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Total Neighborhood Prime Labeling of Some Graphs

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Abstract— Let $G = (V, E)$ be a graph with p vertices and q edges. A bijection $f: V(G) \cup E(G) \rightarrow \{1,2,3, \dots, p + q\}$ is said to be total neighborhood prime labeling if it satisfies that, for each vertex of degree at least two, the gcd of labeling on its neighborhood vertices is one and for each vertex of degree at least two, the gcd of labeling on the induced edges is one. In this paper we investigate total neighborhood prime labeling for comb, disjoint union of paths, disjoint union of sunlet graphs, disjoint union of wheel graphs, graph obtained by one copy of path $P_n(which\ has\ n\ vertices)$ and n copies of $k_{1,m}$ and joining i^{th} vertex of P_n with an edge to fix vertex in the i^{th} copy of $k_{1,m}$, corona product of cycle with m copies of k_1 and subdivision of bistar.

Keywords— Neighborhood prime labeling, Total Neighborhood prime labeling, disjoint union of graphs, corona product of graphs, subdivision of graph

I. INTRODUCTION

The notion of prime labeling was initiated by Roger Entringer and was introduced in 1980's by Tout et al [1]. Motivated by the study of prime labeling S K Patel and N P Shrimali [2] introduced the notion of neighborhood prime labeling and proved many results. Further so many results have been investigated by many researchers. Later Rajesh Kumar T J and Mathew Varkey T K [3] introduce total neighborhood prime labeling, and proved paths and cycles are total neighborhood prime graphs. They also prove that the comb graph is total neighborhood prime graph with two cases.

In this paper we investigate total neighborhood prime labeling for comb in only one case, disjoint union of paths, disjoint union of sunlets, disjoint union of wheels, graph obtained by one copy of path $P_n(which\ has\ n\ vertices)$ and n copies of $k_{1,m}$ and joining i^{th} vertex of P_n with an edge to fix vertex in the i^{th} copy of $k_{1,m}$, corona product of cycle with m copies of k_1 and, subdivision of bistar. Here we consider finite graphs which are simple, connected and undirected. We follow [4] for notations and definitions in graph theory and Gallian [5] for all terminology regrading prime and neighborhood prime labelings. For number theoretical results we follow [6].

In this paper there are four sections in which section I deals with introduction of total neighborhood prime labeling. Section II is about preliminaries and notations which contains some definitions and remarks. Section III includes main results regarding total neighborhood prime labeling. Section IV includes conclusion that indicates future scope of research,

II. PRELIMINERIES AND NOTATIONS

Definition:1. Let $G = (V(G), E(G))$ be a graph with p vertices. A bijection $f: V(G) \rightarrow \{1,2,3, \dots, p\}$ is called prime labeling if for each edge $e = uv, gcd(f(u), f(v)) = 1$. A graph which admits prime labeling is called a **Prime graph.**

Definition:2. Let $G = (V(G), E(G))$ be a graph with p vertices. A bijection $f: V(G) \rightarrow \{1,2,3, \dots, p\}$ is called neighborhood prime labeling if for each vertex $v \in V(G)$ with $deg(v) > 1$, $\gcd\{f(u): u \in N(v)\} = 1$. A graph which admits neighborhood prime labeling is called a **Neighborhood-prime graph.**

Definition:3. Let $G = (V(G), E(G))$ be a graph with p vertices and q edges. A bijection $f: V(G) \cup E(G) \rightarrow \{1,2,3, \dots, p + q\}$ is said to be total neighborhood prime labeling if it satisfies the following two conditions:

1: For each vertex of degree atleast two, the gcd of labeling on its neighborhood vertices is one;

2: For each vertex of degree atleast two, the gcd of labeling on the induced edges is one.

A graph which admits total neighborhood prime labeling is called **Total neighborhood prime graph.**

Definition:4. The $n -$ sunlet graph is a graph of $2n$ vertices is obtained by attaching $n -$ pendant edges to the cycle C_n and it is denoted by S_n.

Definition:5. Bistar is a graph obtained from a path P_2 by joining the root of stars S_m and S_n at the terminal vertices of P_2. It is denoted by $B_{m,n}$.

Definition:6. A subdivision of a graph G is a graph that can be obtained from G by a sequence of edge Subdivisions.

Definition:7. The corona product $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as a graph obtained by one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and joining i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.

Definition 8: Let $G = (V(G), E(G))$ be a graph, $u \in V(G)$

$N_V(u) = \{w \in V(G)/$ uw is an edge$\}$

$N_E(u) = \{e \in E(G)/e = uv$ for some $v \in V(G)\}$

Remark 1. The graphs K_n where $n \neq 3$, W_n, $K_{1,n}$, $F_n = P_n + K_1$ are Total neighborhood prime graph. [5]

III. MAIN RESULTS

The total neighborhood prime labeling of comb graph was discussed in [5] with two cases, n $\not\cong$0(mod6) and n $\cong$ 0(mod6). Here we proved this result without any condition on n.

Theorem 1. The comb graph $P_n \odot K_1$ is total neighborhood prime for all n.

Proof: Let $v_1, v_2, v_3, \ldots\ldots, v_n$ be vertices of the path in G = $P_n \odot K_1$ and $u_1, u_2, u_3, \ldots\ldots, u_n$ be the vertices joined with it.

The edges are, $e_1 = v_1u_1, e_2 = v_1v_2$, $e_3 = v_2u_2, e_4 = v_2v_3$,... $e_{2n-2} = v_{n-1}v_n, e_{2n-1} = v_nu_n$. The total number of vertices are $2n$ and total number of edges are $2n - 1$.

$|V(G) \cup E(G)| = 4n - 1$.

Define a bijection

$f: V(G) \cup E(G) \to \{1,2,3 \ldots . |V(G) \cup E(G)|\}$ as follows.

$f(v_i) = 2i, \quad 1 \leq i \leq n$

$f(u_i) = 2i - 1, 1 \leq i \leq n$

$f(e_i) = 2n + i, 1 \leq i \leq 2n - 1$

We claim that f is a total neighborhood prime labeling.

Note that $u_1, u_2, u_3 \ldots\ldots, u_n$ are vertices of degree one.

Let v be any vertex of ,

For $v = v_1$, deg(v) = 2

Since $u_1 \in N_V(v_1)$, $f(u_1) = 1$ and $e_1, e_2 \in N_E(v_1)$, $f(e_1) = 2n + 1$, , $f(e_2) = 2n + 2$, so we are done.

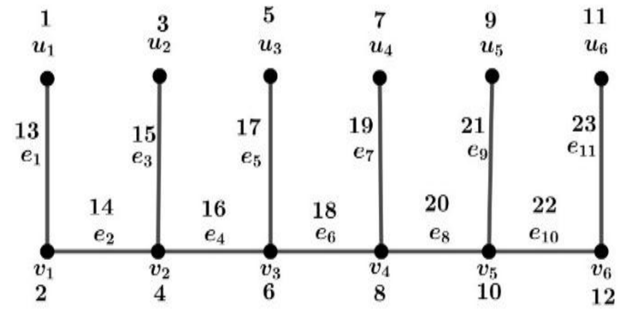
For $v = v_n$, deg(v) = 2

The labels of neighbor vertices of v_n are consecutive integers and the labels of neighbor edges of v_n are also consecutive integers, so conditions are satisfied.

For $v = v_i$, $2 \leq i \leq n - 1$ $\deg(v) = 2$

The neighbor vertices of v_i are v_{i-1} , v_{i+1} and u_i. Since $f(v_{i-1}) = 2i - 2$ and $f(u_i) = 2i - 1$, therefore $\gcd\{f(x)/x \in N_V(v_i)\}$ = 1. The labels of incident edges of v_i are consecutive integers, therefore $\gcd\{f(e)/e \in N_E(v_i)\}$ = 1

Thus f is a total neighborhood prime labeling.



Example:1 Total neighborhood prime labeling of comb $P_6 \odot K_1$

Theorem 2. The disjoint union of path graph $P_m \cup P_n$ is total neighborhood prime for all m and n.

Proof: Let G = $P_m \cup P_n$. $v_1, v_2, v_3, \ldots\ldots, v_m$ are consecutive vertices of path P_m and $u_1, u_2, u_3, \ldots\ldots, u_n$ are consecutive vertices of the path P_n . $e_1, e_2, e_3 \ldots\ldots, e_{m-1}$ are the consecutive edges of the path P_m and $e'_1, e'_2, e'_3, \ldots\ldots, e'_{n-1}$ are consecutive edges of the path P_n.

$|V(G)| = $ n + m , $|E(G)| = $ n + m $- 2$,

$|V(G) \cup E(G)| = 2n + 2m - 2$.

Define a bijection

$f: V(G) \cup E(G) \to \{1,2,3 \ldots . |V(G) \cup E(G)|\}$ as follows.

Case-1: $m =$ even , $n =$ even

$f(e_i) = i, 1 \leq i \leq m - 1$

$f(e'_i) = 2m - 1 + i, 1 \leq i \leq n - 1$

$f(v_{2i}) = 2m - i, 1 \leq i \leq \frac{m}{2}$

$f(v_{2i-1}) = 2m - i - \frac{m}{2}$, $1 \leq i \leq \frac{m}{2}$

$f(u_{2i}) = 2m - 1 - i, 1 \leq i \leq \frac{n}{2}$

$f(u_{2i-1}) = 2m + 2n - 1 - i - \frac{n}{2}$, $1 \leq i \leq \frac{n}{2}$

Case-2: $m =$ even , $n =$ odd

$f(e_i) = i, 1 \leq i \leq m - 1$

$f(e'_i) = 2m - 1 + i, 1 \leq i \leq n - 1$

$f(v_{2i}) = 2m - i, 1 \leq i \leq \frac{m}{2}$

$f(v_{2i-1}) = 2m - i - \frac{m}{2}$, $1 \leq i \leq \frac{m}{2}$

$f(u_{2i}) = 2m + 2n - 1 - i - \frac{n+1}{2}, 1 \leq i \leq \frac{n-1}{2}$

$f(u_{2i-1}) = 2m - 1 + 2n - i$, $1 \leq i \leq \frac{n+1}{2}$

Case-3: $m =$ odd, $n =$ odd

$f(e_i) = i, 1 \leq i \leq m - 1$

$f(e'_i) = 2m - 1 + i, 1 \leq i \leq n - 1$

$f(v_{2i}) = 2m - i - \frac{m+1}{2}, 1 \leq i \leq \frac{m-1}{2}$

$f(v_{2i-1}) = 2m - i$, $1 \leq i \leq \frac{m+1}{2}$

$f(u_{2i}) = 2m + 2n - 1 - i - \frac{n+1}{2}, 1 \leq i \leq \frac{n-1}{2}$

$f(u_{2i-1}) = 2m + 2n - 1 - i\,, 1 \leq i \leq \frac{n+1}{2}$

We claim that f is a total neighborhood prime labeling, in all of the above cases.

Note that $v_1, v_m\, u_1, u_n$ are vertices of degree 1.

Let v be any vertex of G and

$v = v_i\,, 2 \leq i \leq m-1 \quad v = u_i,\ 2 \leq i \leq n-1$ be the vertices whose degree is two.

$N_V\,(v_i)= \{v_{i-1}, v_{i+1}\}$

$N_V\,(u_i)= \{u_{i-1}, u_{i+1}\}$

$N_E\,(v_i)= \{e_{i-1}\,, e_i\}$

$N_E\,(u_i)= \{e'_{i-1}\,, e'_i\}$

We observe in all of the above cases by definition of f that $\{f(v_{i-1})\,, f(v_{i+1})\}$ and $\{f(u_{i-1})\,, f(u_{i+1})\}$ as well as $\{f(e_{i-1})\,, f(e_i)\}$ and $\{f(e'_{i-1})\,, f(e'_i)\}$ are sets of consecutive integers, therefore

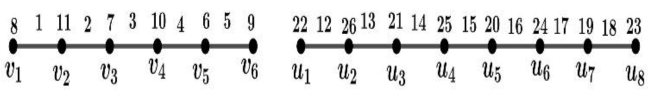
$\gcd\{f(x)/x \in N_V\,(v_i)\} = 1$

$\gcd\{f(x)/x \in N_V\,(u_i)\} = 1$

$\gcd\{f(e)/x \in N_E\,(v_i)\} = 1$

$\gcd\{f(e')/x \in N_E\,(u_i)\} = 1$

Thus G $= P_m \cup P_n$ is total neighborhood prime graph for every m and n.



Example:2 Total neighborhood prime labeling of disjoint union of path $P_6 \cup P_8$.

Theorem 3. The disjoint union of sunlet graphs $S_n \cup S_m$ is total neighborhood prime for all n and m.

Proof: Let G $= S_n \cup S_m$ with $2n + 2m$ vertices and $2n + 2m$ edges. Let $u_1, u_2, u_3 \ldots\ldots, u_n$ are the vertices of the cycle of sunlet graph S_n, and $v_1, v_2, v_3, \ldots\ldots, v_n$ are the pendent vertices of sunlet graph S_n such that u_i is adjacent to v_i. Let $e_\mathrm{i} = u_\mathrm{i}v_\mathrm{i}$, $1 \leq i \leq n$ and starting with e''_1 which is edge between the vertex u_2 and u_3 in clockwise direction $e''_1, e''_2, e''_3, \ldots, e''_4$ are edges of the sunlet graph S_n. $u'_1, u'_2, u'_3, \ldots, u'_m$ are the vertices of the cycle of sunlet graph S_m. $v'_1, v'_2\ v'_3 \ldots v'_m$ are the pendent vertices of sunlet graph S_m such that u'_i is adjacent to v'_i. Let $e'_1 = u'_1v'_1, e'_2 = u'_1u'_2,\ , e'_3 = u'_2v'_2, , e'_4 = u'_2u'_3, \ldots, , e'_{2m} = u'_mu'_1$.

Define a bijection

$f: V(G) \cup E(G) \to \{1,2,3 \ldots.|V(G) \cup E(G)|\}$ as follows.

Case:1: m is even

$f(u_i) = 3\mathrm{i} - 2, 1 \leq i \leq n$

$f(v_i) = 3\mathrm{i}, 1 \leq i \leq n$

$f(e_i) = 3\mathrm{i} - 1, 1 \leq i \leq n$

$f(e''_i) = 3\mathrm{n} + \mathrm{i}, 1 \leq i \leq n$

$f(u'_{2i-1}) = 4n + \mathrm{i}\,, 1 \leq i \leq \frac{m}{2}$

$f(u'_{2i}) = 4n + \frac{m}{2} + 1 + \mathrm{i}\,, 1 \leq i \leq \frac{m}{2}$

$f(v'_i) = 4\mathrm{n} + \mathrm{m} + 1 + \mathrm{i}, 1 \leq i \leq \mathrm{m} - 1$

$f(v'_m) = 4n + \frac{m}{2} + 1$

$f(e'_i) = 4\mathrm{n} + 2\mathrm{m} + \mathrm{i}, 1 \leq i \leq 2m$

We have to show that f is total neighborhood prime labeling, Here $v_iv'_j\ 1 \leq i \leq n, 1 \leq j \leq m$ are vertices of degree 1.

Let v be any vertex of G.

Let $v = u_i\,, 1 \leq i \leq n$ with $\deg(v) \geq 2$ then the set $\{f(w)/ w \in N_V\,(u_i)\ \}$ contains two consecutive numbers or 1, therefore $\gcd\{f(w)/w \in N_V\,(u_i)\} = 1$. The set $\{f(d_i)/d_i \in N_E\,(u_i)\}$ contains either two consecutive integers or numbers of the form $3n + 1, 4n$ and 5, where $n = 3,4,5, \ldots$ and $\gcd\{f(d_i)/d_i \in N_E\,(u_i)\} = 1$.

Let $v = u'_i\,, 1 \leq i \leq m$ with $\deg(v) \geq 2$ then the set $\{f(v)/ v \in N_V\,(u'_i)\ \}$ contains always two consecutive numbers, therefore $\gcd\{f(v)/v \in N_V\,(u'_i)\} = 1$ and the set $\{f(e)/e \in N_E\,(u'_i)\}$ contains two consecutive integers so, $\gcd\{f(e)/e \in N_E\,(u'_i)\} = 1$

Case:2: m is odd

$f(u_i) = 3\mathrm{i} - 2, 1 \leq i \leq n$

$f(v_i) = 3\mathrm{i}, 1 \leq i \leq n$

$f(e_i) = 3\mathrm{i} - 1, 1 \leq i \leq n$

$f(e''_i) = 3\mathrm{n} + \mathrm{i}, 1 \leq i \leq n$

$f(u'_{2i-1}) = 4n + \mathrm{i}\,, 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil$

$f(u'_{2i}) = 4n + \left\lceil \frac{m}{2} \right\rceil + i, 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor$

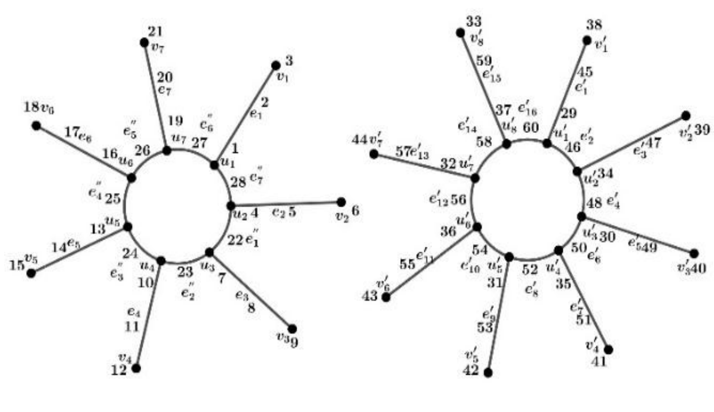
$f(v'_i) = 4\mathrm{n} + \mathrm{m} + 1 + \mathrm{i}, 1 \leq i \leq \mathrm{m} - 1$

$f(v'_m) = 4n + \mathrm{m} + 1$

$f(e'_i) = 4\mathrm{n} + 2\mathrm{m} + \mathrm{i}, 1 \leq i \leq 2m$

By the similar arguments in case-1 we can prove f is total neighborhood prime labeling in this case.

Thus G $= S_n \cup S_m$ is total neighborhood prime graph for every n and m.



Example:3 Total neighborhood prime labeling of disjoint union of sunlet $S_7 \cup S_8$.

Theorem 4. The disjoint union of wheel graphs $W_n \cup W_m$ is total neighborhood prime for all n and m.

Proof: Let $\{u_{0,}u_{1,}u_2, \ldots\ldots, u_n\}$ and $\{v_{0,}v_{1,}v_2, \ldots\ldots, v_m\}$ denote the vertex sets of W_n and W_m respectively, where $u_{0,}$ and v_0 are the central vertices of W_n and W_m respectively. Let $e_{1,} e_2, \ldots\ldots, e_n$ be the outer edges of W_n and $d_{1,} d_2, \ldots\ldots, d_n$ be the inner edges of W_n. Let $e'_1, e'_2, \ldots, e'_n$ be the outer edges of W_m and $d'_1, d'_2, \ldots\ldots, d'_n$ be the inner edges of W_m.

Define a bijection

$f: V(G) \cup E(G) \to \{1,2,3 \ldots . |V(G) \cup E(G)|\}$ as follows.

Case-1: m is odd

$f(u_0) = 1$

$f(u_i) = \mathrm{i} + 1, 1 \le i \le n$

$f(e_i) = \mathrm{n} + 1 + \mathrm{i}, 1 \le i \le n$

$f(d_i) = 2\mathrm{n} + 1 + \mathrm{i}, 1 \le i \le n$

$f(v_0) = 3\mathrm{n} + \mathrm{m} + 2$

$f(v_{2i-1}) = 3\mathrm{n} + 1 + \mathrm{i},, 1 \le i \le \left\lceil \frac{m}{2} \right\rceil$

$f(v_{2i}) = 3\mathrm{n} + 1 + \left\lceil \frac{m}{2} \right\rceil + i, 1 \le i \le \left\lceil \frac{m}{2} \right\rceil$

$f(e'_i) = 3\mathrm{n} + \mathrm{m} + 2 + \mathrm{i}, 1 \le i \le m$

$f(d'_i) = 3\mathrm{n} + 2\mathrm{m} + 2 + \mathrm{i}, 1 \le i \le m$

We claim that f is neighborhood prime labeling.

Let v be any vertex of G.

Let $v = u_0$, then the set $\{f(w)/w \in N_V(u_0)\}$ contains consecutive integers therefore $\gcd\{f(w)/w \in N_V(u_0)\} = 1$ and,the set $\{f(d_i)/d_i \in N_E(u_0)\}$ contains consecutive integers, so $\gcd\{f(d_i)/d_i \in N_E(u_i)\} = 1$.

Let $v = u_i$,$1 \le i \le n$ with $\deg(v) \ge 2$ then the set $\{f(w)/w \in N_V(u_i)\}$ contains 1, therefore $\gcd\{f(w)/w \in N_V(u_i)\} = 1$ and the set $\{f(d_i)/d_i \in N_E(u_i)\}$ contains consecutive integers, so $\gcd\{f(d_i)/d_i \in N_E(u_i)\} = 1$.

Let $v = v_i$,$1 \le i \le m$ with $\deg(v) \ge 2$ then the set $\{f(w)/w \in N_V(v_i)\}$ and $\{f(d_i)/d_i \in N_E(v_i)\}$ contains two consecutive integers, therefore,

$\gcd\{f(w)/w \in N_V(v_i)\} = 1$

$\gcd\{f(d_i)/d_i \in N_E(v_i)\} = 1$.

Case-2: m is even

$f(u_0) = 1$

$f(u_i) = \mathrm{i} + 1, 1 \le i \le n$

$f(e_i) = \mathrm{n} + 1 + \mathrm{i}, 1 \le i \le n$

$f(d_i) = 2\mathrm{n} + 1 + \mathrm{i}, 1 \le i \le n$

$f(v_0) = 3\mathrm{n} + \frac{m}{2} + 2$

$f(v_{2i-1}) = 3\mathrm{n} + 1 + \mathrm{i},, 1 \le i \le \frac{m}{2}$

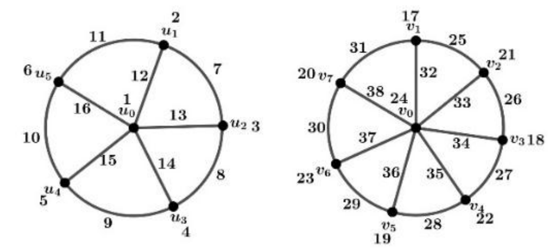
$f(v_{2i}) = 3\mathrm{n} + 2 + \frac{m}{2} + i, 1 \le i \le \frac{m}{2}$

$f(e'_i) = 3\mathrm{n} + \mathrm{m} + 2 + \mathrm{i}, 1 \le i \le m$

$f(d'_i) = 3\mathrm{n} + 2\mathrm{m} + 2 + \mathrm{i}, 1 \le i \le m$

Proof is by similar arguments in case-1.

Thus G = $W_n \cup W_m$ is total neighborhood prime graph for every n and m.



Example:4 Total neighborhood prime labeling of disjoint union of wheel $W_5 \cup W_7$.

Theorem 5. The graph G obtained by one copy of path $P_n(which\ has\ n\ vertices)$ and $n\ copies\ of\ k_{1,m}$ and joining i[th] vertex of P_n with an edge to fix vertex in the i[th] copy of $k_{1,m}$ is total neighborhood prime for all n and m.

Proof:

Case-1: Suppose fix vertex of $K_{1,\mathrm{m}}$ is central vertex, that is graph G obtained by one copy of path $P_n(which\ has\ n\ vertices)$ and n copies of $k_{1,m}$ and joining i[th] vertex of P_n with an edge to central vertex in the i[th] copy of $k_{1,m}$. Let $u_{1,}u_2, \ldots\ldots, u_n$ be the consecutive vertices of P_n and v_i is the central vertex of i[th] copy of $k_{1,m}$,where $1 \le i \le n$ and be v'_1 , v'_2, $\ldots, v'_m, v'_{m+1}, \ldots, v'_{2m}, \ldots, v'_{(n-1)m+1}, \ldots \ldots v'_{mn}$ be the pendent vertices of n copies of $k_{1,m}$.

Here edge set,

$E(G) = \{e_i, e'_j /1 \le i \le 2n - 1, 1 \le j \le mn\}$ such that $e_1 = u_1v_1, e_2 = u_1u_2, e_3 = u_2v_2, e_4 = u_2u_3, \ldots\ldots, e_{2n-1} = u_\mathrm{n}v_\mathrm{n}$, and $e'_1 = v_1\ v'_1,\ e'_2 = v_1\ v'_2, \ldots\ldots,\ e'_m = v_1\ v'_m, \ldots, e'_{\mathrm{mn}} = v_\mathrm{n}\ v'_{mn}$.

$|V(G)| = 2\mathrm{n} + \mathrm{mn}$, $|E(G)| = 2\mathrm{n} + \mathrm{mn} - 1$ thus we have $|V(G) \cup E(G)| = 4\mathrm{n} + 2\mathrm{mn} - 1$.

Define a bijection

$f: V(G) \cup E(G) \to \{1,2,3 \ldots . |V(G) \cup E(G)|\}$ as follows

$f(u_i) = 2\mathrm{i}, 1 \le i \le n$

$f(v_i) = 2\mathrm{i} - 1, 1 \le i \le n$

$f(v'_i) = 4\mathrm{n} - 1 - \mathrm{i}, 1 \le i \le \mathrm{m}n$

$f(e_i) = 2n + \mathrm{i}, 1 \le i \le 2n - 1$

$f(e'_i) = 4\mathrm{n} - 1 + \mathrm{mn} + \mathrm{i}, 1 \le i \le \mathrm{mn}$

Case-2 : Let fix vertex is any pendent vertex

Subcase-1: Let m = 1 that is, graph G obtained by one copy of path $P_n(which\ has\ n\ vertices)$ and n copies of $k_{1,1}$ and joining i[th] vertex of P_n with an edge to pendent vertex in the i[th] copy of $k_{1,1}$. Let $u_{1,}u_2, \ldots\ldots, u_n$ be the consecutive vertices of P_n , u'_1 , u'_2,… u'_n be the pendent vertices of n copies of $k_{1,1}$ joined with u_i . $v_{1,}v_2, \ldots\ldots, v_n$ be the remaining pendent vertices of n copies of $k_{1,1}$ joined with u'_i.

Here,

$e_1 = v_1u'_1, e_2 = u'_1u_1, e_3 = u_1u_2, e_4 = u_2u'_2, e_5 = v_2u'_2, \ldots\ldots, e_{3\mathrm{n}-1} = u'_nv_\mathrm{n}$ are edges of G.

$|V(G)| = 3\mathrm{n}$, $|E(G)| = 3\mathrm{n} - 1$, thus we have $|V(G) \cup E(G)| = 6n - 1$.

Define a bijection

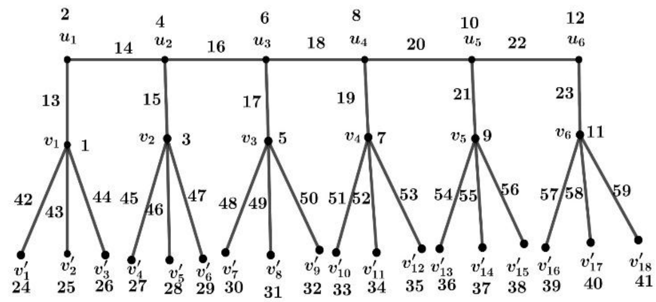
$f: V(G) \cup E(G) \to \{1,2,3 \ldots . |V(G) \cup E(G)|\}$ as follows

$f(u_i) = 3\text{i}, 1 \leq i \leq n$
$f(u_i') = 3\text{i} - 2, 1 \leq i \leq n$
$f(v_i) = 3\text{i} - 1, 1 \leq i \leq n$
$f(e_i) = 3n + \text{i}, 1 \leq i \leq 3n - 1$

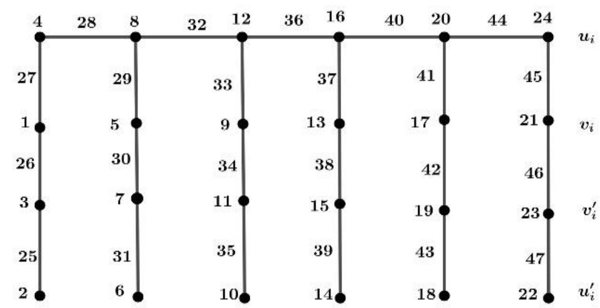
Subcase-2: Let m = 2 that is, graph G obtained by one copy of path $P_n(\textit{which has } n \textit{ vertices})$ and n *copies of* $k_{1,2}$ and joining i^th^ vertex of P_n with an edge to pendent vertex in the i^th^ copy of $k_{1,2}$. Let $u_1, u_2, \ldots, u_n$ be the consecutive vertices of P_n, v_i be the pendent vertex of i^th^ copy of $k_{1,2}$ which is joined with u_i for each i, v_i' be the central vertex of i^th^ copy of $k_{1,2}$ for each i and $u_1', u_2', \ldots u_n'$ be the remaining pendent vertices of n copies of $k_{1,2}$.
Here,
$e_1 = u_1'v_1$, $e_2 = v_1'v_1$, $e_3 = v_1u_1$, $e_4 = u_1u_2$, $e_5 = u_2v_2$, $e_6 = v_2v_2'$, $e_7 = v_2'u_2$, $e_8 = u_2u_3, \ldots, e_{4n-2} = v_nv_n'$, $e_{4n-1} = v_n'u_n'$ are edges of G.
$|V(G)| = 4\text{n}$, $|E(G)| = 4\text{n} - 1$, thus we have
$|V(G) \cup E(G)| = 8n - 1$.
Define a bijection
$f: V(G) \cup E(G) \to \{1,2,3 \ldots |V(G) \cup E(G)|\}$ as follows
$f(u_i) = 4\text{i}, 1 \leq i \leq n$
$f(v_i) = 4\text{i} - 3, 1 \leq i \leq n$
$f(v_i') = 4\text{i} - 1, 1 \leq i \leq n$
$f(u_i') = 4\text{i} - 2, 1 \leq i \leq n$
$f(e_i) = 4n + \text{i}, 1 \leq i \leq 4n - 1$

Subcase-3: Let m ≥2 that is, graph G obtained by one copy of path $P_n(\textit{which has } n \textit{ vertices})$ and n copies of $k_{1,m}$ and joining i^th^ vertex of P_n with an edge to pendent vertex in the i^th^ copy of $k_{1,m}$. Let $u_1, u_2, \ldots, u_n$ be the consecutive vertices of P_n, u_i' be the pendent vertex of i^th^ copy of $k_{1,m}$ which is joined with u_i for each i, v_i be the central vertex of i^th^ copy of $k_{1,m}$ for each i and v_1', v_2', $\ldots, v_{m-1}', v_m', \ldots, v_{2m-2}', v_{2m-1}', \ldots, v_{3m-3}'$, $v_{3m-2}' \ldots, v_{4m-4}', \ldots, v_{nm-n}'$ be the remaining pendent vertices of n copies of $k_{1,m}$.
Here edge set is,
$E(G) = \{e_i, e'_j / 1 \leq i \leq 3n - 1, 1 \leq j \leq nm - n\}$ such that
$e_1 = v_1u_1'$, $e_2 = u_1'u_1$, $e_3 = u_1u_2$, $e_4 = u_2u_2'$, $e_5 = u_2'v_2$, $e_6 = u_2u_3 \ldots, e_{3n-1} = u_n'v_n$, and
$e'_1 = v_1 v_1'$, $e'_2 = v_1v_2'$, $\ldots e'_{nm-n} = v_nv_{nm-n}'$.
$|V(G)| = 2\text{n} + \text{mn}$, $|E(G)| = 2\text{n} + \text{mn} - 1$ thus we have
$|V(G) \cup E(G)| = 4\text{n} + 2\text{mn} - 1$.
Define a bijection
$f: V(G) \cup E(G) \to \{1,2,3 \ldots |V(G) \cup E(G)|\}$ as follows
$f(u_i) = 3\text{i}, 1 \leq i \leq n$
$f(u_i') = 3\text{i} - 2, 1 \leq i \leq n$
$f(v_i) = 3\text{i} - 1, 1 \leq i \leq n$
$f(v_i') = 3\text{n} + \text{i}, 1 \leq i \leq n(m - 1)$
$f(e_i) = 2n + \text{nm} + \text{i}, 1 \leq i \leq 3n - 1$
$f(e_i') = 5\text{n} - 1 + \text{nm} + \text{i}, 1 \leq i \leq \text{n}(\text{m} - 1)$

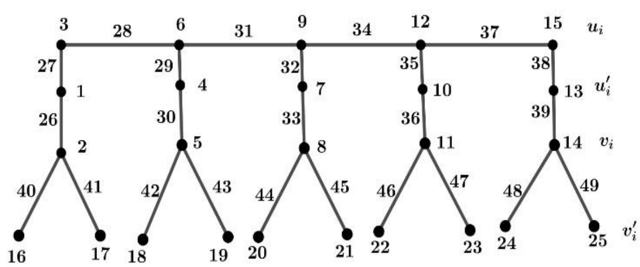
In all of the above cases f is a total neighborhood-prime labeling because if x is an arbitrary vertex of G with deg (x)≥ 2 then the set $\{f(p)/p \in N_V(\text{x})\}$ contains two consecutive integers or 1 and the set $\{f(e)/e \in N_E(\text{x})\}$ contains two consecutive integers.
Thus G is total neighborhood prime graph for all n and m.



Example:5: ***Case*: 1** Total neighbourhood prime labelling of G when $n = 6$ and $m = 3$



Example:6
***Subcase*: 2** Total neighborhood prime labeling of G when $n = 6$ and $m = 2$

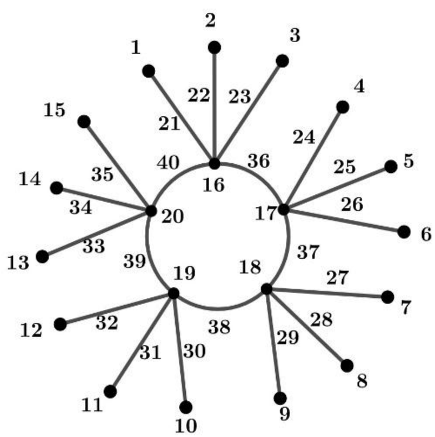


Example:7
***Subcase*: 3** Total neighborhood prime labeling of G when $n = 5$ and $m = 3$

Theorem 6. The graph $C_n \odot mK_1$ is total neighborhood prime graph for all n and m.

Proof: Let $G = C_n \odot mK_1$ be the graph with vertex set $V(G) = \{u_i, v_j / 1 \leq i \leq nm, 1 \leq j \leq n\}$ and edge set $E(G) = \{e_i, e_j / 1 \leq i \leq nm, 1 \leq j \leq n\}$ such that,
$e_1 = v_1u_1$, $e_2 = v_1u_2, \ldots, e_m = v_1u_m, e_{m+1} = v_1u_{m+1}, \ldots, e_{2m} = v_2u_{2m}, \ldots, e_{(n-1)m+1} = v_nu_{(n-1)m+1, \ldots,} e_{nm} = v_nu_{nm}$, and
$e_1' = v_1v_2$, $e_2' = v_2v_3, \ldots e_n' = v_nv_1$
$|V(G)| = \text{nm} + \text{n}$, $|E(G)| = \text{nm} + \text{n}$ thus we have
$|V(G) \cup E(G)| = 2\text{nm} + 2\text{n}$.

If $m = 1$ in $C_n \odot mK_1$ then $G = C_n \odot K_1 = S_n$ which is already proved in [7]
Define a bijection
$f: V(G) \cup E(G) \rightarrow \{1,2,3 \ldots . |V(G) \cup E(G)|\}$ as follows
$f(u_i) = \text{i}, 1 \le i \le nm$
$f(v_i) = \text{nm} + \text{i}, 1 \le i \le n$
$f(e_i) = \text{nm} + \text{n} + \text{i}, 1 \le i \le \text{nm}$
$f(e_i') = 2\text{nm} + \text{n} + \text{i}, 1 \le i \le \text{n}$
f is a total neighborhood-prime labeling because if x is an arbitrary vertex of G with deg(x)≥ 2 then the set $\{f(p)/p \in N_V\ (\text{x})\}$ contains two consecutive integers or 1 and the set $\{f(e)/e \in N_E\ (\text{x})\}$contains two consecutive integers.
Thus $G = C_n \odot mK_1$ is total neighborhood prime graph for all n and m.

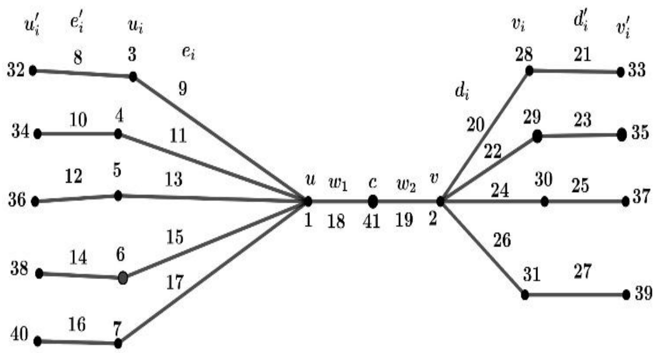


Example: 8 Total neighborhood prime labeling of $C_5 \odot 3K_1$

Theorem 7. The subdivision of bistar graph is total neighborhood prime.
Proof: Let G be the graph obtained by subdivision of bistar.
Here $|V(G)| = 2\text{m} + 2\text{n} + 3$, $|E(G)| = 2\text{m} + 2\text{n} + 2$ thus we have $|V(G) \cup E(G)| = 4\text{m} + 4\text{n} + 5$.
Without loss of generality let $m \ge n$. If $m = n = 1$, then $G = P_7$ is total neighborhood prime labeling by[5].
Let
$V(G) = \{u, v, u_i, u_i', c, v_j, v_j'/1 \le i \le m, 1 \le j \le n\}$ where
$\deg(u) = m + 1, \deg(v) = n + 1$
$\deg(u_i) = 1, \deg(v_i) = 1, 1 \le i \le m,$
$\deg(u_i') = 2, \deg(v_i') = 2, 1 \le i \le n, \deg(c) = 2$
For $i = 1,2, \ldots, m, u_i$ is adjacent to u_i' and u_i' is adjacent u.
For $i = 1,2, \ldots, n, v_i$ is adjacent to v_i' and v_i' is adjacent v.
$E(G) = \{e_i = u_i u_i' / \ i = 1,2, \ldots \ldots m\} \cup \{e'_i = u_i' u / \ i = 1,2, \ldots, m\}\} \cup \{d_i = v_i v_i' / \ i = 1,2, \ldots, n\} \cup \{d_i' = v_i' v / \ i = 1,2, \ldots, n\} \cup \{w_1 = uc\}\} \cup \{w_2 = vc\}$
Define a bijection
$f: V(G) \cup E(G) \rightarrow \{1,2,3 \ldots . |V(G) \cup E(G)|\}$ as follows
We prove f is total neighborhood prime labeling by considering the following cases.
Case-1: m and n with opposite parity.
$f(\text{u}) = 1,$
$f(\text{v}) = 2$
$f(u_i) = 2 + \text{i}, 1 \le i \le m$
$f(u_i') = 3(\text{m} + \text{n}) + 3 + 2\text{i}, 1 \le i \le m$
$f(v_i) = 3\text{m} + 2\text{n} + 4 + \text{i}, 1 \le i \le n$
$f(v_i') = 3(\text{m} + \text{n}) + 4 + 2\text{i}, , 1 \le i \le n$
$f(d_i) = 3\text{m} + 3 + 2\text{i}, 1 \le i \le n$
$f(d_i') = 3\text{m} + 4 + 2\text{i}, 1 \le i \le n$
$f(\text{c}) = 4\text{m} + 4\text{n} + 5$
$f(e_i) = \text{m} + 2 + 2\text{i}, 1 \le i \le \text{m}$
$f(e_i') = \text{m} + 1 + 2\text{i}, 1 \le i \le \text{m}$
$f(w_1) = 3\text{m} + 3, 1 \le i \le \text{m}$
$f(w_2) = 3\text{m} + 4, 1 \le i \le \text{m}$
Note that u_i' and v_i' are vertices of degree 1.
Let x be any vertex of G.
Let $x = u$, deg(x)≥ 2,then $\{u_{i\,,}\text{c}\} \subset N_V(\text{u})$.
Since $f(u_1) = 3$, $f(u_2) = 4 \ldots f(u_m) = 2 + m$, therefore gcd$\{ f(w)/w \in N_v(u)\}$ =1
Also in this case $\{e_1, \ldots ..,e_m,w_1\} \subset N_E(\text{u})$, therefore gcd$\{ f(e_1), \ldots, f(e_m), f(w_1)\}$ = gcd$\{ 3, \ldots, 3m + 2, 3m + 3\}$ =1.
Let $x = v$, deg(x)≥ 2.
In this case $\{v_{i\,,}\text{c}\} \subset N_V(\text{v})$. Since $f(v_1), f(v_2), \ldots, f(v_n)$ are consecutive integers, therefore gcd$\{ f(w)/w \in N_V(v)\}$ =1
Also in this case $\{w_2, d_1, \ldots ..,d_m\} \subset N_E(\text{v})$, and $f(w_2) = 3m + 4$, $f(d_1) = 3m + 5$ therefore gcd$\{f(w_2), f(d_1), \ldots .., , f(d_m)\} = 1$.
Let $x = u_i$, deg(x)= 2
In this case $\{u, u_i'\} \subset N_V\ (u_i)$. Since $f(u)$ =1, therefore gcd$\{ f(w)/w \in N_V(u_i)\}$ =1
Also in this case $\{e_i', e_i\} \subset N_E(u_i)$, and $f(e_i') = m + 2i + 1$, $f(e_i) = m + 2i + 2$, therefore gcd$\{ f(e)/e \in N_E(u_i)\}$ =1
Let $x = v_i$, deg(x)= 2
In this case $\{v, v_i'\} \subset N_V(v_i)$ $f(v)$=2 , and $f(v_i')$=3(m+n)+4+2i therefore gcd$\{ 2, 3(m + n) + 4 + 2i\} = 1$ And $\{d_i, d_i'\} \subset N_E(v_i)$, and $f(d_i) = 3m + 2i + 3$, $f(d_i') = 3m + 2i + 4$, therefore gcd$\{ f(d_i), f(d_i')\} = 1$
thus f is a total neighborhood prime labeling.
Case-2: m and n with same parity
$f(\text{u}) = 1,$
$f(\text{v}) = 2$
$f(u_i) = 2 + \text{i}, 1 \le i \le m$
$f(u_i') = 3(\text{m} + \text{n}) + 4 + 2\text{i}, , 1 \le i \le m$
$f(v_i) = 3\text{m} + 2\text{n} + 4 + \text{i}, 1 \le i \le n$
$f(v_i') = 3(\text{m} + \text{n}) + 3 + 2\text{i}, 1 \le i \le n$
$f(d_i) = 3\text{m} + 3 + 2\text{i}, 1 \le i \le n$
$f(d_i') = 3\text{m} + 4 + 2\text{i}, 1 \le i \le n$
$f(\text{c}) = 4\text{m} + 4\text{n} + 5$

$f(e_i) = \mathrm{m} + 2 + 2\mathrm{i}, 1 \le i \le \mathrm{m}$
$f(e_i') = \mathrm{m} + 1 + 2\mathrm{i}, 1 \le i \le \mathrm{m}$
$f(w_1) = 3\mathrm{m} + 3, 1 \le i \le \mathrm{m}$
$f(w_2) = 3\mathrm{m} + 4, 1 \le i \le \mathrm{m}$

Proof is similar to case-1 and hence f is a total neighborhood prime labeling.



Example:9 Total neighborhood prime labeling of $B_{5,4}$

IV. CONCLUSION

We have shown that comb, disjoint union of paths, disjoint union of sunlet graphs, disjoint union of wheel graphs, graph obtained by one copy of path $P_n(which\ has\ n\ vertices)$ and n copies of $k_{1,m}$ and joining i^{th} vertex of P_n with an edge to fix vertex in the i^{th} copy of $k_{1,m}$, corona product of cycle with m copies of k_1 and subdivision of bistar admit total neighborhood prime labeling. It is an open problem to find a total neighborhoood prime labeling for some new class of graphs.

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