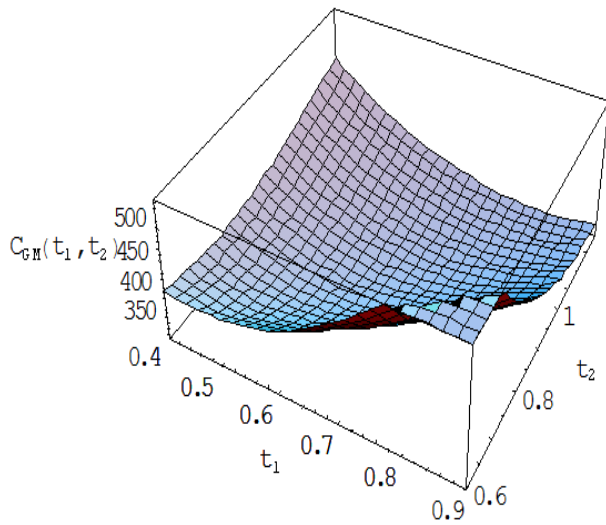


Figure 2. Total average cost $C_{GM}(t_1, t_2)$ vs. t_1 and t_2

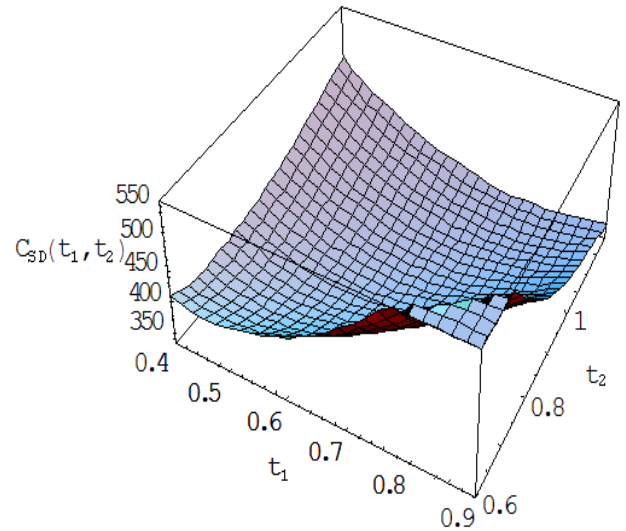


Figure 3. Total average cost $C_{SD}(t_1, t_2)$ vs. t_1 and t_2

VI. SENSITIVITY ANALYSIS

Table 6. (Sensitivity analysis by graded mean integration method)

Parameters	% change in parameters	t_1^*	t_2^*	$C^*(t_1, t_2)$	% change in $C^*(t_1, t_2)$
$A = 200$	+50	0.830305	1.30126	411.648	+25.63382
	+25	0.781606	1.19592	371.605	+13.41281
	-25	0.658268	0.946732	278.308	-15.0612
	-50	0.573077	0.788821	220.713	-32.639
$\alpha = \begin{pmatrix} 60, 80, 100, \\ 120, 140 \end{pmatrix}$	+50	0.632417	0.897612	390.353	+19.13464
	+25	0.672831	0.97487	360.895	+10.14414
	-25	0.798552	1.23212	289	-11.798
	-50	0.91248	1.48828	241.668	-26.2436
$\beta = \begin{pmatrix} 0.1, 0.3, 0.5, \\ 0.7, 0.9 \end{pmatrix}$	+50	0.637383	0.903344	358.561	+9.431814
	+25	0.672094	0.971667	346.691	+5.809124
	-25	0.770198	1.1595	298.876	-8.78388
	-50	0.975775	1.64098	250.088	-23.6738
$\theta = \begin{pmatrix} 0.1, 0.3, 0.5, \\ 0.7, 0.9 \end{pmatrix}$	+50	0.641688	1.01273	354.689	+8.250091
	+25	0.679193	1.04235	342.086	+4.40369

	-25	0.784668	1.12742	310.76	-5.15692
	-50	0.865169	1.19399	290.313	-11.3973
$h = (1,3,5,7,9)$	+50	0.685016	1.04685	340.093	+3.795432
	+25	0.704186	1.0622	334.073	+1.958145
	-25	0.749159	1.09862	320.786	-2.09701
	-50	0.77601	1.12065	313.383	-4.35638
$c = \begin{pmatrix} 10,15,20, \\ 25,30 \end{pmatrix}$	+50	0.643902	1.01491	354.308	+8.133811
	+25	0.68069	1.04382	341.845	+4.330138
	-25	0.781464	1.12429	311.195	-5.02416
	-50	0.854565	1.18365	291.596	-11.0057
$S = \begin{pmatrix} 11,13,15, \\ 17,19 \end{pmatrix}$	+50	0.741458	0.988534	347.766	+6.137211
	+25	0.734754	1.02562	339.266	+3.543034
	-25	0.711461	1.16439	310.788	-5.14837
	-50	0.688065	1.32127	283.783	-13.3902

Table 7 (Sensitivity analysis by signed distance method)

Parameters	% change in parameters	t_1^*	t_2^*	$C^*(t_1, t_2)$	% change in $C^*(t_1, t_2)$
$A = 200$	+50	0.754444	1.16167	424.914	+25.86949
	+25	0.710118	1.0684	383.255	+13.52912
	-25	0.59798	0.847489	286.365	-15.172
	-50	0.520609	0.707241	226.694	-32.8479
$\alpha = \begin{pmatrix} 60,80,100, \\ 120,140 \end{pmatrix}$	+50	0.574495	0.803887	401.443	+18.91683
	+25	0.611212	0.872456	371.451	+10.0325
	-25	0.725539	1.10046	298.149	-11.6813
	-50	0.829314	1.32712	249.796	-26.0046
$\beta = \begin{pmatrix} 0.1,0.3,0.5, \\ 0.7,0.9 \end{pmatrix}$	+50	0.591209	0.830664	365.744	+8.341948
	+25	0.617686	0.882715	355.256	+5.235157
	-25	0.727595	1.10662	308.671	-8.56441
	-50	0.855209	1.38921	261.432	-22.5577

$\theta = \begin{pmatrix} 0.1, 0.3, 0.5, \\ 0.7, 0.9 \end{pmatrix}$	+50	0.580899	0.900222	365.179	+8.174582
	+25	0.615841	0.929105	352.328	+4.367815
	-25	0.714501	1.01179	320.269	-5.12881
	-50	0.790143	1.07627	299.245	-11.3566
$h = (1, 3, 5, 7, 9)$	+50	0.622644	0.984716	349.318	+3.476182
	+25	0.639931	0.949118	343.919	+1.876872
	-25	0.680252	0.982943	330.808	-2.00691
	-50	0.70414	1.00314	323.522	-4.1652
$c = \begin{pmatrix} 10, 15, 20, \\ 25, 30 \end{pmatrix}$	+50	0.582507	0.901713	364.821	+8.068534
	+25	0.616935	0.930121	352.1	+4.300276
	-25	0.712116	1.00957	320.683	-5.00618
	-50	0.782127	1.06877	300.474	-10.9926
$S = \begin{pmatrix} 11, 13, 15, \\ 17, 19 \end{pmatrix}$	+50	0.675797	0.890949	358.303	+6.13775
	+25	0.668744	0.92128	349.539	+3.541648
	-25	0.644644	1.03424	320.229	-5.14066
	-50	0.620993	1.1612	292.49	-13.3576

VII. RESULTS AND DISCUSSION

The effects of changes in the system of parameters $A, \alpha, \beta, \theta, h, c$ and S on the fuzzy costs derived by the proposed methods are now studied. The sensitivity analysis is performed by keeping all but one system parameters fixed at a time and study the change in the identified variable by fluctuating it from 25 % to 50 %. The analysis is based on the results which are shown in Table -5.6 and Table-5.7. The sensitivity analysis for some parameters is presented in Table (5.6 & 5.7) and some interesting findings are summarized as follows

- (i) t_1^*, t_2^* and fuzzy costs increase (decrease) drastically with the increase (decrease) with respect to the parameter A .
- (ii) t_1^* and t_2^* drop down (move up) gradually while fuzzy costs move up (drop down) normally with the increase (decrease) of parameter α .
- (iii) t_1^* and t_2^* decelerate (accelerate) regularly while fuzzy costs accelerate (decelerate) slightly with the increase (decrease) of parameter β .
- (iv) t_1^* and t_2^* descend (ascend) slightly while fuzzy costs ascend (descend) gradually with the increase (decrease) of parameter θ .
- (v) t_1^*, t_2^* decelerate (accelerate) and fuzzy costs accelerate (decelerate) moderately with the increases (decreases) of parameter h .
- (vi) t_1^* and t_2^* drop down (move up) slowly while fuzzy costs move up (drop down) gradually with the increase (decrease) of parameter c .
- (vii) t_1^* rises (falls) insensitively and t_2^* falls (rises) gradually while fuzzy costs accelerate (decelerate) strictly with the increase (decrease) with respect to parameter S .

Among all the parameters, fuzzy total cost is especially sensitive to the variation of demand rate, deterioration rate, ordering cost, unit cost, holding cost, shortage cost and total fuzzy cost in both methods are represented by the following figures

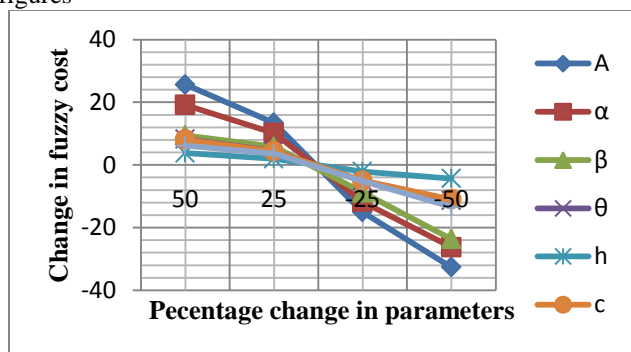


Figure 3. Behavior of fuzzy cost in graded mean integration method

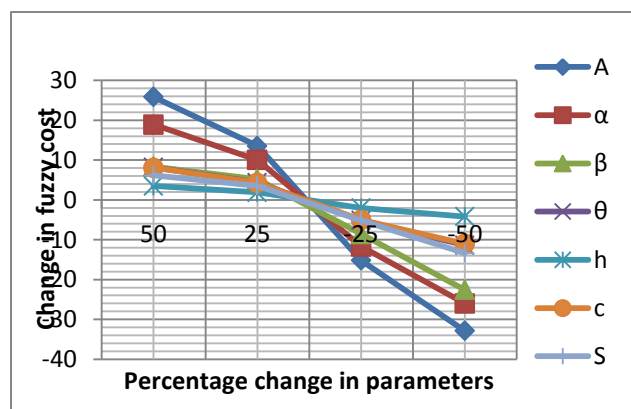


Figure 4. Behavior of fuzzy cost in signed distance method

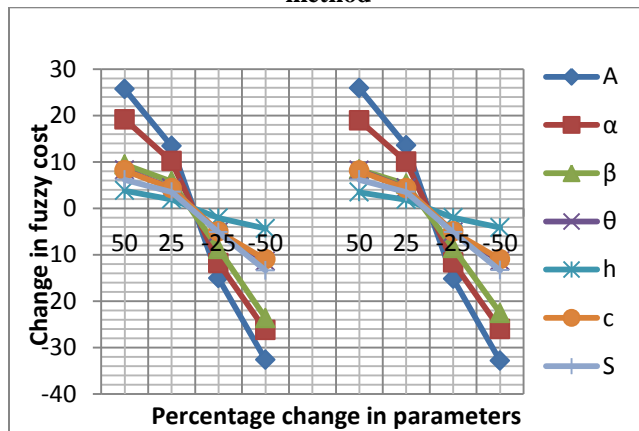


Figure 5. Comparison of fuzzy cost in both GMIM and SDM

VIII. CONCLUSION

This paper deals with optimal fuzzy replenishment model with Weibull demand rate and time dependent proportional

deterioration rate. Here holding cost varies per unit of time. The demand rate, deterioration rate and various costs are taken as pentagonal fuzzy numbers. Graded mean integration method and signed distance method are used for defuzzification of total inventory cost under fuzzy sense to take proper decision. After comparison, it is concluded that graded mean integration method provides optimum inventory cost as compared to signed distance method for this model.

The general method can be applied to the inventory problem of mixture in inventory system. Numerical illustrations suggest the total fuzzy costs are significantly sensitive to the variation of length of planning horizon and system parameters. The results show that the proposed method can be an acceptable model. This model can be induced in industry dealing with production of deteriorating commodity vis-a-vis imprecision. This proposed model can be improved by introducing trade credit, non-instantaneous deterioration, discounted selling price, salvage cost, stock dependent demand and freezing technology. The results of the study give managerial insights to decision maker developing an optimal ordering decision for deterioration products. Compensation mechanism should also be included as a scope of future research to induce collaboration in the supply chain of deterioration products with limited expiration period.

ACKNOWLEDGEMENT

The second author would like to thank anonymous referees for their valuable and constructive comments and suggestions that have led to improvement on the earlier version of the paper.

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Note:

This paper has been presented at National Seminar on "Statistics in Research: Trends & Applications" organized by the P.G. Dept. of Statistics, Utkal University, Bhubaneswar in association with ORSI, Bhubaneswar chapter, on 15th April, 2018.