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Parametric Generalizations of 'Useful' R-Norm Fuzzy Information Measures

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***Abstract*—** In this paper, we introduce a family of 'useful' R-norm fuzzy information measures. These measures include no parameter, one parameter and two parameters. We show that these proposed measures are valid fuzzy information measures and are the generalizations of various 'useful' R-norm fuzzy information measures. Further, to establish its validity we take a numerical data and obtain the results by using R-software. In addition, we study the monotonic behaviour of the proposed measures with respect to the parameters.

***Keywords*—** Fuzzy Set, Fuzzy Information Measure, 'Useful' Information Measure, R-Norm Fuzzy Information Measure.

I. INTRODUCTION

Information theory basically deals with the problems related to communication systems. In recent years of literature, various measures of information were proposed by several authors. The first measure of information was given by Shannon [1]. Later, in order to overcome certain limitations of this measure, various generalized measures were studied by Reyni [2], Sharma and Taneja [3], Safeena et al. [4], Saima et al. [5] etc.

Later in 1965, L. A. Zadeh [6] developed the idea of fuzzy set to measure the ambiguity or vagueness or fuzziness in human decision making. In this regard, De Luca and Termini [7] defined fuzzy information measure (FIM) corresponding to Shannon's entropy for the first time. He also proposed four basic properties for any information measure to be valid FIM. These properties are sharpness, maximality, resolution and symmetry.

Consider a set of probability distributions $\Delta_n\left(n \geq 2\right)$, associated with discrete random variable X taking values $x_1, x_2, \ldots, x_n$, where

$$\Delta_n = \left\{ P = \left(p_1, p_2, \ldots, p_n\right), 0 \leq p_i \leq 1, i = 1,2,\ldots,n \ \& \ \sum_{i=1}^{n} p_i = 1 \right\}.$$

For distribution P and for $R \in \Re^+$ $\left(\Re^+ = \left\{R : R > 0\left(\neq 1\right)\right\}\right)$, Boekee and Lubbe [8] defined a new information measure called R-norm information measure that is given as

$$H_R\left(P\right) = \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right]; R > 0, R \neq 1. \tag{1}$$

Further, many researchers which include Hooda and Ram [9], Tomar and Ohlan [10], Safeena et al. [11] etc. proposed their measures of R-norm information. Hooda [12] gave the following R-norm fuzzy information measure (RFIM)

$$H_R\left(A\right) = \frac{R}{R-1}\sum_{i=1}^{n}\left[1 - \left\{\mu_A^R\left(x_i\right) + \left(1 - \mu_A\left(x_i\right)\right)^R\right\}^{\frac{1}{R}}\right];$$

$$R > 0, R \neq 1. \tag{2}$$

The measure given in (2) is the equivalent fuzzy measure of Boekee and Lubbe [8].

Belis and Guiasu [13] were the first to consider the attribute of the events. They added the concept of utility to the already existing definition of entropy. Longo [14] characterized a qualitative-quantitative measure and named it as 'useful' measure of information. The 'useful' measure is given as

$$H(P;U) = -\sum_{i=1}^{n} u_i\, p_i \log p_i ; u_i > 0,\ \ 0 < p_i \le 1\ \ \& \sum_{i=1}^{n} p_i = 1.$$

The first 'useful' R-norm information measure (RIM) was given by Singh et al. [15]. After that, various researchers such as Kumar [16], Hooda et al. [17], Saima et al. [18] contributed in this field. In this paper, the 'useful' concept is added to the measures of RFIM.

In Section II, three new 'useful' RFIMs are proposed. In Section III, the properties of these measures are studied and a numerical example is given for this purpose. In Section IV, the monotonic behaviour of the measures is studied and finally in the Section V, the conclusion of the whole research paper is presented.

II. 'Useful' R-norm Fuzzy Information Measures

Here, we describe a family of 'useful' RFIMs. Initially, we define a 'useful' RFIM as:

$$H_R(A;U) = \frac{R}{R-1}\left[\frac{n\sum_{i=1}^{n} u_i\left[1-\left\{\mu_A^R(x_i)+(1-\mu_A(x_i))^R\right\}^{\frac{1}{R}}\right]}{\sum_{i=1}^{n} u_i}\right];$$

$$R > 0(\neq 1) \& u_i > 0 \qquad (3)$$

Further, we introduce a one parametric 'useful' RFIM as:

$$H_R^{\beta}(A;U) = \frac{R-\beta+1}{R-\beta}\left[\frac{n\sum_{i=1}^{n} u_i\left[1-\left\{\mu_A^{R-\beta+1}(x_i)+(1-\mu_A(x_i))^{R-\beta+1}\right\}^{\frac{1}{R-\beta+1}}\right]}{\sum_{i=1}^{n} u_i}\right]$$

$$R > 0(\neq 1);\ 0 < \beta \le 1;\ R > \beta\ \&\ u_i > 0 \qquad (4)$$

In order to increase the flexibility, we introduce a two parametric 'useful' RFIM as:

$$H_R^{\alpha,\beta}(A;U) = \frac{R-\beta+\alpha}{R-\beta}\left[\frac{n\sum_{i=1}^{n} u_i\left[1-\left\{\mu_A^{\frac{R-\beta+\alpha}{\alpha}}(x_i)+(1-\mu_A(x_i))^{\frac{R-\beta+\alpha}{\alpha}}\right\}^{\frac{\alpha}{R-\beta+\alpha}}\right]}{\sum_{i=1}^{n} u_i}\right]$$

$$R > 0(\neq 1);\ 0 < (\alpha,\beta) \le 1\ \& u_i > 0 \qquad (5)$$

III. Properties of Proposed 'Useful' R-Norm Fuzzy Information Measures

With the help of a numerical example, we demonstrate that the proposed 'useful' RFIMs satisfy all the four properties.

1. Sharpness: Using the R-software, we observe that all the measures defined in (3), (4) and (5) give minimum value if and only if the set A is crisp. That is, for $\mu_A(x_i) = 0\, or\, 1$ the measures (3), (4) and (5) yield the value zero.

2. Maximality: This property states that fuzzy entropy is maximum if and only if A is most fuzzy set. To show that measures defined in (3), (4) and (5) satisfy maximality property, we first differentiate $H_R(A;U)$, $H_R^{\beta}(A;U)$ and $H_R^{\alpha,\beta}(A;U)$ with respect to $\mu_A(x_i)$ We get the following three expressions and obtain the results numerically in the subsequent tables:

$$\frac{\partial H_R(A;U)}{\partial \mu_A(x_i)} = -\frac{nR}{R-1}\left[\frac{\sum_{i=1}^{n} u_i\left\{\mu_A^R(x_i)+(1-\mu_A(x_i))^R\right\}^{\frac{1-R}{R}}}{\sum_{i=1}^{n} u_i}\left\{\mu_A^{R-1}(x_i)-(1-\mu_A(x_i))^{R-1}\right\}\right] \qquad (6)$$

$$\frac{\partial H_R^{\beta}(A;U)}{\partial \mu_A(x_i)} = -n\left(\frac{R-\beta+1}{R-\beta}\right)\left[\frac{\sum_{i=1}^{n} u_i\left\{\mu_A^{R-\beta+1}(x_i)+(1-\mu_A(x_i))^{R-\beta+1}\right\}^{\frac{\beta-R}{R-\beta+1}}}{\sum_{i=1}^{n} u_i}\left\{\mu_A^{R-\beta}(x_i)-(1-\mu_A(x_i))^{R-\beta}\right\}\right] \qquad (7)$$

$$\frac{\partial H_R^{\alpha,\beta}(A;U)}{\partial \mu_A(x_i)} = -n\left(\frac{R-\beta+\alpha}{R-\beta}\right)\left[\frac{\sum_{i=1}^{n} u_i\left\{\mu_A^{\frac{R-\beta+\alpha}{\alpha}}(x_i)+(1-\mu_A(x_i))^{\frac{R-\beta+\alpha}{\alpha}}\right\}^{\frac{\beta-R}{R-\beta+\alpha}}}{\sum_{i=1}^{n} u_i}\left\{\mu_A^{\frac{R-\beta}{\alpha}}(x_i)-(1-\mu_A(x_i))^{\frac{R-\beta}{\alpha}}\right\}\right] \qquad (8)$$

Table 1: Values of expression (6), (7) & (8) at $0 \le \mu_A(x_i) < 0.5$, $\alpha = 0.62$ **and** $\beta = 0.31$

u_i	R	$\mu_A(x_i)$	$\frac{\partial H_R(A;U)}{\partial \mu_A(x_i)}$	$\frac{\partial H_R^{\beta}(A;U)}{\partial \mu_A(x_i)}$	$\frac{\partial H_R^{\alpha,\beta}(A;U)}{\partial \mu_A(x_i)}$
5	0.81	0.10	7.4068	7.1459	7.0260
2		0.12			
4		0.19			
1		0.23			
6		0.36			
3		0.49			

From the Table 1, we find that the values of expressions (6), (7) and (8) are greater than zero in the interval $0 \le \mu_A(x_i) < 0.5$. This shows that the measures defined in (3), (4) and (5) are increasing functions w.r.t. $\mu_A(x_i)$.

Table 2: Values of expression (6), (7) & (8) for $0.5 < \mu_A(x_i) \le 1$, $\alpha = 0.62$ **and** $\beta = 0.31$

u_i	R	$\mu_A(x_i)$	$\frac{\partial H_R(A;U)}{\partial \mu_A(x_i)}$	$\frac{\partial H_R^{\beta}(A;U)}{\partial \mu_A(x_i)}$	$\frac{\partial H_R^{\alpha,\beta}(A;U)}{\partial \mu_A(x_i)}$
5	0.81	0.52	-7.3836	-6.5208	-6.2362
2		0.57			
4		0.62			
1		0.75			
6		0.89			
3		0.94			

From Table 2, we see for all the defined 'useful' RFIMs, the expressions (6), (7) and (8) are less than zero in the interval $0.5 < \mu_A(x_i) \le 1$. This implies that defined measures are decreasing functions in terms of $\mu_A(x_i)$.

For $\mu_A(x_i) = 0.5, R = 0.81, \alpha = 0.62 \ \& \ \beta = 0.31$, we get

$$\frac{\partial H_R(A;U)}{\partial \mu_A(x_i)} = \frac{\partial H_R^{\beta}(A;U)}{\partial \mu_A(x_i)} = \frac{\partial H_R^{\alpha,\beta}(A;U)}{\partial \mu_A(x_i)} = 0.0 \qquad (9)$$

Thus, from Tables 1, 2 and equation (9), we conclude that the proposed 'useful' RFIMs are concave functions with global maximum at $\mu_A(x_i) = 0.5$.

3. Resolution: With the help of following tables, we show that the proposed measures satisfy the resolution property i.e., fuzzy entropy decreases if the set is sharpened.

Table 3: Numerical values of sharpened version and actual version in the interval $0 \le \mu_A(x_i) < 0.5$ **for** $\alpha = 0.62$, $\beta = 0.31$ **&** $R = 0.81$.

u_i	5	2	4	1	6	3
$\mu_A(x_i)$	0.23	0.12	0.36	0.49	0.09	0.19
$H_R(A;U)$	3.2382					
$H_R^{\beta}(A;U)$	2.3218					
$H_R^{\alpha,\beta}(A;U)$	2.1345					
$\mu_{A^*}(x_i)$	0.20	0.0	0.32	0.41	0.02	0.14
$H_R(A^*;U)$	2.4428					
$H_R^{\beta}(A^*;U)$	1.7247					
$H_R^{\alpha,\beta}(A^*;U)$	1.5849					

Table 4: Numerical values of sharpened version and actual version in the interval $0.5 < \mu_A(x_i) \le 1$ **for** $\alpha = 0.62$, $\beta = 0.31$ **&** $R = 0.81$.

u_i	5	2	4	1	6	3
$\mu_A(x_i)$	0.89	0.69	0.60	0.76	1.0	0.52
$H_R(A;U)$	2. 6340					
$H_R^{\beta}(A;U)$	2. 0067					
$H_R^{\alpha,\beta}(A;U)$	1. 8853					
$\mu_{A^*}(x_i)$	0.92	0.71	0.62	0.80	1.0	0.56
$H_R(A^*;U)$	2. 4939					
$H_R^{\beta}(A^*;U)$	1. 8760					
$H_R^{\alpha,\beta}(A^*;U)$	1. 7578					

We observe from the Tables 3 and 4 that the 'useful' RFIMs have lesser values if the membership function is less fuzzy. This holds in both the intervals i.e., $0 \le \mu_A(x_i) < 0.5$ and $0.5 < \mu_A(x_i) \le 1$.

4. Symmetry: Using the R-software, we find that all the measures defined in (3), (4) and (5) give the same value when we change A to A' .

A. Particular Cases:

- The 'useful' RFIM (3) reduces to the measure given by Hooda [12] when utilities are ignored.

- When $u_i = 1$ the 'useful' RFIM (4) reduces to the RFIM given by Tomar and Ohlan [10]

    i. It reduces to measure defined in (3) when $\beta = 1$.

    ii. The measure (4) further reduces to Hooda's [12] RFIM when $\beta = 1$ and $u_i = 1$.

- When utilities are ignored the two parametric 'useful' RFIM (5) reduces to the measure given by Safeena et al. [11].

    i) Expression (5) reduces to the single parametric measure defined in (3) when $\alpha = 1$.
    ii) Again, (5) tends to the measure defined in (3) when $\alpha = 1$ and $\beta = 1$.
    iii) Further, (5) reduces to Hooda's [12] RFIM when $\alpha = 1$, $\beta = 1$ and $u_i = 1$.

Since, all the three 'useful' RFIMs reduce to Hooda's [12] RIM which is the fuzzy version of Boekee and Lubbe [8]. Thus, the defined measures also reduce to Shannon's [1] measure of information for $R \to 1$.

IV. Monotonic Behaviour of 'Useful' R-Norm Fuzzy Information Measures

In this section, we study the monotonic behaviour of the proposed 'useful' RFIMs defined in (3), (4) and (5) at different values of α , β & R .

Let $A = (0.23, 0.65, 0.91, 0.43, 0.39, 0.82)$ be a fuzzy set and $u_i = (5, 2, 4, 1, 6, 3)$. By using these values, we construct the following tables and figures. In these figures, red, green and blue lines represent the values of (3), (4) and (5) respectively.

Table 5: Value of Measures for α =0.50 and β =0.80

R	$H_R(A;U)$	$H_R^{0.80}(A;U)$	$H_R^{0.50,0.80}(A;U)$
0.89	3.4278	3.1022	2.9927
6	1.8711	1.8614	1.7275
20	1.6601	1.6592	1.6182
34	1.6249	1.6246	1.6008
57	1.6053	1.6052	1.5911
79	1.5974	1.5973	1.5872
108	1.5919	1.5918	1.5844
154	1.5875	1.5873	1.5822
213	1.5846	1.5845	1.5808
220	1.5843	1.5843	1.5807

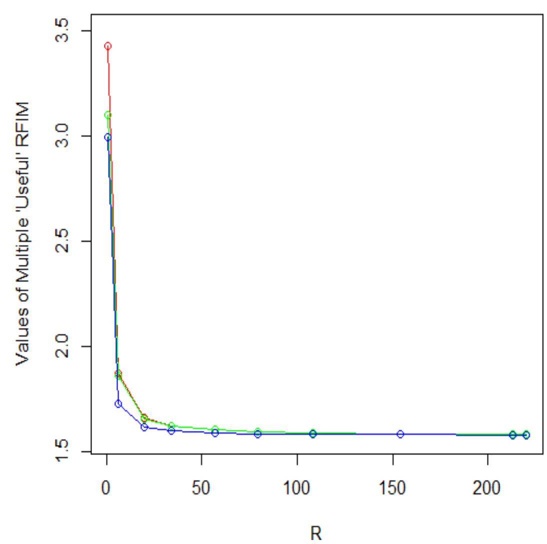

Figure 1: Behaviour of Measures for alpha=0.50 and beta=0.80

From Table 5, we conclude that the proposed 'useful' RFIMs show decreasing trend as the value of R increases. We further observe that $H_R(A;U) \ge H_R^{\beta}(A;U) \ge H_R^{\alpha,\beta}(A;U)$ for fixed alpha and beta. Same is depicted in the Figure 1.

Table 6: Value of Measure (5) for R=12 and β =0.80

α	$H_{12}^{\alpha,0.80}(A;U)$
0.1	1.5912
0.2	1.6053
0.3	1.6193
0.4	1.6334
0.5	1.6475
0.6	1.6615
0.7	1.6755
0.8	1.6895
0.9	1.7033
1.0	1.7170

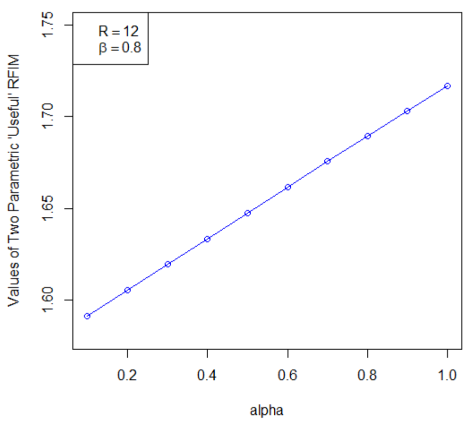

Figure 2: Behaviour of Measure (5) for R=12 and beta=0.80

Table 7: Value of Measures (4) & (5) at R=12 and $\alpha = 0.50$

β	$H_{12}^{\beta}(A;U)$	$H_{12}^{0.50,\beta}(A;U)$
0.1	1.7090	1.6434
0.2	1.7101	1.6439
0.3	1.7112	1.6445
0.4	1.7123	1.6451
0.5	1.7135	1.6457
0.6	1.7146	1.6463
0.7	1.7158	1.6469
0.8	1.7170	1.6475
0.9	1.7183	1.6481
1.0	1.7195	1.6488

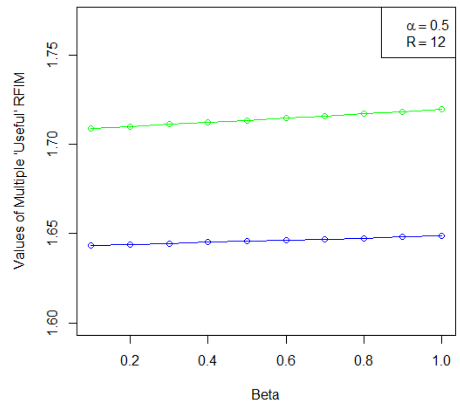

Figure 3: Behaviour of Measures (4) & (5) at R=12 and alpha=0.50

From Table 6 and 7, we observe that there is a positive relation between parameters and the measures. This clearly shows that the behaviour of measures is monotonically increasing with respect to parameters alpha and beta. This is shown graphically in the Figures 2 and 3.

V. CONCLUSION and Future Scope

In this research paper, three 'useful' RFIMs are presented. We have shown that these measures are valid 'useful' R-norm fuzzy information measures with the help of numerical data. For this purpose, we have used R-software. These proposed 'useful' RFIMs are also the generalized measures of previously given RIMs. Further, the monotonic behaviour of all these proposed measures and the relation between no, single and double parametric 'useful' RFIMs has been studied.

In future, these measures can be used to develop the 'useful' R-norm fuzzy directed divergence measures and their applications can be studied.

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