International Journal of Scientific Research in Mathematical and Statistical Sciences
Research Paper
Vol.6, Issue.2, pp.124-130, April (2019)
E-ISSN: 2348-4519
DOI: https://doi.org/10.26438/ijsrmss/v6i2.124130

Construction of Some Block Structured (Complex) Hadamard Matrices

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Available online at: www.isroset.org

Received: 29/Mar/2019, Accepted: 14/Apr/2019, Online: 30/Apr/2019

Abstract— This article investigates some methods to construct Hadamard matrices, made up of other Hadamard blocks of lower orders. Some methods are presented to construct families of real and complex block structuted Hadamard matrices using real and complex orthogonal designs together with some suitable matrices. Other new arrays are also introduced to construct block structured (complex) Hadamard matrices, along with a few methods for their constructions. Block structured (complex) Hadamard matrices have further resulted in (block structured) weighing matrices. Also infinite families of orthogonal design of order 4*t* and type (*2t, 2t*) are also constructed which depend upon the existence of Williamson matrices and Turyn-type Williamson matrices.

Keywords—*Hadamard matrix, weighing matrix, orthogonal design, plug-in arrays, Kronecker product.*

I. INTRODUCTION

Hadamard matrices were first studied by Sylvester in 1867 [13]. He gave a recursive formula for Hadamard matrices. i.e., given a Hadamard matrix of order 2^n a new Hadamard matrix of order 2^{n+1} can be constructed. These Hadamard matrices have each $2^n \times 2^n$ blocks Hadamard matrices. These matrices are used in Walsh-Hadamard transform extensively [17]. However, significance of these matrices were later discovered by Jacques Hadamard in 1893. He had shown that Hadamard matrices are the extremal solutions of maximum determinant problem [4]. Later on such matrices were named after J. Hadamard. Hadamard matrices exist only for orders 1, 2 or a multiple of 4. It is conjectured that these matrices exist for each multiple of 4. In 1980s S. S. Agaian studied block circulant Hadamard (BCH) matrices [1]. These matrices can be regarded as the generalization of Sylvester's matrices, as BCH matrices are constructed from Hadamard blocks of order 4. Sylvester's matrices have number of Hadamard blocks even in each row (column), whereas BCH matrices have number of Hadamard blocks odd in a row (column). In 2014 Singh and Topno [11] discovered the infinite families of such matrices whose existence is dependent upon existence of Williamson matrices. Such matrices are constructed from a set of five Hadamard matrices of order 4. Although, these matrices are block circulant, we prefer to call them block structured Hadamard (BSH) matrices for the purpose of generalization. Later in 2017, Topno and Singh reduced the number of constructing blocks to be 3 [14]. But in this case Turyn's method [15] was employed, which is a special case of Williamson's matrix. Moreover they produced block structured complex Hadamard matrices. Also, they constructed families of block structured half-full weighing matrices using such matrices.

In the present paper authors have generalized the result using complex orthogonal designs. Block structured half-full weighing matrices arose as a biproduct. Apart from orthogonal designs, block structured complex Hadamard matrices are also constructed using certain arrays having special properties. Symmetric and skew-symmetric Hadamard matrices are of special interest among the authors [9, 7]. The Hadamard matrices constructed in this paper are block-wise symmetric or skew-symmetric according as the array (OD or other) used is symmetric or skew.

Section I contains the introduction of the research work presented in this paper. Section II contains the basic terminologies used in the main construction. Section III contains the main result of the paper in the forms of many theorems and corollaries. Section IV is the concluding section.

II. PRELIMINARIES

We introduce certain elementary concepts here, which are useful in constructing BSH matrices. A *Hadamard matrix* (H-matrix) *H* is a (1, -1)-matrix of order *n* such that

$HH^T=nI_n$. Here H^T is transpose of H and I_n is identity matrix of order n. This property implies that every two distinct rows (columns) of Hadamard matrix have inner product zero. An H-matrix of order mn (n, a multiple of 4) will be called *block structured Hadamard matrix* (BSH matrix) if its $n\times n$ blocks are H-matrices. A matrix H of order n is said to be a *complex Hadamard matrix* if it has entries from {±1, ±i} and $HH^*=nI_n$. Here H^* is conjugate transpose of H and I_n is identity matrix of order n. A complex Hadamard matrix of order mn (n, a multiple of 2) will be called *block structured complex Hadamard matrix* (BSCH matrix) if its $n\times n$ blocks are complex Hadamard matrices. A (0, 1, -1)- matrix W of order n is said to be a *weighing matrix* if $WW^T=kI_n$, ($k\leq n$ is a positive integer). k is called the weight of W. In fact Hadamard matrix is a special case of weighing matrix where weight is equal to the order of the matrix. A *complex weighing matrix* of weight k and order n is an $n\times n$ matrix A, with entries from {±1, ±i} satisfying $AA^*=kI_n$ ($k\leq n$). A (complex) weighing matrix of order mn (n, order of a (complex) weighing matrix) will be labeled as 'block structured' if each $n\times n$ block is a (complex) weighing matrix. An *orthogonal design* (OD) of order n and type $(s_1, s_2, ..., s_l)$ (s_i positive integers) on the commuting variables is an $n\times n$ matrix X, with entries chosen from the set $\{\pm x_1, \pm x_2, ..., \pm x_l\}$ such that $XX^T=(\sum_{i=1}^{l} s_i x_i^2) I_n$ [4]. *A complex orthogonal design* (COD) of order n and type $(s_1, s_2, ..., s_l)$ (s_i positive integers) on the real commuting variables $x_1, x_2, ..., x_l$ is an $n\times n$ matrix X, with entries chosen from $\varepsilon_1 x_1, \varepsilon_2 x_2, ..., \varepsilon_l x_l \mid \varepsilon_i$ is a fourth root of 1 satisfying $XX^*=(\sum_{i=1}^{l} s_i x_i^2) I_n$ [5]. If $A=(a_{ij})$ is an $m\times m$ matrix and $B=(b_{rs})$ is an $n\times n$ matrix, then the *Kronecker product* $A\times B$ is the $mn\times mn$ matrix given by $A\times B=(a_{ij}B)$. It is easy to see that $(A\times B)(C\times D)=AC\times BD$ and $(A\times B)^T=A^T\times B^T$. *Hadamard product* of two matrices $A=(a_{ij})$ and $B=(b_{ij})$ of order n is a matrix of order n given by $A*B=(a_{ij}b_{ij})$. The binary operation '$*$' here should not be confused with the symbol for conjugate transpose above.

Let the elements z_i of an additive abelian group G be ordered in a fixed way. Let $X\subset G$. Then the matrix $M=(m_{ij})$ defined by

$$m_{ij}=\psi(z_j-z_i),\quad \text{where}\quad \psi(z_j-z_i)=\begin{cases}1 & \text{if} \quad z_j-z_i\in X,\\ 0 & \text{otherwise},\end{cases}$$

is called *type 1* incidence matrix of X in G. A *Circulant* matrix $M=(m_{ij})$ defined by $m_{ij}=m_{1,i-j+1}$ is a special case of type 1 matrix. A circulant matrix A with its first row R_1 is denoted by A= $circ(R_1)$. Following proposition is useful in development of the results in this paper.

Proposition 2.1 *[16, p. 288] If X and Y are type 1 matrices then* $XY=YX$, $X^TY=YX^T$, $XY^T=Y^TX$, $X^TY^T=Y^TX^T$.
If $L_1, L_2, \ldots, L_n$ be n type 1 (or circulant) $(0,\pm1)$-matrices of order m which satisfy (i) $L_i * L_j=0, i\neq j$ (ii) $\sum_{i=1}^{n} L_i L_i=kI_m$ where $*$ denotes the Hadamard product, then these are called *L-matrices* of weight k. Two matrices A and B of order n are said to be *amicable* if $AB^T=BA^T$ and *anti-amicable* if $AB^T+BA^T=O$. For details of these notions authors refer to [5, 8, 6, 16].

III. MAIN RESULT

3.1 Construction from orthogonal designs

These construction theorems are dependent upon existence of certain matrices, special cases for which are known to exist.

Theorem 3. *Existence of* $(\pm1,\pm i)$ *-matrices* $A_1, A_2, ..., A_n$ *of order* m *which satisfy (i)* $A_iA_j^*=A_jA_i^*; 1\leq i\neq j\leq k$ *(ii)* $\sum_{i=1}^{n} A_iA_i^*=nmI_m$ *and orthogonal design OD(nt;t,t,…,t) implies the existence of BSCH matrix of order nmt.*

Proof. Let X be an OD($nt;t,t,\ldots,t$) on the commuting variables $0,\pm x_i, i=1,2,\ldots n$. We replace these variables with A_i's above. Then X can be written as $X=\sum_{i=1}^{n} W_i\times A_i$ where W_i are $(0,1,-1)$-matrices such that (i) $W_i * W_j=0$ if $i\neq j$. (ii) $W_iW_i^T=tI_{nt}, i=1,2,\ldots,n$. (iii) $W_iW_j^T+W_jW_i^T=0$, $1\leq i\neq j\leq n$ [4]. Let $Y=\sum_{i=1}^{n} A_i\times W_i$ then

$$\begin{aligned}YY^* &= \sum_{i=1}^{n}(A_iA_i^*\times W_iW_i)+\sum_{1\leq i\neq j\leq n}\{(A_iA_j^*\times W_iW_j)\\ &\quad +(A_jA_i^*\times W_jW_i)\}\\ &=(\sum_{i=1}^{n}A_iA_i^*)\times tI_{nt}+\sum_{1\leq i\neq j\leq n}A_iA_j^*\times(W_iW_j+W_jW_i)\\ &=nmI_m\times tI_{nt}+0\\ &=mntI_{mnt}\end{aligned}$$

i.e., Y is a complex Hadamard matrix of order mnt. Also each $nt\times nt$ block H_{ij} ; $i,j=1,2,\ldots m$ is a linear combination of W_i's over $\{1,-1,i,-i\}$ i.e., $H_{ij}=\sum_{k=1}^{n}\rho_kW_k$; $\rho_k\in\{1,-1,i,-i\}$ then

$$\begin{aligned}H_{ij}H_{ij}^* &= \sum_{k=1}^{n}(\rho_kW_k)(\rho_kW_k)^*\\ &\quad \pm(1+i)\sum_{1\leq k\neq l\leq n}(W_kW_l^T+W_lW_k^T)\\ &= ntI_{nt}\pm 0\\ &= ntI_{nt}\end{aligned}$$

i.e., each $nt\times nt$ block is a complex Hadamard matrix of order nt. Hence Y is the required BSCH matrix.

Cor. 3.2 *Existence of ($\pm 1, \pm i$)-matrices $A_1, A_2, \ldots, A_n$ of order m which satisfy (i) $A_i A_j^* = A_j A_i^*; 1 \le i \ne j \le k$ (ii) $\sum_{i=1}^{n} A_i A_i^* = nmI_m$ and orthogonal design OD(nt;t,t,…,t) implies the existence of block structured half-full weighing matrix of order nmt.*

Proof. By matrix representation of complex numbers $a+ib = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for 1 and i in above construction.

Cor. 3.3 *Existence of ($0, \pm 1, \pm i$)-matrices $A_1, A_2, \ldots, A_k$ of order m which satisfy (i) $A_i A_j^* = A_j A_i^*; 1 \le i \ne j \le k$ (ii) $\sum_{i=1}^{k} A_i A_i^* = rmI_m$ ($r \le m$) and (complex) orthogonal design OD(nt;t,t,…,t) implies the existence of block structured complex weighing matrix of order nmt and weight rmt each block having weight rt.*

Lemma 3.4 *[5] Let X be an COD of order nt and type (t,t,t,t) on the commuting variables $A_i, i = 1,2,\ldots,n$ such that $\sum_{i=1}^{n} W_i \times A_i$ where W_i are ($0, \pm 1, \pm i$)-matrices of order n then (i) $W_i * W_j = 0$ if $i \ne j$ (ii) $W_i W_i^* = tI_{nt}, i = 1,2,\ldots,n$. (iii) $W_i W_j^* + W_j W_i^* = 0$, $1 \le i \ne j \le n$, where W^* is transpose conjugate of W.*

Theorem 3.5 *Existence of ($\pm 1, \pm i$)-matrices $A_1, A_2, \ldots, A_n$ of order m which satisfy (i) $A_i A_j^* = A_j A_i^*; 1 \le i \ne j \le k$ (ii) $\sum_{i=1}^{n} A_i A_i^* = nmI_m$ and COD ($nt; t, t, \ldots, t$) implies the existence of BSCH matrix of order nmt.*

Proof. Let X be an COD(*nt;t,t,…,t*) on the commuting variables $0, \pm x_i, i = 1,2,\ldots n$. We replace these variables with A_i's above. Then X can be written as $X = \sum_{i=1}^{n} W_i \times A_i$ where W_i are ($0, \pm 1, \pm i$)-matrices satisfying conditions of lemma (3.4). Define $Y = \sum_{i=1}^{n} A_i \times W_i$ then using same argument as theorem (3.1) and employing the lemma (3.4) above we get $YY^* = nmtI_{nmt}$. i.e., Y is a complex Hadamard matrix of order *nmt*. Now, each $nt \times nt$ block of Y is $H_{rs} = \sum_{k=1}^{4} \rho_k W_k;\ r,s = 1,2,\ldots,m,$ which is a COD(*nt;t,t,t,t*) on commuting variables $\pm 1, \pm i$ since Hadamard product W_i and W_j is zero matrix. Hence each H_{rs} is a complex H-matrix.

Theorem 3.6 *Existence of ($1, -1$)-matrices $A_1, A_2, \ldots, A_n$ of order m which satisfy (i) $A_i A_j^T = A_j A_i^T; 1 \le i \ne j \le k$ (ii) $\sum_{i=1}^{n} A_i A_i^T = nmI_m$ and COD(nt;t,t,…,t) implies the existence of BSCH matrix of order nmt.*

Proof. Proof is Similar to theorem (3.1) and (3.5).

Theorem 3.7 *Existence of ($1, -1$)-matrices $A_1, A_2, \ldots, A_n$ of order m which satisfy (i) $A_i A_j^T = A_j A_i^T; 1 \le i \ne j \le k$ (ii) $\sum_{i=1}^{n} A_i A_i^T = nmI_m$ and OD(nt;t,t,…,t) implies the existence of BSCH matrix of order nmt.*

Proof. Proof is Similar to theorem (3.1) and (3.5) using lemma (3.4).

Remark 3.8 *For $n = 4$ and 8 matrices A_i are known to exist for certain orders. These matrices are called Williamson-type matrices and Eight- Williamson-type matrices respectively.*

Cor. 3.9 *[11] Existence of Williamson matrices A, B, C, D of order n implies the existence of BCH matrix of order 4n, whose each 4×4 block is Hadamard matrix.*

Proof. Use Williamson's array as OD.

Cor. 3.10 *[12] Existence of symmetric, circulant ($0, \pm 1$)-matrices $A_1, A_2, \ldots, A_K$ of order m satisfying $\sum_{i=1}^{k} A_i^2 = rtI_n, r \le m$ and orthogonal design OD(nt;t,t,…,t) implies the existence of block-circulant weighing matrix of order nmt and weight rmt each $nt \times nt$ block having weight rt.*

Remark 3.11 *These weighing matrices can be used to accelerate and compress deep neural networks as shown by Ding et al. [3].*

So far block structured (complex) Hadamard matrices are constructed using real and complex orthogonal designs. Now we construct infinite families of an orthogonal design.

3.1.1 Infinite families of OD(4t;2t,2t)

Theorem 3.12 *Existence of Williamson matrices imply the existence of OD(4t;2t,2t).*

Proof. The proof is constructive. Let a and b be commuting variables.

Define $h_1 = \begin{bmatrix} -a & b \\ b & a \end{bmatrix}, h_2 = \begin{bmatrix} b & a \\ a & -b \end{bmatrix}$ then

$$h_1^2 = h_2^2 = (a^2 + b^2) I_2$$
$$h_1 h_2^T + h_2 h_1^T = O$$
$$h_1^T = h_1, h_2^T = h_2.$$

Again define

$$H_1 = \begin{bmatrix} h_1 & h_1 \\ -h_1 & h_1 \end{bmatrix}, H_2 = \begin{bmatrix} h_1 & -h_1 \\ h_1 & h_1 \end{bmatrix}, H_3 = \begin{bmatrix} h_2 & h_2 \\ h_2 & -h_2 \end{bmatrix}, H_4 = \begin{bmatrix} -h_2 & h_2 \\ h_2 & h_2 \end{bmatrix}$$

then
T
Now using following Hadamard transform we obtain matrices $\overline{H_i}$, $i = 1,2,3,4$.

$$\begin{bmatrix} \overline{H}_1 \\ \overline{H}_2 \\ \overline{H}_3 \\ \overline{H}_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{bmatrix}$$

such that

$$H_i \overline{H}_j^T + \overline{H}_j H_i^T = \pm 2(a^2 + b^2) I_4; 1 \le i \ne j \le 4$$
$$\overline{H}_i \overline{H}_i^T = 2(a^2 + b^2) I_4; 1 \le i \le 4$$
$$\overline{H}_i \overline{H}_j^T + \overline{H}_j \overline{H}_i^T = O; 1 \le i \ne j \le 4.$$

Replacing $\hat{1}$ by $\overline{H}_1$ and $\underline{1}, \underline{2}, \underline{3}, \underline{4}$ by H_1, H_2, H_3, H_4 respectively in the result of Singh and Topno [11] and in Topno and Singh [14] we get the infinite family of $OD(4t; 2t, 2t)$. Former depends upon the existence of Williamson matrices and the later upon Turyn-type Williamson matrices.

3.2 Construction from certain arrays

In this section BSCH matrices are constructed using certain arrays having special properties. Some of their construction methods are also discussed.

Theorem 3.13 *Let M and N be two amicable ($\pm 1, \pm i$)-matrices of order m such that $MM^* + NN^* = 2mI_m$, and $(M \pm N)/2$ is a $(\pm i, \pm 1)-$matrix, then there exists a BSCH matrix of order mn whose blocks are complex H-matrices of order n, where n is the order of a complex H-matrix.*

Proof. Define $A = (M + N)/2$ and $B = (M - N)/2$ and let C be a complex Hadamard matrix of order n. Then there exists a complex Hadamard matrix D of order n such that $CD^* + DC^* = 0$ [16, p. ~296]. Define another matrix $H = A \times C + B \times D$.Then

$$\begin{aligned}
HH^* &= (AA^* \times CC^* + BB^* \times DD^*) + (AB^* \times CD^*) + (BA^* + DC^*) \\
&= (\frac{M+N}{2})(\frac{M^*+N^*}{2}) \times nI_n + (\frac{M-N}{2})(\frac{M^*-N^*}{2}) \times nI_n \\
&\quad + (\frac{M+N}{2})(\frac{M^*-N^*}{2}) \times CD^* + (\frac{M-N}{2})(\frac{M^*+N^*}{2}) \times DC^* \\
&= \frac{1}{4}\{(MM^* + NN^* + MN^* + NM^*) \times nI_n \\
&\quad + (MM^* + NN^* - MN^* - NM^*) \times nI_n \\
&\quad + (MM^* - NN^* - MN^* + NM^*) \times CD^* \\
&\quad + (MM^* - NN^* + MN^* - NM^*) \times DC^*\} \\
&= \frac{1}{4}(2mI_m \times nI_n + 2mI_m \times nI_n) \\
&= mnI_{mn}
\end{aligned}$$

i.e., H is a complex Hadamard matrix. Let $(i, j)^{th}$ element of A, B, M and N be denoted by a_{ij}, b_{ij}, m_{ij}, and n_{ij} respectively. Now, each $n \times n$ block $H_{ij}; i, j = 1,2,\ldots,m$ of H is given by

$$\begin{aligned}
H_{ij} &= a_{ij} \times C + b_{ij} \times D \\
&= \frac{1}{2}(m_{ij} + n_{ij}) \times C + \frac{1}{2}(m_{ij} - n_{ij}) \times D \\
&= \frac{1}{2}\{m_{ij} \times (C + D) + n_{ij} \times (C - D)\}
\end{aligned}$$

and

$$\begin{aligned}
H_{ij}H_{ij}^* &= \frac{1}{4}\{m_{ij} \times (C+D) + n_{ij} \times (C-D)\}\{\overline{m}_{ij} \times (C^* + D^*) + \overline{n}_{ij} \times (C^* - D^*)\} \\
&= \frac{1}{4}\{m_{ij}\overline{m}_{ij} \times (C+D)(C^*+D^*) + n_{ij}\overline{n}_{ij} \times (C-D)(C^*-D^*)\} \\
&\quad + m_{ij}\overline{n}_{ij} \times (C+D)(C^*-D^*) + n_{ij}\overline{m}_{ij}(C-D)(C^*+D^*) \\
&= \frac{1}{4}\{m_{ij}\overline{m}_{ij} \times (CC^* + DD^* + CD^* + DC^*) \\
&\quad + n_{ij}\overline{n}_{ij} \times (CC^* + DD^* - CD^* - DC^*) \\
&\quad + m_{ij}\overline{n}_{ij} \times (CC^* - DD^* - CD^* + DC^*) \\
&\quad + n_{ij}\overline{m}_{ij} \times (CC^* - DD^* + CD^* + DC^*)\} \\
&= \frac{1}{4}\{m_{ij}\overline{m}_{ij} \times (2nI_n + 0) + n_{ij}\overline{n}_{ij} \times (2nI_n - 0) \\
&\quad + m_{ij}\overline{n}_{ij} \times 2(CC^* - DD^*)\} \quad [m_{ij}\overline{n}_{ij} = n_{ij}\overline{m}_{ij} \quad as \quad MN^* = NM^*] \\
&= \frac{1}{4}((m_{ij}\overline{m}_{ij} + n_{ij}\overline{n}_{ij}) \times 2nI_n)
\end{aligned}$$

Now $m_{ij}\overline{m}_{ij} + n_{ij}\overline{n}_{ij} = \begin{cases} 2 & if \quad i = j \\ 0 & otherwise \end{cases}$

[Since $MN^* + NM^* = 2mI_m$]

Therefore

$$\begin{aligned}
H_{ij}H_{ij}^* &= \frac{1}{4}\{2 \times 2nI_n\} \\
&= nI_n
\end{aligned}$$

Hence H_{ij} is a complex Hadamard matrix of order *n*.

Remark 3.14 *If matrices C and D in above theorem are symmetric or skew-symmetric then resulting matrix is block-wise symmetric or skew.*

Following result from Geramita et. al. [5] is a corollary to the theorem (3.13) .

Cor. 3.15 *Let M and N be circulant* $(0,\pm1,\pm i)$*-matrices of order n such that* $MM^*+NN^*=fI_n$, $f\le n$ *and* $(M\pm N)/2$ *is a* $(\pm i,\pm1)-$ *matrix, then there exists a block structured complex weighing matrix of weight* f *and order* $2n$.

In the above constructions BSCH matrices depend upon the existence of matrices *M* & *N*. Now we discuss some of the methods to construct *M* & *N*.

3.2.1 Construction of matrices M & N

Theorem 3.16 *If A, B, C, D are Williamson-type matrices of order n (odd) then* $M=\frac{1}{2}(1+i)A+\frac{1}{2}(1-i)B$ *and* $M=\frac{1}{2}(1+i)C+\frac{1}{2}(1-i)D$.

Proof. Straightforward verification.

Theorem 3.17 *If there exist complex Hadamard matrix* H_{2n} *of the form* $\begin{bmatrix} A & B \\ -B^* & A^* \end{bmatrix}$ *or* $\begin{bmatrix} C & D \\ D^* & -C^* \end{bmatrix}$ *such that* $(A\pm B)/2\ \&(C\pm D)/2$ *are* $(\pm i,\pm1)-$ *matrices, then M=A and N=B are required M & N matrices of (3.13).*

Theorem 3.18 *[10] Existence of BIBD with* $v=2k^2-2k+1, b=2v, r=2v (k\ge 2), \lambda=1$ *implies the existence of matrices* M & N *of orders* $2(2k^2-2k+1)$.

Now we discuss another method of construction for BSCH matrix.

Theorem 3.19 *If there exist four* $(0,\pm1)$ *-matrices* $X_i; 1\le i\le 4$ *of order 2t such that*

$$
\begin{aligned}
&(i)\ \ X_1X_1^T=X_3X_3^T,\ X_2X_2^T=X_4X_4^T\\
&\qquad X_1X_1^T+X_2X_2^T=tI_{2t},\\
&(ii)\ X_1X_4^T=-X_3X_2^T,\ X_4X_1^T=-X_2X_3^T\\
&(iii)\ X_1X_2^T=X_2X_1^T=X_1X_3^T=X_3X_1^T=X_2X_4^T\\
&\qquad =X_4X_2^T=X_3X_4^T=X_4X_3^T=0
\end{aligned}
$$

and two L-matrices L_1 and L_2 of order m and weight m

(i) then there exists a complex Hadamard matrix of order *2mt*

(ii) and if two of X_i's have same sign then there exists a BSCH matrix of order *2mt* with complex Hadamard blocks of order *2t*.

Proof. Define

$$
\begin{aligned}
A_1 &= L_1+L_2,\ A_2=L_1+L_2\\
A_3 &= -L_1+L_2,\ A_4=L_1-L_2
\end{aligned}
\tag{3.1}
$$

then $A_i; 1\le i\le 4$ are type 1 (or circulant) matrices of order m such that

$$
\begin{aligned}
&(i)\ A_1A_1^T+A_3A_3^T=2mI_m=A_2A_2^T+A_4A_4^T\\
&(ii)\ A_iA_j=A_jA_i,\ A_i^TA_j=A_jA_i^T,\ A_iA_j^T=A_j^TA_i,\\
&\qquad A_i^TA_j^T=A_j^TA_i^T;\ i,j\in\{1,2,3,4\}\\
&(iii)\ A_1A_2^T-A_2A_1^T+A_3A_4^T-A_4A_3^T=0.
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
\textit{Let}\quad H = &\ \frac{1}{2}(1+i)\{A_1\times X_1+A_2^T\times X_2+A_3\times X_3+A_4^T\times X_4\}\\
&+\frac{1}{2}(1-i)\{A_1^T\times X_2+A_2\times X_1+A_3^T\times X_4+A_4\times X_3\}
\end{aligned}
\tag{3.3}
$$

Then

.

$$
\begin{aligned}
HH^* = &\ \sum_{i=1}^{4}A_iA_i\times X_iX_i^T\\
&+\frac{1}{2}[\{(A_1A_4-A_4A_1)+(A_2A_3-A_3A_2)\}\times X_1X_4^T\\
&+\{(A_4^TA_1^T-A_1^TA_4^T)+(A_3^TA_2^T-A_2^TA_3^T)\}\times X_4X_1^T]\\
&\frac{i}{2}[\{(A_1A_2^T-A_2A_1^T)+(A_3A_4^T-A_4A_3^T)\}\times X_1X_1^T\\
&+\{(A_2^TA_1-A_1^TA_2)+(A_4^TA_3-A_3^TA_4)\}\times X_2X_2^T\\
&+\{(A_1A_3-A_3A_1)+(A_4A_2-A_2A_4)\}\times X_1X_4^T\\
&+\{(A_4^TA_2^T-A_2^TA_4^T)+(A_3^TA_1^T-A_1^TA_3^T)\}\times X_4X_1^T]
\end{aligned}
$$

then by virtue of (3.2)

$$
\begin{aligned}
HH^* = &\ \sum_{i=1}^{4}A_iA_i^T\times X_iX_i^T\\
= &\ (A_1A_1^T+A_3A_3^T)\times X_1X_1^T+(A_2A_2^T+A_4A_4^T)\times X_2X_2^T\\
= &\ 2mI_m\times(\sum_{i=1}^{2}X_iX_i^T)\\
= &\ 2mI_m\times tI_{2t}\\
= &\ 2mtI_{2mt}.
\end{aligned}
$$

$\Rightarrow H$ is a complex Hadamard matrix of order *2mt*. Also in *2t*×*2t* partition of H, each *pq*th-block H_{pq} is of the form

$$
H_{pq}=\frac{1}{2}(1+i)(\pm X_1\pm X_2\pm X_3\pm X_4)+\frac{1}{2}(1-i)(\pm X_2\pm X_1\pm X_4\pm X_3).
$$

If two of X_i*'s* have same sign then $H_{pq}H^*_{pq}=2tI_{2t}$, which can be verified directly.

Cor. 3.20 *If there exist L-matrices and* X_i*'s-matrices of theorem (3.19) then there exists a Hadamard matrix of order 4mt.*

Proof. Render the matrix H obtained in theorem (3.19) in the form *H=P+iQ* then, following the result of Craigen et. al. [2] the required matrix is

$$P \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + Q \times \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Cor. 3.21 *If there exist X_i's -matrices and anti-amicable L-matrices of theorem (3.19)*
(i) then there exists a BSH matrix of order *2mt* with Hadamard blocks of order *2t*
(ii) and if two of X_i*'s* have same sign then there exists a BSH matrix of order *2mt* with Hadamard blocks of order *2t*.

Proof. *Define* $H = \frac{1}{2}\{(A_1 + A_2)\times X_1 + (A_1^T + A_2^T)\times X_2 + (A_3 - A_4)\times X_3 + (A_3^T - A_4^T)\times X_4\}$

and follow the argument of (3.19).

Theorem 3.22 *If there exist X_i-matrices of theorem (3.19) and two L-matrices L_1 and L_2 of order m and weight k, ($k \le m$)(i) then there exist complex weighing matrix of order 2mt and weight 2kt and (ii) if two of X_i's have same sign then there exist complex weighing matrix of order 2mt and weight 2kt each 2t×2t blocks being a weighing matrices of weight k.*

Matrices X_i*'s* used in this section exist for certain orders. Following section deals with some of the methods of construction of these matrices.

3.2.2 Construction of matrices X_i *;* $1 \le i \le 4$

Theorem 3.23 *If* X_{i_n} *;* $1 \le i \le 4$ *be the matrices of order n satisfying theorem (3.19) then*

$$X_{1_{2n}} = \begin{bmatrix} X_{1_n} & X_{1_n} \\ X_{2_n} & -X_{2_n} \end{bmatrix}, X_{2_{2n}} = \begin{bmatrix} X_{2_n} & X_{2_n} \\ X_{1_n} & -X_{1_n} \end{bmatrix}, X_{3_{2n}} = \begin{bmatrix} X_{3_n} & X_{3_n} \\ X_{4_n} & -X_{4_n} \end{bmatrix},$$
$$X_{4_{2n}} = \begin{bmatrix} X_{4_n} & X_{4_n} \\ X_{3_n} & -X_{3_n} \end{bmatrix}$$

Proof. Straightforward verification.

Remark 3.24 *For n=2 following matrices satisfy the equation:*

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, X_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, X_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
.

Following section deals with the construction of *L*-matrices.

3.2.3 Construction of L-matrices

Theorem 3.25 *If there exist two Golay pairs A & B of order k then L_1=circ(A+B)/2 and L_2=circ(A-B)/2 are two L-matrices of same order.*

Theorem 3.26 *If there exist two L-matrices of order m and two L-matrices of order n then there exist two L-matrices of order mn.*

Proof. Let l_1 & l_2 be *L*-matrices of order *m* and L_1 & L_2 be *L*-matrices of order *n*. Define

$$P = \frac{1}{2}\{(l_1 + l_2)\times(L_1 + L_2) + (l_1 - l_2)\times(L_2 - L_1)\}$$
$$Q = \frac{1}{2}\{(l_1 + l_2)\times(L_1 - L_2) + (l_1 - l_2)\times(L_2 + L_1)\}.$$

Then it can be directly verified that P & Q are required *L*-matrices of order *mn*.

Remark 3.27 *So far only one pair of anti-amicable L-matrices are found viz circ(1 -1 0 0) and circ(0 0 1 1).*

IV. CONCLUSION & FUTURE SCOPE

We have extended the results of Agaian and other authors in this paper by constructing block structured (complex) Hadamard matrices of orders *nmt*, where *nt* is order of a (complex) Orthogonal Design and *m* is order of some suitable matrices. Moreover BSCH matrices of orders *mn* and *2mt* are also constructed, where *m, n, t* are defined in theorems 3.13 and 3.19. These methods use matrices X_i, *M*, *N* and *L*. Their methods of consructions are also discussed. In addition to this infinite families of OD(*4t; 2t, 2t*) are constructed. Block structured weighing matrices are also part of the above results. Orthogonal Designs and other arrays X_i, *M*, *N* and *L* of new orders will produce BSCH matrices of new orders.

REFERENCES

[1] S. S. Agaian, *"Hadamard matrices and their Applications"*, Springer- Verlag, Berlin Heidelberg, New York, Tokyo, pp.**78-102**, **1985**.

[2] R. Craigen, W. Holzmann and H. Kharaghani, "*Complex Golay sequences: structure and applications*" Discrete Math., Vol. **252** ,pp 73–89, **2002**.

[3] C. Ding, S. Liao, Y. Wang, et. al. "*CircNN: Accelerating and compressing deep neural networks using block-circulant weight matrices*", International symposium on microarchitecture (MICRO) pp.**395-408**, **2017**..

[4] J. Hadamard, "*Résolutiond'une question relative aux déterminants*", Bull. Sci. Math., Vol. **17, pp.** 240-246, **1893**.

[5] A. V. Geramita and J. M. Geramita, "*Complex orthogonal designs*" J. Combin. Theory Ser. A, Vol. **25** , pp.211–225, **1978**.

[6] M. Hall (Jr.), *"Combinatorial Theory"*, Wiley- Interscience, 2nd edition, pp.**238-263**, **1988**.

[7] O. Mateo, D. Ž. Doković and I. S. Kotsireas, "*Symmetric Hadamard matrices of order 116 and 172 exist*", Spec. Matrices, Vol. **3**, pp.227–234, **2015**.

[8] J. Seberry, *"Orthogonal Designs: Hadamard Matrices, Quadratic Forms and Algebras"*, Springer International Publishing **AG 2017**.

[9] J. Seberry and N. A. Balonin, "*Two infinite families of symmetric Hadamard matrices*", Australas. J. Combin. Vol. **69,** No. **3**, pp.349–357, **2017**.

[10] M. K. Singh, K. Sinha and S. Kageyama, "*A construction of Hadamard matrices from BIBD(*$2k^2 - 2k + 1, k, 1$*)*", Australas. J. Combin., Vol. **26**, pp.93–97, **2002**.

[11] M. K. Singh, and S. N. Topno, "*On the construction of Hadamard matrices of order 4n (n odd, n*$\geq$*3) with Hadamard blocks of order 4*", Acta Cient. Indica, Vol **XL M**, No. **3,** pp.309–313, **2014**.

[12] M. K. Singh, S. N. Topno and T. Paswan, "*Anticirculant structured block weighing matrices from Williamson matrices*", Int. J. Math. Trends Tech., Vol. **52**, No. **4**, pp.43-47, **2017**.

[13] J. J. Sylvester, "*Thoughts on inverse orthogonal matrices, simultaneous sign successions, and tessellated pavements two or more colours, with applications to Newton's rule, ornamental tile-work, and the theory of numbers*", Phil. Mag. Vol. **34**, No. **1**, pp. 461–475, **1867**.

[14] S. N. Topno and M. K. Singh, "*Construction of Block Structured Complex Hadamard Matrices*", Acta Cient. Indica, Vol. **XLIII M**, No. **2** , pp.109–115, **2017**.

[15] R. J. Turyn,"*An infinite Class of Williamson Matrices*", J. Combin. Theory Ser. A, Vol. **12**, pp. 319–321, **1972**.

[16] W. D. Wallis, A. P. Street and J. S. Wallis, *Combinatorics: Room Squares, Sum-Free Sets, Hadamard matrices*, Springer-Verlag, Berlin-Heidelberg, New York, pp.**279-299**, **1972**.

[17] P. Mazumder, R. Middya and M. K. Naskar, "*Hardware Implementation of Fast Recursive Walsh-Hadamard Transform*", International Journal of Computer Sciences and Engineering, Vol.**7,** Issue.**1,** pp.28-32**, 2019.**

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