International Journal of Scientific Research in Mathematical and Statistical Sciences
Research Paper
Volume-5, Issue-5, pp.179-184, October (2018)
E-ISSN: 2348-4519

Convergent point of G-type nonexpansive mapping with graph

D.P. Shukla[1] and Vivek Tiwari[2*]

[1]Govt. Science College Rewa (M.P.) 486001, India
[2]Sharda Mahavidyalya Sarlanagar Maihar satna (M.P.) 485772, India
Affiliated to A.P.S. University, Rewa (M.P.) 486003, India

Available online at: www.isroset.org

Received: 18/Sept/2018, Accepted:14/Oct/2018, Online: 31/Oct/2018

Abstract- In this paper, we prove some weak and strong convergence of a sequence $\{x_n\}$ generated by the Abbas at el. techniques to some common fixed points of G-type nonexpansive mappings defined on Banach space with graph.

Keywords- Fixed point, G-nonexpansive mappings, Abbas et al., Banach space, directed graph.

I. INTRODUCTION

Banach proved a noteworthy and instance outcomes result is said to be a Banach contraction principle. Because of it's fruitful results, it has been generalized in various areas. The present version of the theorem was given in Banach space by using graph and it's property. Jachymski [1] provide a generalization of the Banach contraction principle to metric space with a graph. Aleomraninejad et al. [2] also give various iterative processes for G-nonexpansive and G-contractive mappings on graphs. Alfuraidan [4] provide a new definition of G-contraction and got sufficient conditions for existence of fixed points on a metric space for multivalued function included graphs property. Tiammee et al. [6] proved Browder's convergence theorem for G-nonexpansive mapping in a Banach space with graphs and their property.

Inspired by all references, the author proves strong and weak convergence theorems for G- type nonexpansive mapping using Abbas at el. iteration generated from any arbitrary x_0 in closed convex subset of a uniformly convex Banach space X with a graph.

II. PRELIMINARIES

In this part, we overview some basic graph notations, terminology and required outcomes.

Let (X,d)be a metric space and $\omega = \{(x,x): x \in X\}$. Now assume a directed graph

$G = (V,E)$ where V is the set of vertices of graph and E is the set of edges of graph contains all loops. Now let the graph G has no parallel edges. Then $G = (V,E)$ by assigning to each edge the distance between it's vertices, G treated as weighted graph.

Definition 2.1 The inverse of a graph G is obtained from G by reversing the direction of edges and it's represented by G^{-1}, and numerically defined as follows:

$$E(G^{-1}) = \{(x,y) \in X \times X: (y\,,x) \in E(G)\}$$

Definition 2.2 Let x and y be the vertices of graph G. A path in G form x to y of length D ($D \in \boldsymbol{D} \cup \{0\}$) is sequence $\{x_n\}_{i=0}^{N}$ of $D+1$ vertex for which

$$x_0 = x, x_N = y \; and \; (x_{i,}x_{i+1}) \in E \;\; for \; i = 0,1, \dots D-1.$$

Definition 2.3 A graph G is called connected graph iff there is a path between any two vertices of the graph G.

Definition 2.4 A graph $G = (V,E)$ is called transitive if, for any $p,q,r \in V$ such that (p,q) and $(q,r) \in E$ then we have $(p,r) \in E$.

Definition 2.5 Let C be nonempty convex subset of a Banach space X and $G = (V, E)$ is a directed graphs such that $V = C$, then a mapping $T: C \to C$ is G-nonexpansive ([3], Definition 2.3 (iii)) if it satisfies following conditions:

(i) T is edge preserving.
(ii) $\|Tx - Ty\| \leq \|x - y\|$, where $(x, y) \in E$ for any $x, y \in C$.

Definition 2.6 [7] Let C be nonempty closed convex subset of a real uniformly convex Banach space X. The mappings T_i $(i = 1, 2)$ on C are said to satisfy condition B if there exists increasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0 \ \forall \ r > 0$ such that $\forall \ x \in C$.

$$\max\{\|x - T_1\|, \|x - T_2 x\|\} \geq f(d(x, F)),$$

Where $F = F(T_1) \cap F(T_2)$ and $F(T_i)$ $(i = 1, 2)$ are the sets of fixed points of T_i.

Definition 2.7: A Banach space X is said to satisfy Opial property if the following inequality holds for any distinct elements x and y in X and for each sequence $\{x_n\}$ weakly convergent to x:

$$\lim_{n\to\infty} lim\|x_n - x\| < \lim_{n\to\infty} inf\|x_n - y\|.$$

Definition 2.8 Let X be a Banach space, A mapping T with domain D and range R in X is demiclosed at 0 if, for any sequence $\{x_n\}$ in D such that $\{x_n\}$ converges weakly to $x \in D$ and $\{Tx_n\}$ converges to strongly to 0, we have T_x= 0.

Lemma 2.9 [8] Let X be a uniformly convex Banach space and $\{\alpha_n\}$ a sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0,1)$. Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $\limsup_{n\to\infty}\|x_n\| \leq c$, $\limsup_{n\to\infty}\|y_n\| \leq c$ and $\limsup_{n\to\infty}\|\alpha x_n + (1 - \alpha_n)y_n\| = c$ for some $c \geq 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 2.10 ([9]) Let X be a Banach space, and R >1 be a fixed number, then X is uniformly convex iff there exists a continuous, strictly increasing and convex function g: $[0, \infty] \to [0, \infty)$ with g (0) = 0 such that

$$\|\lambda x + (1 - \lambda)xy\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|\lambda\|^2 - \lambda(1 - \lambda)\mathrm{g}(\|x - y\|)$$

For all $x, y \in B_R(0) = \{x \in X: \|x\| \leq R\}$ and $\lambda \in [0, 1]$.

Lemma 2.11([10]) Let X be a Banach space that satisfies Opial condition, and let $\{x_n\}$ be a sequence in X. Let x, y in X be such that $\lim_{n\to\infty}\|x_n - x\|$ and $\lim_{n\to\infty}\|x_n - y\|$ exist. If $\{x_n\}$ and $\{x_{n_k}\}$ are subsequence of $\{x_n\}$ that converge weakly to x and y then $x = y$.

III. MAIN RESULTS

In this section, we let C be a nonempty closed convex subset of a Banach space X endowed with a directed graph $G = (V, E)$ such that $V = C$ and E is convex. We also take that the graph $G = (V, E)$ is satisfies transitive property. The mapping Ti $(i = 1, 2, 3)$ are G-nonexpansive mapping from C to C with $F = F(T_1) \cap F(T_2) \cap F(T_3)$. Let $\{x_n\}$ be a sequence generated from arbitrary element $x_0 \in C$.

$$x_{n+1} = (1 - \alpha_n)T_1 y_n + \alpha_n T_1 z_n$$

$$y_n = (1 - \beta_n)T_2 x_n + \beta_n T_2 z_n$$

$$z_n = (1 - \gamma_n)x_n + \gamma_n T_3 x_n$$

Where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequence in [0, 1].

Theorem 3.1 Let $w_0 \in F$ such that $(x_0, w_0), (y_0, w_0), (z_0, w_0), (w_0, x_0), (w_0, y_0)$ and (w_0, z_0) are in E (G). Then$(x_n, w_0), (y_n, w_0), (z_n, w_o)$,$(w_0, x_n), (w_0, y_n)$ and$(w_0, z_n,)$, are in $E(G)$.

Proof: We will proof this theorem by three section, with assumption $(x_0, w_0), (y_0, w_0)$ *and* $(z_0, w_0) \in E(G)$, then we will show by induction that$(x_n, w_0), (y_n, w_0)$ *and* $(z_n, w_0) \in E(G)$. Then with the assumption$(w_0, x_0), (w_0, y_0), (w_0, z_0) \in E(G)$, we will again prove by induction method that$(w_0, x_n), (w_0, y_n)$ *and* $(w_0, z_n) \in E(G)$. In last step we combine above two results than by using transitive property of G to get the statement. Let $(x_0, w_0), (y_0, w_0)$ and $(z_0, y_0) \in E(G)$. Then $(T_1 y_0, w_0), (T_2 z_0, w_0)$*and*$(T_3 x_0, w_0) \in E(G)$ then since $(i = 1, 2, 3)$ are edge preserving. By the convexity of (G) $(T_1 y_0, w_0), (x_0, w_0) \in E(G)$, we have $(x_1, z_0) \in E(G)$. Then by edge- preserving of T_2, $(T_2 z_1, w_0) \in E(G)$. Simiarly for T_3. Again by convexity of $E(G)$ and $(T_2 y_1, w_0), (y_1, w_0)$ and $(T_3 x_1, w_0), (x_1, w_0) \in E(G)$. Next we suppose that$(x_k, w_0)$, (y_k, w_0) and $(z_k, w_0) \in E(G)$.Then $(T_3 x_k, w_0)$, $(T_2 z_k, w_0)$and $(T_1 y_k, w_0) \in E(G)$. Since $T_i (i = 1,2,3)$ are edge-preserving. Since $E(G)$ is convex, $(x_{k+1}, w_0) \in E(G)$. Indeed

$$\alpha(T_1 y_k, w_0) + (1 - \beta)(x_k, w_0) = (\alpha T_1 y_k + (1 - \alpha) x_k, w_0) = (y_{k+1}, w_0) \in E(G).$$

Similarly, since T_2 and T_3 are edge-preserving therefore by the convexity of $E(G)$, we get

$$\beta(T_2 z_k, w_0) + (1 - \beta)(x_{k+1}, w_0) = (\beta T_2 z_{k+1} + (1 - \beta) x_{k+1}, w_0) = (z_{k+1}, w_0) \in E(G)$$

$$\gamma(T_3 x_k, w_0) + (1 - \gamma)(x_{k+2}, w_0) = (\gamma T_3 x_{k+1} + (1 - \gamma) x_{k+2}, w_0) = (x_{k+1}, w_0) \in E(G)$$

Hence by the induction (x_n, w_0), (y_n, w_0) and $(z_n, w_0) \in E(G)$. now using similar argument we can show that (w_0, x_n), (w_0, y_n) and $(w_0, z_n) \in E(G)$. Therefore $(x_n, z_n) \in E(G)$ by the transitivity of G.

Theorem 3.2 Let $w_0 \in F$. Suppose $(x_0, w_0), (y_0, w_0), (z_0, w_0), (w_0, x_0), (w_0, y_0)$ and

$(w_0, z_0) \in E(G)$ for arbitrary x_0 in C. Then $\lim_{n\to\infty} \|x_n - z_0\|$ exists.

Proof:

$$\begin{aligned}
\|x_{n+1} - w_0\| &= \|(1 - \alpha_n) T_1 y_n + \alpha_n T_1 z_n - w_0\| \\
&\leq (1 - \alpha_n)\|T_1 y_n - w_0\| + \alpha_n \|T_1 z_n - w_0\| \\
&\leq (1 - \alpha_n)\|(1 - \beta_n) T_2 x_n + \beta_n T_2 z_n - z_0\| + \alpha_n \|(1 - \gamma_n) x_n + \gamma_n T_3 x_n - w_0\| \\
&\leq (1 - \alpha_n)(1 - \beta_n)\|T_2 x_n - w_0\| + (1 - \alpha_n)\beta_n \|T_2 z_n - w_0\| + \\
&\qquad \alpha_n (1 - \gamma_n)\|x_n - w_0\| + \alpha_n \gamma_n \|T_3 x_n - w_0\| \\
&\leq (1 - \alpha_n)(1 - \beta_n)\|T_2 x_n - w_0\| + (1 - \alpha_n)(1 - \gamma_n)\beta_n \|x_n - w_0\| + \\
&\qquad (1 - \alpha_n)\beta_n \gamma_n \|x_n - x_0\| + \alpha_n (1 - \gamma_n)\|x_n - x_0\| + \alpha_n \gamma_n \|T_3 x_n - w_0\| \\
&= \|x_n - w_0\|
\end{aligned}$$

Thus $\lim_{n\to\infty} \|x_n - w_0\|$ exists. Hence the sequence $\{x_n\}$ is bounded.

Theorem 3. 3 If X is uniformly convex $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\} \subset [\delta; 1 - \delta]$ for some $\delta \in (0, 1/3)$

and $(x_0, w_0), (y_0, w_0), (z_0, w_0), (w_0, x_0)(w_0, y_0)$*and* $(w_0, z_0) \in E(G)$ for arbitrary x_0 in c and $z_0 \in F$, then

$\lim_{n\to\infty} \|x_n - T_1 x_n\| = \lim_{n\to\infty} \|x_n - T_2 x_n\| = \lim_{n\to\infty} \|x_n - T_3 x_n\| = 0$.

Proof: Let $w_0 \in F$. Then by the boundedness of $\{x_n\}$, $\{T_2 z_n\}$and $\{T_3 x_n\}$ there exists $r > 0$, such that$(x_n - w_0)$, $(y_n - w_0)$ and $(z_n - w_0) \in B_r(0)$ for all $n \geq 1$. Put $c = \lim_{n\to\infty} \|x_n - w_0\|$. If c = 0, then by the G-nonexpansiveness of $T_i (i = 1,2,3)$ we have

$\|x_n - T_i x_n\| \le \|x_n - w_0\| + \|w_0 - T_i x_n\| \le \|x_n - w_0\| + \|w_0 - x_n\|$

Therefore the result follows that $c > 0$. Hence by lemma 2.10 together with the G-nonexpansiveness of T_2 and T_3, we have

$$\|y_n - w_0\|^2 = \|(1-\beta_n)T_2 x_n + \beta_n T_2 z_n - w_0\|^2$$
$$= \|\beta_n(T_2 z_n - w_0) + (1-\beta_n)(T_2 x_n - w_0)\|^2$$
$$\le \beta_n\|T_2 z_n - w_0\|^2 + (1-\beta_n)\|(T_2 x_n - w_0)\|^2 - \beta_n(1-\beta_n)\mathrm{g}(\|T_2 z_n - z_n\|$$
$$\le \beta_n\|z_n - w_0\|^2 + (1-\beta_n)\|x_n - w_0\|^2$$
$$\le \beta_n\{\|(1-\gamma_n)x_n + \gamma_n T_3 x_n - w_0\|^2\} + (1-\beta_n)\|x_n - w_0\|^2$$
$$\le \beta_n\|x_n - w_0\|^2 + (1-\beta_n)\|x_n - w_0\|^2$$
$$= \|x_n - w_0\|^2.$$

and

$$\|z_n - w_0\|^2 = \|(1-\gamma_n)x_n + \gamma_n T_3 x_n - w_0\|^2$$
$$= \|\gamma_n(T_3 x_n - w_0) + (1-\gamma_n)(x_n - w_0)\|^2$$
$$\le \gamma_n\|T_3 x_n - w_0\|^2 + (1-\gamma_n)\|(x_n - w_0)\|^2 - \gamma_n(1-\gamma_n)\mathrm{g}(\|T_3 x_n - x_n\|$$
$$\le \gamma_n\|x_n - w_0\|^2 + (1-\gamma_n)\|x_n - w_0\|^2$$
$$= \|x_n - w_0\|^2.$$

Thus we have

$$\lim_{n\to\infty}\|y_n - w_0\| \le \lim_{n\to\infty}\|x_n - w_0\| \le \lim_{n\to\infty}\|z_n - w_0\|$$

Now we have also

$$\|x_{n+1} - w_0\|^2 = \|(1-\alpha_n)T_1 y_n + \alpha_n T_1 z_n - w_0\|^2$$
$$= \|\alpha_n(T_1 z_n - w_0) + (1-\alpha_n)(T_1 y_n - w_0)\|^2$$
$$\le \alpha_n\|T_1 z_n - w_0\|^2 + (1-\alpha_n)\|(T_1 y_n - w_0)\|^2 - \alpha_n(1-\alpha_n)\mathrm{g}(\|T_2 z_n - T_1 y_n\|$$
$$\le \|x_n - w_0\|^2 - \delta^2\mathrm{g}(\|T_1 y_n - x_n\|)$$

Thus

$\delta^2\mathrm{g}\,(\|T_1 y_n - x_n\|) \le \|x_n - x_0\|^2 - \|x_{\mathrm{n}+1} - w_0\|^2.$

Implies that $\lim_{n\to\infty}\mathrm{g}(\|T_1 y_n - w_0\|) = 0$ and g is strictly increasing and continuous at 0.

$\lim_{n\to\infty}\|T_1 y_n - w_0\| = 0.$ (i)

Since T_1 is G-nonexpansive, then we have

$\|x_n - w_0\| \le \|x_n - T_1 y_n\| + \|T_1 y_n - T_1 w_0\| \le \|x_n - T_1 y_n\| + \|y_n - w_0\|$

Now taking lim inf yields

$\underline{\lim}_{n\to\infty}\|y_n - w_0\| \ge c.$

Hence we have

$\lim_{n\to\infty}\|y_n - w_0\| = c.$

Since

$\lim_{n\to\infty}\|\beta_n(T_2x_n - w_0) + (1-\beta_n)(x_n - w_0)\| = \lim_{n\to\infty}\|y_n - w_0\| = c$

and

$\lim_{n\to\infty}\|\gamma_n(T_3x_n - w_0) + (1-\gamma_n)(x_n - w_0)\| = \lim_{n\to\infty}\|z_n - w_0\| = c.$

and $\lim_{n\to\infty} sup\|T_2x_n - w_0\| \leq c,$

Similarly

$\lim_{n\to\infty} sup\|T_3x_n - w_0\| \leq c.$

So, by lemma 2.9, we have

$\lim_{n\to\infty} sup\|T_2x_n - x_n\| = 0.$ (ii)

$\lim_{n\to\infty} sup\|T_3z_n - z_n\| = 0.$ (iii)

by the G −nonexpansiveness of T_1, T_2 and T_3together with

$\|x_n - y_n\| \leq \|T_2z_n - x_n\| \leq \|T_3x_n - x_3\|$, we have

$$\|T_1x_n - x_n\| \leq \|T_1x_n - T_1z_n\| + \|T_1z_n - T_1y_n\| + \|T_1y_n - x_n\|$$

$$\leq \|x_n - z_n\| + \|z_n - y_n\| + \|y_n - x_n\|$$

$$\leq \|T_3x_n - z_n\| + \|T_2z_n - y_n\| + \|T_1y_n - x_n\|$$

From (i), (ii) and (iii), we have

$$\|T_1x_n - x_n\| = 0.$$

Hence Theorem is proved.

Theorem 3.4 Let X satisfies the opial's condition that $(x_0, w_0), (y_0, w_0), (z_0, w_0) \in E(G)$ for $w_0 \in F$ and arbitrary $x_0 \in C$. Then $I - T_i$ $(i = 1,2,3)$ are demiclosed.

Proof: Assume that $\{x_n\}$ is a sequence in C that converges weakly to q from theorem 3.3, we have $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0$.

Then by contradiction that $q \neq T_iq$, then by opial's condition we have

$$\lim_{n\to\infty} sup\|x_n - q\| < \lim_{n\to\infty} sup\|x_n - T_iq\|$$

$$\leq \lim_{n\to\infty} \sup(\|x_n - T_ix_n\| + \|T_ix_n - T_iq\|)$$

$$\leq \lim_{n\to\infty} sup\|x_n - q\|, \text{ a contradiction.}$$

Hence $T_iq = q$.

So the conclusion holds.

IV. CONCLUSION

This paper presents the calculation of fixed point by the use of graph and it's properties. It also helps to calculate fixed point by various properties of graphs in the future with different aspects.

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AUTHORS PROFILE:

Prof. Dr. D.P.Shukla is a professor and Head, department of Mathematics and computer science , Govt. P.G. Science college Rewa (M.P.) affiliated with Awadhesh Pratap Singh university Rewa (M.P.), India. He has more than 25 year Research and teaching experience in the field of pure mathematics. His major research interest is in fixed point theory, functional analysis, measure theory etc. and has more than 150 articles in the national and international journals.

Mr. Vivek Tiwari is research scholar of mathematics under the supervision of Prof. D.P. Shukla. He is assistant professor of Mathematics at Sharda Mahavidyalaya, Sarlangar, Maihar, disst. Satna (M.P.), India runs by B.K. Birla group of companies. His area of research in fixed point theory and he has presented research paper in various national and international conferences.

