International Journal of Scientific Research in Mathematical and Statistical Sciences
Volume-5, Issue-6, pp.152-156, December 2018

**Research Paper**

**E-ISSN:** 2348-4519

# Harmonious and Pell Labeling for Some Extended Duplicate Graph

**R. Avudainayaki[1*], B. Selvam[2]**

[1] Department of Mathematics,Sri Sairam Institute of Technology, Chennai, India
[2] Department of Mathematics, S.I.V.E.T.College,Gowrivakkam, Chennai, India

*Corresponding author: avuramesh2004@gmail.com, Telephone: 7708001593*

**Available online at: www.isroset.org**

Received: 27/Nov/2018, Accepted: 24/Dec/2018, Online: 31/Dec/2018

***Abstract*—** A Graph G is said to be harmonious if there exist an injection f : V(G)→ Zq such that the induced function f *: E(G) → Zq defined by f *(uv) = (f(u) + f(v)) (mod q) is a bijection. Let G be a (p, q) graph. If there exists a mapping f: V(G) →{ 0,1, …., p-1} such that the induced function f* : E(G) → N given by f*(uv) = f(u) +2 f(v) for every uv E(G) are all distinct where u,v ≥ 0, then the function f is called a Pell labeling. In this paper, we proved that the extended duplicate graph of arrow graph and splitting graph of path admits harmonious and pell labeling.

***Keywords*—** Arrow graph, Splitting graph of path, Duplicate graph, Harmonious , Pell labeling.

## I. INTRODUCTION

For an extensive survey on graph labeling and bibliographic references we refer to J.A.Gallian[1]. Graham and Sloane[2] proved that Km,n is harmonious if and only if m or n = 1. Harmonious graphs naturally arose in the study of modular version of error-correcting codes and channel assignment problems. Lots of research work is been carried out in the labeling of graphs in past few work since the first initiated by A.Rosa[3]. The concept of duplicate graph was introduced by E.Sampthkumar and he proved many results on it [4]. Shiama [5] defined a new labeling called Pell labeling. K.Thirusangu,B.Selvam and P.P.Ulaganathan[6] have proved that the extended duplicate graph of twig graphs is cordial and total cordial .

## II. PRELIMINARIES

Let G(V,E) be a finite, simple and undirected graph with p vertices and q edges.

**Definition 2.1: Arrow Graph:** An arrow graph $A^n_m$ with width 'n' and length 'm' is obtained by joining a vertex 'v' with superior vertices of $P_t \times P_m$ by ' t' new edges from one end. Clearly it has 2m+1 vertices and 3m edges.

**Illustration 1: Arrow Graph**

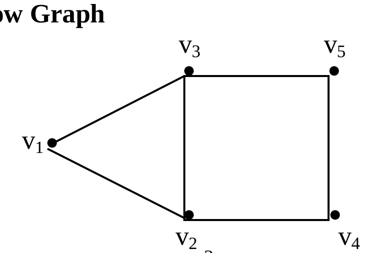
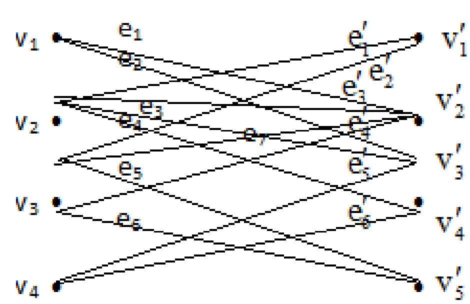



Fig: $A^2{}_2$

**Definition 2.2: Duplicate Graph:** Let G (V,E) be a simple graph and the duplicate graph of G is DG ($V_1$, $E_1$) ,where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \phi$ and $f : V \to V'$ is bijective (for $v \in V$, we write $f(v) = v'$ for convenience) and the edge set $E_1$ of DG is defined as the edge *ab* is in E if and only if both *ab′* and *a′b* are edges in $E_1$.

**Definition 2.3:Extended duplicate graph of Arrow graph:** Let DG ($V_1$, $E_1$) be a duplicate graph of the arrow graph G(V,E). Extended duplicate graph of arrow graph is obtained by adding the edge $v_2 v'_2$ to the duplicate graph. It is denoted by EDG ( $A^2_m$ ). Clearly it has 4m+2 vertices and 6m+1 edges, where 'm' is the number of length.

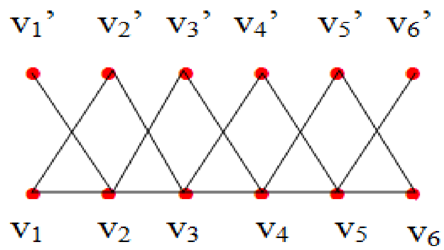
**Illustration 2: Extended Duplicate Graph of Arrow Graph**




**EDG ( $A_2{}^2$ )**

**Definition 2.4: Splitting graph:** For each vertex v of a graph G, take a new vertex *v′*. Join *v′* to all the vertices of G adjacent to v. The graph Spl(G) thus obtained is called splitting graph of G.

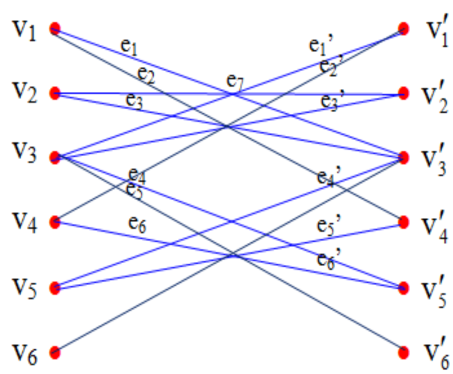
**Illustration 3: The Splitting Graph of Path SPL($P_6$)**




*Spl($P_6$)*

**Definition 2.5: Extended duplicate graph of splitting graph of path:** Let DG = ($V_1$,$E_1$) be a duplicate graph of splitting graph of path G(V,E). Extended duplicate graph of splitting graph of path is obtained by adding the edge $v_2v'_2$ to the duplicate graph. It is denoted by EDG Spl($P_m$). Clearly it has 4m vertices and 6m-5 edges, where $m \geq 2$ is the number of length.

**Illustration 4: Extended Duplicate Graph of Splitting Graph of a Path**




*EDG(Spl($P_3$))*

**Definition 2.6: Harmonious Labelling:** Let G be a graph with q edges. A function f is called harmonious labeling of graph G if f: V → {0, 1, 2,,,,q-1} is injective and the induced function f*: E → {0, 1, 2,,,,,,q} defined as f*( uv) = (f(u) + f(v))(mod q) is bijective. A Graph which admits harmonious labelling is called harmonious graph

**Definition 2.7: Pell Labeling :**Let G be a graph with vertex set V and edge set E and let f be function from V to {0,1,2,…….p-1}. Define f* : E→ N such that for any $u_iu'_i$∈E, f* ($u_iu'_i$) = f($u_i$)+2f($u'_i$).

## III. MAIN RESULTS

### 3.1 Harmonious labeling

***Algorithm: 3.1.1***

***Procedure [Harmonious labeling for EDG ($A_m^2$), m≥2]***

V ← {$v_1, v_2, \ldots, v_{2m}, v_{2m+1}, v'_1, v'_2, \ldots, v'_{2m}, v'_{2m+1}$}

E ← {$e_1, e_2, \ldots, e_{3m}, e_{3m+1}, e_1', e_2', \ldots, e_{3m}'$. }

$v_1 \leftarrow 0$ , $v_2 \leftarrow 2$ , $v_3 \leftarrow 6m$

$v_1' \leftarrow 6m-3$ , $v_2' \leftarrow 6m+1$, $v_3' \leftarrow 1$

for i = 0 to [(m-2)/2] do

$v_{4+4i} \leftarrow 6m-2-6i$
$v_{5+4i} \leftarrow 6+6i$
$v'_{4+4i} \leftarrow 3+6i$
$v'_{5+4i} \leftarrow 6m-5-6i$

end for

for i = 0 to [(m-3)/2] do

$v_{6+4i} \leftarrow 8+6i$
$v_{7+4i} \leftarrow 6m-6-6i$
$v'_{6+4i} \leftarrow 6m-7-6i$
$v'_{7+4i} \leftarrow 7+6i$

end for

**end procedure**

**Theorem 3.1.1:** The extended duplicate graph of arrow graph $A_m^2$ , m ≥ 2 is harmonious.

**Proof:** Let $A_m^2$ , m ≥ 2 be a arrow graph**.** Let EDG ($A_m^2$), m ≥ 2 be the extended duplicate graph of arrow graph. In order to label the vertices, define a function

f : V (G) → {0,1,2,.... q} as given in algorithm 3.1.1 as follows:

f ($v_1$) = 0 , f ($v_2$) = 2 , f ($v_3$) = 6m ,

f ($v_1'$) = 6m-3 , f ($v_2'$) = 6m+1 , f ($v_3'$) = 1

For 0 ≤ i ≤ [(m-2)/2],

f ($v_{4+4i}$) = 6m-2-6i
f ($v_{5+4i}$) = 6+6i
f ($v'_{4+4i}$) = 3+6i
f ($v'_{5+4i}$) = 6m-5-6i

For 0 ≤ i ≤ [(m-3)/2],
f ($v_{6+4i}$) = 8+6i
f ($v_{7+4i}$) = 6m-6-6i
f ($v'_{6+4i}$) = 6m-7-6i
f ($v'_{7+4i}$) = 7+6i

Thus the entire 4m+2 vertices are labeled.

Now to compute the edge labeling, we define the induced function $f^*$: E(G) → Zq defined by
$f^*(uv) = (f(u) + f(v)) \pmod q$

The edge functions are as follows:

$f^*(e_1) = 0$, $f^*(e_2) = 1$, $f^*(e_3) = 3$,
$f^*(e'_1) = 6m-1$, $f^*(e'_2) = 6m-4$,
$f^*(e'_3) = 6m$, $f^*(e'_4) = 6m-2$, $f^*(e_{3m+1}) = 2$
For $0 \le i \le [(m-2)/2]$,

$f^*(e_{4+6i}) = 5+12i$, $f^*(e_{5+6i}) = 6m-6-12i$,

$f^*(e_{6+6i}) = 6m-8-12i$, $f^*(e'_{5+6i}) = 7+12i$

$f^*(e'_{6+6i}) = 9+12i$
For $0 \le i \le [(m-3)/2]$,
$f^*(e_{7+6i}) = 6m-10-12i$, $f^*(e_{8+6i}) = 13+12i$,
$f^*(e_{9+6i}) = 15+12i$, $f^*(e'_{7+6i}) = 11+12i$,
$f^*(e'_{8+6i}) = 6m-12-12i$, $f^*(e'_{9+6i}) = 6m-14-12i$
For $0 \le i \le [(m-4)/2]$,
$f^*(e'_{10+6i}) = 6m-16-12i$
Thus the entire 6m+1 edges are labeled 0, 1, 2,...,6m which are all distinct.

Hence the extended duplicate graph of arrow graph $A^2_m$, $m \ge 2$ admits harmonious labeling.

**Algorithm: 3.1.2**

***Procedure* [*Harmonious labeling for EDG(Spl($P_m$))*, m ≥ 2]**
$V \leftarrow \{v_1, v_2, \ldots, v_{2m}, v_{2m}, v'_1, v'_2, \ldots, v'_{2m}, v'_{2m}\}$

$E \leftarrow \{e_1, e_2, \ldots, e_{3m-3}, e_{3m-2}, e'_1, e'_2, \ldots, e'_{3m-3}\}$

$v_1 \leftarrow 2$; $v_2 \leftarrow 0$;

$v'_1 \leftarrow 6m-7$; $v'_2 \leftarrow 6m-5$;

for i = 0 to [(m-2)/2] do
$v_{3+4i} \leftarrow 6m-6-6i$
$v_{4+4i} \leftarrow 6m-8-6i$
$v'_{3+4i} \leftarrow 1+6i$
$v'_{4+4i} \leftarrow 3+6i$
end for
for i = 0 to [(m-3)/2] do
$v_{5+4i} \leftarrow 8+6i$
$v_{6+4i} \leftarrow 6+6i$
$v'_{5+4i} \leftarrow 6m-7-6i$
$v'_{6+4i} \leftarrow 6m-5-6i$
end for

**end procedure**

**Theorem 3.1.2:** The extended duplicate graph of splitting graph of path graph *Spl($P_m$)*, $m \ge 2$ is harmonious.

**Proof:** Let *EDG(Spl($P_m$))*, $m \ge 2$ be the extended duplicate graph of splitting graph of path graph. In order to label the vertices, define a function f: V → {0,1,2,.... q} as given in algorithm 3.1.2 as follows:

$f(v_1) = 2$, $f(v_2) = 0$, $f(v'_1) = 6m-7$, $f(v'_2) = 6m-5$
For $0 \le i \le [(m-2)/2]$,

$f(v_{3+4i}) = 6m-6-6i$
$f(v_{4+4i}) = 6m-8-6i$
$f(v'_{3+4i}) = 1+6i$
$f(v'_{4+4i}) = 3+6i$
For $0 \le i \le [(m-3)/2]$,
$f(v_{5+4i}) = 8+6i$
$f(v_{6+4i}) = 6+6i$
$f(v'_{5+4i}) = 6m-7-6i$
$f(v'_{6+4i}) = 6m-5-6i$
Thus the entire 4m vertices are labeled.
Now to compute the edge labeling, we define the induced function $f^*$: E(G) → Zq defined by
$f^*(uv) = (f(u) + f(v)) \pmod q$
The edge functions are as follows:
$f^*(e_1) = 3$, $f^*(e_2) = 5$, $f^*(e_3) = 1$,
$f^*(e'_1) = 6m-8$, $f^*(e'_2) = 6m-10$,
$f^*(e'_3) = 6m-6$, $f^*(e_{3m-2}) = 0$
For $0 \le i \le [(m-3)/2]$,

$f^*(e_{4+6i}) = 6m-14-12i$

$f^*(e_{5+6i}) = 6m-12-12i$

$f^*(e_{6+6i}) = 6m-16-12i$

For $0 \le i \le [(m-4)/2]$,

$f^*(e_{7+6i}) = 15+12i$

$f^*(e_{8+6i}) = 17+12i$

$f^*(e_{9+6i}) = 13+12i$

For $0 \le i \le [(m-3)/2]$,

$f^*(e'_{4+6i}) = 9+12i$

$f^*(e'_{5+6i}) = 7+12i$

$f^*(e'_{6+6i}) = 11+12i$

For $0 \le i \le [(m-4)/2]$,

$f^*('e'_{7+6i}) = 6m-20-12i$

$f^*(e'_{8+6i}) = 6m-22-12i$

$f^*(e'_{9+6i}) = 6m-18-12i$
Thus the entire 6m-5 edges are labeled 0,1,2,3,...,6m-6 which are all distinct.
Hence the extended duplicate graph of splitting graph of path graph *EDG(Spl($P_m$))*, $m \ge 2$ admits harmonious labeling.

### 3.2 Pell labeling

***Algorithm:* 3.2.1**

***Procedure [Pell labeling for EDG ( $A_m^2$ ), m≥ 2]***

$V \leftarrow \{v_1, v_2, \ldots, v_{2m}, v_{2m+1}, v'_1, v'_2, \ldots, v'_{2m}, v'_{2m+1}\}$

$E \leftarrow \{e_1, e_2, \ldots, e_{3m}, e_{3m+1}, e_1', e_2', \ldots, e_{3m}'\}$

$v_1 \leftarrow 0; v_2 \leftarrow 5; v_3 \leftarrow 2$

$v'_1 \leftarrow 4; v'_2 \leftarrow 3; v'_3 \leftarrow 1$

for i = 0 to [(m-2)/2] do

$v_{4+2i} \leftarrow 8+4i$

$v_{5+2i} \leftarrow 7+4i$

$v'_{4+2i} \leftarrow 6+4i$

$v'_{5+2i} \leftarrow 9+4i$

end for

**end procedure**

**Theorem 3.2.1 :** The extended duplicate graph of arrow graph $A_m^2$ , m ≥ 2 admits pell labeling.

**Proof:** Let $A_m^2$ , m ≥ 2 be a arrow graph**.** Let EDG ( $A_m^2$ ), m ≥ 2 be the extended duplicate graph of arrow graph.

In order to label the vertices, define a function f : V(G) → {0,1,2,……,p-1} as given in algorithm 3.2.1 as follows:

$f(v_1) = 0$, $f(v_2) = 5$, $f(v_3) = 2$, $f(v'_1) = 4$, $f(v'_2) = 3$, $f(v'_3) = 1$

For $0 \le i \le [(m-2)/2]$,
$f(v_{4+2i}) = 8+4i$,

$f(v_{5+2i}) = 7+4i$,

$f(v'_{4+2i}) = 6+4i$,

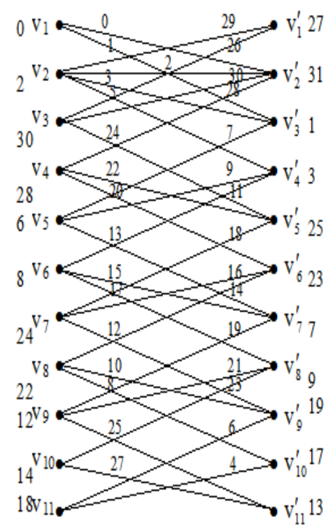
$f(v'_{5+2i}) = 9+4i$

Thus the entire 4m+2 vertices are labeled.
Now to compute the edge labeling, we define the induced function f* : E→ N such that for any $u_iu'_i \in E$,
$f^*(u_iu'_i) = f(u_i)+2f(u'_i)$.
The edge functions are as follows:

$f^*(e_1) = 6$ , $f^*(e_2) = 2$ , $f^*(e_3) = 7$ ,
$f^*(e_4) = 17$ , $f^*(e_5) = 20$ ,
$f^*(e'_1) = 13$ , $f^*(e'_2) = 10$, $f^*(e'_3) = 8$ ,
$f^*(e'_4) = 14$ , $f^*(e'_5) = 9$ , $f^*(e_{3m+1}) = 11$
For $0 \le i \le [(m-2)/2]$,
$f^*(e_{6+3i}) = 26+12i$ , $f^*(e'_{6+3i}) = 19+12i$
For $0 \le i \le [(m-3)/2]$,
$f^*(e_{7+3i}) = 28+12i$, $f^*(e_{8+3i}) = 33+12i$,
$f^*(e'_{7+3i}) = 24+12i$ , $f^*(e'_{8+3i}) = 29+12i$

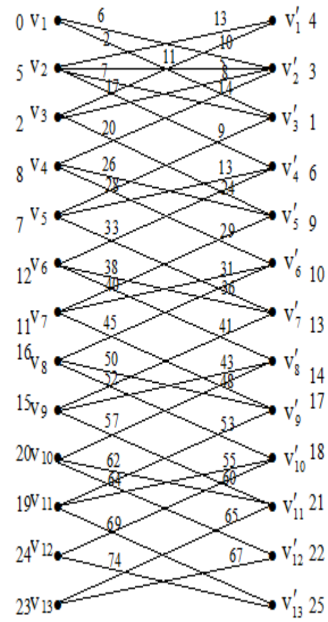
Thus the entire 6m+1 edges are all distinct. Hence the extended duplicate graph of arrow graph $A_m^2$ , m ≥ 2 admits pell labeling.

**Illustration 5: Harmonius and Pell Labeling for the Extended Duplicate Graph of Arrow Graph**




***EDG($A_5^2$ )***

***(HARMONIOUS LABELING)***




***EDG($A_6^2$)***

***(PELL LABELING)***

**Algorithm: 3.2.2**

**Procedure [Pell labeling for *EDG(Spl($P_m$))*, m ≥ 2]**

$V \leftarrow \{v_1, v_2, \ldots, v_{2m}, v_{2m}, v'_1, v'_2, \ldots, v'_{2m}, v'_{2m}\}$

$E \leftarrow \{ e_1, e_2, \ldots, e_{3m-3}, e_{3m-2}, e_1', e_2', \ldots, e_{3m-3}' \}$

$v_1 \leftarrow 0$ , $v_2 \leftarrow 3$
$v'_1 \leftarrow 2$ , $v'_2 \leftarrow 1$
for i = 0 to [(m-2)/2] do

$v_{3+2i} \leftarrow 4+4i$

$v_{4+2i} \leftarrow 5+4i$

$v'_{3+2i} \leftarrow 6+4i$

$v'_{4+2i} \leftarrow 7+4i$

end for

**end procedure**

**Theorem 3.2.2 :** The extended duplicate graph of splitting graph of path graph *Spl($P_m$)*, m ≥ 2 admits pell labeling.

**Proof:** Let *EDG (Spl($P_m$))*, m ≥ 2 be the extended duplicate graph of splitting graph of path graph. In order to label the vertices, define a function f: V(G) $\rightarrow$ {0,1,2,......,p-1} as given in algorithm 3.2.2 as follows:

$f(v_1) = 0$, $f(v_2) = 3$, $f(v'_1) = 2$, $f(v'_2) = 1$

For $0 \le i \le [(m-2)/2]$,

$f(v_{3+2i}) = 4+4i$, $f(v_{4+2i}) = 5+4i$, $f(v'_{3+2i}) = 6+4i$, $f(v'_{4+2i}) = 7+4i$

Thus the entire 4m vertices are labeled.Now to compute the edge labeling, we define the induced function f* : E→ N such that for any $u_i u'_i \in E$, $f^*(u_i u_i') = f(u_i)+2f(u_i')$.

The edge functions are as follows:

$f^*(e_1) = 12$ , $f^*(e_2) = 14$ , $f^*(e_3) = 15$ ,

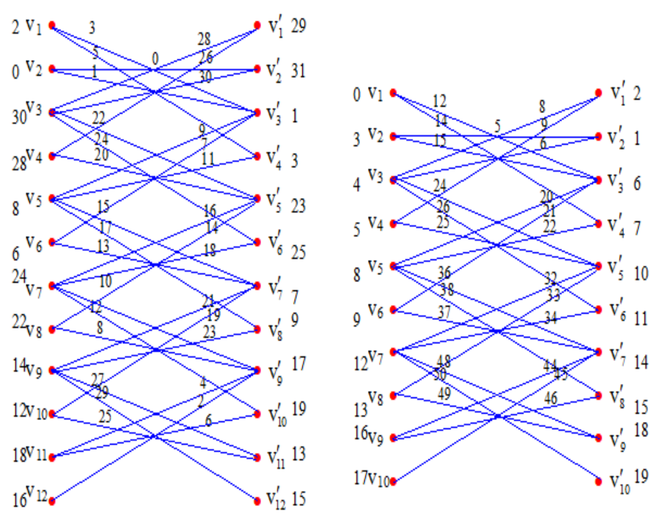
$f^*(e'_1) = 8$ , $f^*(e'_2) = 9$ ,

$f^*(e'_3) = 6$ , $f^*(e_{3m-2}) = 5$

For $0 \le i \le [(m-3)/2]$,
$f^*(e_{4+3i}) = 24+12i$
$f^*(e_{5+3i}) = 26+12i$
$f^*(e_{6+3i}) = 25+12i$
$f^*(e'_{4+6i}) = 20+12i$
$f^*(e'_{5+6i}) = 21+12i$
$f^*(e'_{6+6i}) = 22+12i$

Thus the entire 6m-5 edges are all distinct. Hence the extended duplicate graph of splitting graph of path graph *EDG(Spl(P$_m$))*, $m \ge 2$ admits pell labeling.

**Illustration 6: Harmonius and Pell Labeling for Extended Duplicate Graph of Splitting Graph of Path**



**EDG(Spl(P$_6$))** ***(HARMONIOUS LABELING)***

***EDG(Spl(P$_5$))*** ***(PELL LABELING)***

## IV. CONCLUSION

In this paper, we presented algorithms and prove that the extended duplicate graph of arrow graph and splitting graph of path admits harmonious and pell labeling.

## V. ACKNOWLEDGEMENT

The authors would like to thank the editors and the referees for their comments and suggestions.

## REFERENCES

[1] Gallian J.A, "A Dynamic Survey of graph labeling", the Electronic Journal of combinatories, 19, # DS6 (2015).Pp. 189-198.
[2] R. L. Graham and N.J.A. Sloane, on additive bases and harmonious graphs, SIAM, J. Alg. Disc. Meth. 1, (1980), 382 – 404.
[3] Rosa A, On certain Valuations of the vertices of a graph, Theory of graphs (Internat. Symposium, Rome, July 1966), Gordon and Breech, N.Y. and Dunod paris, 1967.pp. 349- 355.
[4] E.Sampath kumar, "On duplicate graphs", Journal of the Indian Math. Soc. 37 (1973), 285 – 293.
[5] J.Shiama, Pell labeling for some graphs, Asian Journal of Current Engineering and Maths, 2(4) (2013).
[6] Thirusangu,K, Selvam.B. and Ulaganathan.P.P, Cordial labelings in extended duplicate twig graphs ,International Journal of computer, mathematical sciences and applications, Vol.4, Nos 3-4, (2010), pp.319-328.