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On χ_s -Orthogonal of Type I Matrices

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Abstract— In this paper, we introduced a new type of matrices, we called it χ_S -orthogonal of type I matrix. Also we have extended some results of [1] in the context of χ_S -orthogonal of type I matrices.

Keywords— *Secondary transpose, χ_S -orthogonal matrices, χ_S -orthogonal of type I matrices.*

I. INTRODUCTION

Throught this paper we use this following notations.

Notation 1.1 [1,2]. The secondary transpose (conjugate secondary transpose) of A is defined by $A^s = VA^T V (A^\theta = VA^* V)$, where V is the fixed disjoint permutation matrix with in its secondary diagonal.

Definition 1.2 [4]. An $n \times n$ non–singular matrix A is said to χ_S -orthogonal, if $\chi_S(A) = A^{-1}$, where $\chi_s(A) = S^{-1} A^s S$ and S satisfies the condition $S^2 = \pm 1$.

Notation 1.3. Let □ be the set of real numbers, □ be the set of all natural numbers, and $M_n(□)$ be the $n \times n$ matrices. Let $O^n_{\chi_s}$ be the set of all $n \times n$, χ_S -orthogonal matrices and $O^n_{\chi_s,1}$ be the set of all $n \times n$, χ_S -orthogonal of type I matrices.

II. MAIN RESELTS

Definition 2.1. A square matrix $A_{n\times n}$ is called an χ_S -orthogonal of type I matrix if $A^K \left(A^\#\right)^K = I$, $A^\# = S^{-1} A^S S$, for some $K \in □$.

$$A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$$ is χ_S -orthogonal of type I matrices.

Definition 2.2. Let ‘ A ’ be an χ_S -orthogonal of type I matrix. The smallest positive integer‘ K ’with $A^K \left(A^\#\right)^K = I$ is called the index of ‘ A ’. In such case, we say that ‘ A ’ is an χ_S -orthogonal of type I of period ‘ K ’ or K -period of χ_S -orthogonal of type I matrix and we denote it by $ind(A)$.

Example 2.3

$$A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$$ is an χ_S -orthogonal of type I matrix.

$$A^\# = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^\# = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$$

K=1, $A^1 \left(A^\#\right)^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

K=2, $A^2 \left(A^\#\right)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

K=3, $A^3 \left(A^\#\right)^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

K=4, $A^4 \left(A^\#\right)^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

K=5, $A^5\left(A^\#\right)^5=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

K=6, $A^6\left(A^\#\right)^6=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Theorem 2.4. If ‘ A ’ is an χ_S -orthogonal of type I matrix of index K, then $\det\left(A^K\right)=\pm 1$.

Proof. Let A be an χ_S -orthogonal of type I matrix of index K, then

$$\Rightarrow \det\left(A^K\left(A^\#\right)^K\right)=I_n$$

$$\Rightarrow \det\left(A^K\left(A^\#\right)^K\right)=\det\left(I_n\right)$$

$$\Rightarrow \det\left(A^K\left(A^\#\right)^K\right)=1$$

$$\Rightarrow \det\left(A^K\right)\det\left(A^\#\right)^K=1$$

$$\Rightarrow \det\left(A^K\right)\det\left(A^K\right)=1$$

$$\Rightarrow \left(\det A^K\right)^2=1$$

Hence $\det\left(A^K\right)=\pm 1$.

Theorem 2.5. If ‘ A ’ is an χ_S -orthogonal of type I matrix of index K, then it is invertible with $A^{-1}=A^{K-1}\left(A^\#\right)^K$.

Proof. By Theorem 2.4, we have $\det(A)\neq 0$, then A is invertible.

$\left(A^K\left(A^\#\right)^K\right)=I_n$ for some $K\in$. This equivalent to,

$$\left(A^K\right)^{-1}=\left(A^\#\right)^K$$

$$\Rightarrow \left(A^{-1}\right)^K=\left(A^\#\right)^K$$

$$\Rightarrow \left(A^{-1}\right)^{K-1}\left(A^{-1}\right)=\left(A^\#\right)^K$$

Hence, $A^{-1}=A^{K-1}\left(A^\#\right)^K$

Theorem 2.6. Let $A\in M_n(\,)$ matrix, then the following statements are equivalent

(1) A is an χ_S -orthogonal of type I matrix.

(2) A^{-1} is an χ_S -orthogonal of type I matrix.

(3) A^T is an χ_S -orthogonal of type I matrix.

(4) $\overline{A}$ is an χ_S -orthogonal of type I matrix.

(5) A^* is an χ_S -orthogonal of type I matrix.

Proof.

$(1)\Rightarrow(2)$. Suppose that A is an χ_S -orthogonal of type I matrix.

So $\left(A^K\left(A^\#\right)^K\right)=I_n$ for some $K\in$

$$\Rightarrow \left(A^K\left(A^\#\right)^K\right)^{-1}=I_n^{-1}$$

$$\Rightarrow \left(A^K\right)^{-1}\left(\left(A^\#\right)^K\right)^{-1}=I_n$$

$$\Rightarrow \left(A^{-1}\right)^K\left(\left(A^{-1}\right)^\#\right)^K=I_n$$

Hence A^{-1} is an χ_S -orthogonal of type I matrix.

$(2)\Rightarrow(3)$. Suppose that A^{-1} is an χ_S -orthogonal of type I matrix. So $\left(A^{-1}\right)^K\left(\left(A^{-1}\right)^\#\right)^K=I_n$ for some $K\in$

$$\Rightarrow \left(\left(A^{-1}\right)^K\left(\left(A^{-1}\right)^\#\right)^K\right)^{-1}=I_n^{-1}$$

$$\Rightarrow \left(\left(\left(A^{-1}\right)^\#\right)^K\right)^{-1}\left(\left(A^{-1}\right)^K\right)^{-1}=I_n$$

$$\Rightarrow \left(\left(\left(A^{-1}\right)^{-1}\right)^\#\right)^K\left(\left(A^{-1}\right)^{-1}\right)^K=I_n$$

$$\Rightarrow \left(A^\#\right)^K A^K=I_n$$

$$\Rightarrow \left(\left(A^\#\right)^K A^K\right)^\#=I_n^\#$$

$$\Rightarrow \left(A^\#\right)^K\left(\left(A^\#\right)^\#\right)^K=I_n.$$

Hence A^T is an χ_S -orthogonal of type I matrix.

$(3)\Rightarrow(4)$. Suppose that A^T is an χ_S -orthogonal of type I matrix. So $\left(\left(A^T\right)^T\right)^K\left(A^\#\right)^K=I_n$ for some $K\in$

$$\Rightarrow \overline{A^K\left(A^\#\right)^K}=\overline{I_n}$$

$$\Rightarrow \left(\overline{A}\right)^K\left(\overline{A^\#}\right)^K=I_n$$

$$\Rightarrow \left(\overline{A}\right)^K\left(\left(\overline{A}\right)^\#\right)^K=I_n$$

Hence $\overline{A}$ is an χ_S -orthogonal of type I matrix.

$(4) \Rightarrow (5)$. Suppose that $\overline{A}$ is an χ_S -orthogonal of type I matrix. So $\left(\overline{A}\right)^K \left(\left(\overline{A}\right)^{\#}\right)^K = I_n$ for some $K \in \mathbb{N}$

$$\Rightarrow \left(\left(\overline{A}\right)^K \left(\left(\overline{A}\right)^{\#}\right)^K\right)^T = I_n{}^T$$

$$\Rightarrow \left(\left(\left(\overline{A}\right)^{\#}\right)^K\right)^T \left(A^*\right)^K = I_n$$

$$\Rightarrow \left(\left(\left(\overline{A}\right)^{\#}\right)^T\right)^K \left(A^*\right)^K = I_n$$

$$\Rightarrow \left(\left(\left(\overline{A}\right)^{*}\right)^{\#}\right)^K \left(A^*\right)^K = I_n$$

Hence A^* is an χ_S -orthogonal of type I matrix.

$(5) \Rightarrow (1)$. Suppose that A^* is an χ_S -orthogonal of type I matrix. So $\left(A^*\right)^K \left(\left(A^*\right)^{\#}\right)^K = I_n$ for some $K \in \mathbb{N}$

$$\Rightarrow \left(\left(A^{-1}\right)^K \left(\left(A^{-1}\right)^{\#}\right)^K\right)^{-1} = I_n{}^{-1}$$

$$\Rightarrow \left(\left(\left(A^*\right)^{\#}\right)^K\right)^* \left(\left(A^*\right)^K\right)^* = I_n$$

$$\Rightarrow \left(\left(\left(A^*\right)^{*}\right)^{\#}\right)^K A^K = I_n$$

$$\Rightarrow \left(A^{\#}\right)^K A^K = I_n$$

Hence ‘ A ’ is an χ_S -orthogonal of type I matrix.

Theorem 2.7. If $A_{n\times n}$ and $B_{n\times n}$ are commute χ_S -orthogonal of type I matrix, then AB is χ_S -orthogonal of type I matrix.

Proof. Let A and B be χ_S -orthogonal of type I matrices with the same index K .

Then $\left(A^K \left(A^{\#}\right)^K\right) = I_n$

$$\left(B^K \left(B^{\#}\right)^K\right) = I_n$$

$$\Rightarrow \left(AB\right)^K \left(\left(AB\right)^{\#}\right)^K$$

$$= A^K B^K \left(B^{\#}\right)^K \left(A^{\#}\right)^K$$

$$= A^K B^K \left(\left(A^{\#}\right)^T\right)^K \left(\left(B^{\#}\right)^T\right)^K$$

$$= A^K B^K \left(\left(A^{\#}\right)^T \left(B^{\#}\right)^T\right)^K$$

$$= A^K B^K \left(\left(B^{\#} A^{\#}\right)^T\right)^K$$

$$= A^K B^K \left(\left(B^{\#}\right)^T\right)^K \left(\left(A^{\#}\right)^T\right)^K$$

$$= I_n$$

Hence AB is χ_S -orthogonal of type I matrix. If A and B be χ_S -orthogonal of type I matrices with the same indices K_1 and K_2 respectively, then $\left(AB\right)^K \left(\left(AB\right)^{\#}\right)^K = I_n$, where K is the least common multiple of K_1 and K_2 .

Theorem 2.8. If $A \in O^n_{\chi_s,1}$ of index ‘ K ’if and only if A^m is χ_S -orthogonal of type I matrix of index ‘ K ’ for each $m \in \mathbb{N} \setminus \{1\}$.

Proof. Suppose that A is an χ_S -orthogonal of type I matrix. So $\left(A^K \left(A^{\#}\right)^K\right) = I_n$ for some $K \in \mathbb{N}$

$$\Rightarrow \left(A^K \left(A^{\#}\right)^K\right)^m = \left(I_n\right)^m$$

$$\Rightarrow \left(A^K\right)^m \left(\left(A^{\#}\right)^K\right)^m = I_n$$

$$\Rightarrow \left(A^m\right)^K \left(\left(A^m\right)^{\#}\right)^K = I_n$$

Hence A^m is an χ_S -orthogonal of type I matrix.

$$ind\left(A^m\right) = lcm\left\{\underbrace{ind\left(A\right), ind\left(A\right), \cdots ind\left(A\right)}_{m-times}\right\}$$

$$= lcm\left\{\underbrace{K, K, \cdots K}_{m-times}\right\}$$

$$= K$$

Now, suppose that A^m is an χ_S -orthogonal of type I matrix for each $m \in \mathbb{N} \setminus \{1\}$; especially, each of A^2 and A^3 if χ_S -orthogonal of type I matrix of index K .

So, $I_n = \left(A^3\right)^K \left(\left(A^3\right)^{\#}\right)^K = \left(A^2\right)^K \left(\left(A^2\right)^{\#}\right)^K$

$$= \left(A\right)^K \left(A^{\#}\right)^K$$

Hence A is an χ_S -orthogonal of type I matrix of index K .

Theorem 2.9. If $A \in O^{n}_{\chi_s,1}$ of index K, then each of A^T, A^{-1}, $\overline{A}$ and A^* are orthogonal of type I of index K.

Theorem 2.10. If λ is an eigen value of an χ_S -orthogonal of type I matrix A with index K, then λ is of modulus 1.

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