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On Asymptotic Testing of Equality of Parameters of K Exponential Family of Distributions

Patel S. R.

Sardar Patel University, Department of Statistics, Vallabh Vidyanagar-388120, Gujarat, India

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Abstract— In this paper attempt has been made to obtain the asymptotic tests for testing the equality of parameters involved in several exponential family of distributions. Any exponential family has been reduced to naturalized form if it is not so. The naturalized exponential family is asymptotically reduced to multivariate normal distribution with the use of well-known multivariate central limit theorem and the test for equality of components of multivariate normal gives equivalently the test for equality of parameters of the family, which turns out as Uniformly Most Powerful (UMP) Invariant (UMPI) test. This general result has been applied to some specific distributions such as Binomial, Poisson, Exponential, Gamma distributions as particular cases of exponential family.

Keywords— Exponential family of distributions, asymptotic distribution, Asymptotic UMPI test.

I. INTRODUCTION

A fair size of literature is available (see [5]) for testing the parameters of regular family of distributions whose ranges are free from parameters. Amongst many authors some of them are ; (i) Engelhardt at el.[2] have derived UMP unbiased (UMPU) test for testing the scale parameter of a gamma distribution with a nuisance shape parameter, (ii) Keating at el. [4] derived UMPU test for testing the shape parameter of a gamma distribution with scale parameter as nuisance, (iii) Kambo at el.[3] developed test for testing equality of location parameters of K exponential distributions, (iv) Bayound at el.[1] derived the test for testing equality of two exponential distributions. For non-regular family of distributions (containing parameters in its ranges) asymptotic tets for testing equality of parameters of K such families have been obtained by Patel [7].

The asymptotic test for testing equality of parameters involved in several exponential family of distributions is the main purpose of this paper. Section II is devoted for the derivation of asymptotic tests, which turns out as UMPI tests. Section III describes the applications of these tests to Binomial, Poison, Exponential, and Gamma distributions. Section IV deals with the conclusion of the paper and points out the future direction.

II. ASYMPTOTIC TESTS

Let $X_1, X_2, \dots, X_k$ be K independently distributed exponential family with probability function (p.f.), $f_{X_i}(x_i;\theta_i)$ as

$$f_{X_i}(x_i;\theta_i) = a(\theta_i)b(x_i)e^{-\theta_i x_i}, \quad -\infty < x_i < \infty,\ i = 1,2,\dots,k \tag{2.1}$$

Let $X_{i1}, X_{i2}, \dots, X_{in}$ be a random sample of size n from each X_i, $i = 1,2,\dots,k$ and let $w_i = \sum_{j=1}^{n} X_{ij}$, $i = 1,2,\dots,k$. Then the joint p.f. of $X_{i1}, X_{i2}, \dots, X_{ik}$ can be written as

$$f_X(x;\theta) = q(\theta)h(x)e^{-\sum_{i=1}^{k}\theta_i w_i(x)},$$

where $x = (x_1, x_2, \dots\dots, x_k), \theta = (\theta_1, \theta_2, \dots.\theta_k),\ w_i(x) = wi, say\ i = 1,2, \dots k$

Let $\underline{w} = (w_{1,}w_2, \dots\dots.w_k)', \overline{w}_i = \frac{1}{n}wi,\ \underline{\overline{w}} = (\overline{w}_1, \overline{w}_2, \dots.., \overline{w}_k)'$.

Theorem 1:

With the above setup of observations, the asymptotic UMPI level-α test for testing the hypothesis, H

$H{:}\ \theta_1 = \theta_2 = \cdots \dots = \theta_k$ Vs K: no two $\theta's$ are equal. is to reject H whenever

$$\frac{n-k+1}{k-1}\left(C\underline{\overline{w}}\right)'\left(C\sum C'\right)^{-1}\left(C\underline{\overline{w}}\right) \geq F_{k-1,\ n-k+1\,;\,\alpha} \tag{2.2}$$

Where $F_{k-1,\ n-k+1\,;\,\alpha}$ is upper α % point of Snedekar F-distribution with $(k-1,\ n-k+1)$ d.f. and C is any $(k-1)\times k$ contrast matrix of constants and Σ is variance -covariance matrix of $w_{1,}w_2, \dots \dots . w_k$.

Proof:

We note that w_i is complete sufficient statistic for $\theta_i,\ i = 1,2, \dots \theta_k$ and the test depends on function of w_i. Further it is easy to show that

$$E(w_i) = q(\theta)\frac{\partial}{\partial\theta_i}[q(\theta)]^{-1} = q(\theta)di, i = 1,2, \dots . k \tag{2.3}$$

$$Var(w_i) = q(\theta)\frac{\partial^2}{\partial{\theta_i}^2}[q(\theta)]^{-1} - \left\{q(\theta)\frac{\partial}{\partial\theta_i}[q(\theta)]^{-1}\right\}^2$$

$$= q(\theta)d_{ii} - (q(\theta)d_i)^2, i = 1,2, \dots . k \tag{2.4}$$

where $d_i = \frac{\partial}{\partial\theta_i}[q(\theta)]^{-1}, d_{ii} = \frac{\partial^2}{\partial{\theta_i}^2}[q(\theta)]^{-1}, i = 1,2, \dots . . k.$

Since $X_{ij,} i = 1,2, \dots . k;\ \ j = 1,2, \dots . n$ are all independently distributed. $Cov(w_i, w_j) = 0$ for i$\neq j$.

Due to multivariate central limit theorem, $\underline{w} = (w_1, w_2, \dots \dots w_k)'$ is asymptotically distributed as k-variate normal distribution with mean vector

$$q(\theta)(d_1, d_2, \dots, d_k)' = q(\theta)D_1'$$

and variance -covariance matrix, Σ as

$$\sum = q(\theta)D - [q(\theta)]^2 D_1'\ D_1$$

where $D_1 = (d_1, d_2, \dots, d_k), D = diag\ (d_{11}, d_{22}, \dots, d_{kk})$

Thus

$$\underline{w} \xrightarrow{d} N_k(q(\theta)D_1', q(\theta)D - (q(\theta)^2 D_1' D_1)$$

and

$$C\underline{w} \xrightarrow{d} N_{k-1}(q(\theta)CD_1', C[q(\theta)D - [q(\theta)]^2 D_1' D_1]C')$$

Where C is $(k-1)\times k$ contrast matrix and let it be

$$C = \begin{pmatrix} 1\ 0\ 0\ \dots \dots 0 & -1 \\ 0\ 1\ 0\ \dots \dots 0 & -1 \\ \dots \dots \dots \dots \dots \dots \dots & \\ 0\ 0\ 0\ \dots \dots 1 & -1 \end{pmatrix}$$

As

$$H{:}\, \theta_i = \theta_k \quad \Leftrightarrow \frac{\partial}{\partial\theta_i}[q(\theta)]^{-1} = \frac{\partial}{\partial\theta_k}[q(\theta)]^{-1}, i = 1,2, .. k-1$$

$$\Leftrightarrow q(\theta)\frac{\partial}{\partial\theta_i}[q(\theta)]^{-1} = q(\theta)\frac{\partial}{\partial\theta_k}[q(\theta)]^{-1}$$

$$\Leftrightarrow\ q(\theta)CD_1' = 0, \text{ say } H_1$$

Thus $H{:}\, \theta_1 = \theta_2 = \dots = \theta_k \Leftrightarrow H_1{:}\, q(\theta)CD_1' = 0$ where $\theta = (\theta_1, \theta_2, \dots . \theta_k)$ and for testing H_1, we need the distribution of $C\underline{w}$ which is asymptotically $N_{k-1}(q(\theta)CD_1', C\Sigma C^1)$. By the usual multivariate theory of testing the mean vector of multivariate normal distribution equal to zero vector is to reject H_1, in this case, whenever

$$\frac{(n-1)-(k-1)+1}{k-1}\left(C\underline{\overline{w}}\right)'\left(C\sum C'\right)^{-1}\left(C\underline{\overline{w}}\right) \geq F_{k-1,\ n-k+1\,;\,\alpha}$$

Where $F_{k-1,\ n-k+1\,;\,\alpha}$ is upper α % point of Snedekar F-distribution with $(k-1,\ n-k+1)$ d.f. Since the test depends on normal variates, the text book theory of testing equality of location parameters of k normal variates, the test turns out as UMPI test and hence in this case also the test for testing equality of parameters of k exponential family is asymptotic UMPI test.

The computation of quadratic form $\left(C\underline{\overline{w}}\right)'(C\sum C')^{-1}\left(C\underline{\overline{w}}\right)$ can be simplified as under. When H is true we have

(i) $C\underline{\overline{w}} = (\overline{w}_1 - \overline{w}_k, \overline{w}_2 - \overline{w}_k, \dots, \overline{w}_{k-1} - \overline{w}_k)$

(ii) $C' = I_{k-1} + \underline{1}\,\underline{1}'$, where $\underline{1}$ is $(k-1)$ vector of unity

(iii) $(CwC')^{-1} = I_{k-1} + \frac{1}{k}\underline{1}\,\underline{1}'$

Using above three results in the quadratic form and simplifying one gets

$$\left(C\underline{\overline{w}}\right)'(C\sum C')^{-1}\left(C\underline{\overline{w}}\right) = \sum_{i=1}^{k}\overline{w}_i^{\ 2} - \frac{1}{k}[(k-1)\overline{\overline{w}} + \overline{w}_k]^2,\ \text{Where } \overline{\overline{w}} = \frac{1}{k}\sum_{i=1}^{k}\overline{w}_i. \qquad (2.5)$$

Particularly when $k = 2$, $c = (1 \quad -1)$, $C\sum C' = 2$, $C\underline{\overline{w}} = (\overline{w}_1 - \overline{w}_2)$ $\left(C\underline{\overline{w}}\right)'(C\sum C')^{-1}\left(C\underline{\overline{w}}\right) = \frac{(\overline{w}_1-\overline{w}_2)^2}{2}$.

As $\frac{\overline{w}_1-\overline{w}_2}{\sqrt{2}} \xrightarrow{d} N(0,1)$,

$\left(C\underline{\overline{w}}\right)'(C\sum C')^{-1}\left(C\underline{\overline{w}}\right) \xrightarrow{d} \chi_1^{\ 2}$.

Thus we reject H at level α when

$$(n-2)\left(C\underline{\overline{w}}\right)'(C\sum C')^{-1}\left(C\underline{\overline{w}}\right) = \frac{(n-2)}{2}(\overline{w}_1 - \overline{w}_2)^2 \geq (n-2)\ \chi^2_{\ 1;\alpha}$$

Where $\chi^2_{\ 1;\alpha}$ is upper α% point of χ^2 on one d.f.

III. Applications to some specific distributions

(i) Binomial distribution

Consider K independently distributed binomial distribution with p.f.

$$P(X_i = x_i; p_i) = \binom{n}{x_i} p_i^{\,x_i}(1-p_i)^{n-x_i},\quad i = 1,2,\dots.k,\ \ x_i = 1,2,\dots.n$$
$$= (1-p_i)^n \binom{n}{x_i} e^{x_i log\frac{p_i}{1-p_i}}$$
$$= a(p_i)b(x_i)e^{x_i log\frac{p_i}{1-p_i}},\ \text{where } a(p_i) = (1-p_i)^n,\ b(x_i) = \binom{n}{x_i}$$

Making the naturalization of the family by taking $log\,\frac{p_i}{1-p_i} = \theta_i$, that is,

$p_i = \frac{e^{\theta_i}}{1+e^{\theta_i}}$, $a(p_i) = (1-p_i)^n = (1+e^{\theta_i})^{-n} = q(\theta_i)$, say and using the results (2.3) and (2.4) and simplifying them we get

$w_i = \sum_{i=1}^{n} x_{ij}$, $i = 1,2,\dots.k$, $E(w_i) = np_i$, $Var(w_i) = np_i(1-p_i)$, $i = 1,2,\dots.k$

$Cov(w_i, w_j) = 0$ for $i \neq j$.

$\underline{w} \xrightarrow{d} N_k(\underline{\mu}, \Sigma)$ where $\underline{\mu} = (np_1, np_2, \dots, np_k)'$,

$\Sigma = \text{diag}(np_1(1-p_1), np_2(1-p_2), \dots, np_k(1-p_k))$

and

$C\underline{w} \xrightarrow{d} N_k(C\underline{\mu},\ C\Sigma C')$, where C is any $(k-1)\times k$ constant matrix such that

$$C = \begin{pmatrix} 1\ 0\ 0\ \dots\dots 0 & -1 \\ 0\ 1\ 0\ \dots\dots 0 & -1 \\ \dots\dots\dots\dots\dots\dots\dots & \\ 0\ 0\ 0\ \dots\dots 1 & -1 \end{pmatrix}$$

The asymptotic UMPI level-α test for testing $H: p_1 = p_2 = \cdots = p_k$ is given by (2.2) and the quadratic form involved in it can be computed by using (2.5), in which

$w_i = \sum_{i=1}^{n} x_{ij}$, $i = 1,2, \dots .k$, $\overline{w}_i = \frac{w_i}{n}$, $i = 1,2, \dots .k$, $\overline{\overline{w}} = \frac{1}{nk}\sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}$.

(ii) Poisson distribution

Let $X_1, X_2, \dots, X_k$ be independently distributed as Poisson with parameters $\lambda_1, \lambda_2, \dots, \lambda_k$. Then the joint p.f. is

$$P(X_1 = x_1, X_2 = x_2, \dots, X_k = x_k;\ \lambda_1, \lambda_2, \dots, \lambda_k) = \frac{e^{-\sum\lambda_i}\prod_{i=1}^{k}\lambda_i^{x_i}}{x_1!\,x_2!\,\dots .x_k!}$$

$$= e^{-\sum\lambda_i}.\frac{1}{x_1!x_2!\dots .x_k!}.e^{\sum x_i log\lambda_i}$$

$$= g(\lambda)h(x)e^{\sum x_i\theta_i}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, $x = (x_1, x_2, \dots, x_k)$

by naturalizing this family by taking $\theta_i = log\lambda_i$, $i = 1, 2, \dots, k$, the joint p.f. is rewritten as

$$P_W(w;\theta) = a(\theta)h(w_i)e^{\sum\theta_i w_i},\ \ \theta = (\theta_1, \theta_2, \dots, \theta_k), w = (w_1, w_2, \dots, w_k)$$

where $a(\theta) = e^{-\sum_{i=1}^{k} e^{\theta_i}} = q(\theta)$ as $\lambda_i = e^{\theta_i}$, $i = 1, 2, \dots, k$

and $Cov(w_i, w_j) = 0$ for $i \neq j$, the variance covariance matrix, Σ of $w_1, w_2, \dots, w_k$ will be

$$\Sigma = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$$

By multivariate central limit theorem,

$\underline{w} \xrightarrow{d} N_k(\underline{\lambda}, \Sigma)$ where $\underline{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_k)$, $\Sigma = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$

Using $(k-1) \times k$ constant matrix C as

$$C = \begin{pmatrix} 1\ 0\ 0\ \dots\dots 0 & -1 \\ 0\ 1\ 0\ \dots\dots 0 & -1 \\ \dots\dots\dots\dots\dots\dots\dots & \\ 0\ 0\ 0\ \dots\dots 1 & -1 \end{pmatrix}$$

$C\underline{w} \xrightarrow{d} N_{k-1}(C\underline{\lambda}, C\Sigma C')$

The asymptotic UMPI level-α test for testing $H: \lambda_1 = \lambda_2 = \cdots = \lambda_k$ is given by (2.2) and the quadratic form involved in it can be computed by using (2.5) in which

$w_i = \sum_{i=1}^{n} x_{ij}$, $i = 1,2, \dots .k$, $\overline{w}_i = \frac{w_i}{n}$, $i = 1,2, \dots .k$, $\overline{\overline{w}} = \frac{1}{nk}\sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}$.

(iii) Exponential distribution

Let $X_i (i = 1,2, \dots, k)$ be k independently distributed exponential distributions with p.d.f.

$$f_{X_i}(x_i, \lambda_i) = \lambda_i e^{-\lambda_i x_i},\ \ x_i > 0, \lambda_i > 0,\ \ i = 1,2, \dots, k$$

$$= a(\lambda_i)h(x)\ e^{-\lambda_i x_i}$$

where $a(\lambda_i) = \lambda_i$, $h(x) = 1$

For a random sample $X_{i1}, X_{i2}, \dots, X_{in} (i = 1,2, \dots k)$ on X_i

let $w_i = \sum_{j=1}^{n} x_{ij}$, $i = 1,2, \dots .k$.

Using results (2.3) and (2.4);

$E(w_i) = -\lambda_i \frac{\partial}{\partial \lambda_i}\left(\frac{1}{\lambda_i}\right) = \frac{1}{\lambda_i}, i = 1,2, \dots, k$

$Var(w_i) = \frac{1}{\lambda_i^2}, , i = 1,2, \dots, k,\ Cov(w_i, w_j) = 0$ for $i \neq j$

$\underline{w} = (w_1, w_2, \dots, w_k) \xrightarrow{d} N_k(\underline{\lambda}, \Sigma)$ where $\underline{\lambda} = (\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_k})$,

$\Sigma = \text{diag}(\frac{1}{\lambda_1^2}, \frac{1}{\lambda_2^2}, \dots, \frac{1}{\lambda_k^2})$

Under $H: \lambda_i = \lambda_2 = \cdots \lambda_k$, $\underline{\lambda} = \frac{1}{\lambda_1}(1,1, \dots, 1)' = \frac{1}{\lambda_1}1'$, $\Sigma = \frac{1}{\lambda_1^2} I_k$ and

$\underline{w} \xrightarrow{d} N_k(\frac{1}{\lambda_1}1, \frac{1}{\lambda_1^2} I_k)$

Also under H,

$C\underline{w} \xrightarrow{d} N_{k-1}(\frac{1}{\lambda_1} C\underline{1}, \frac{1}{\lambda_1^2} CC')$

The asymptotic UMPI level-α test is given by (2.2) and the quadratic form involved in it can be computed by using (2.5) in which

$w_i = \sum_{i=1}^{n} x_{ij}, i = 1,2, \dots . k, \overline{w}_i = \frac{w_i}{n}, i = 1,2, \dots . k, \overline{\overline{w}} = \frac{1}{nk}\sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}$.

(iv) Gamma distribution

Let $X_i (i = 1,2, \dots, k)$ be k independently distributed as gamma with p.d.f. as

$$f_{X_i}(x_i; \theta_i) = \frac{1}{\Gamma(\theta_i + 1)} x_i^{\theta_i} e^{-x_i}, x_i > 0, \theta_i > 0,\ i = 1,2, \dots, k$$

$$= a(\theta_i). e^{-x_i}. e^{\theta_i log x_i}, \text{ where } a(\theta_i) = \frac{1}{\Gamma(\theta_i + 1)}$$

For a random sample $X_{i1}, X_{i2}, \dots, X_{in} (i = 1,2, \dots k)$ from X_i

let $w_i = \sum_{j=1}^{n} x_{ij}, i = 1,2, \dots . k$.

Using results (2.3) and (2.4);

$E(w_i) = \frac{1}{\Gamma(\theta_i+1)} \frac{\partial}{\partial \theta_i}(\Gamma(\theta_i + 1)), i = 1,2, \dots, k$ and

$Var(w_i) = \frac{1}{\Gamma(\theta_i+1)} \frac{\partial^2}{\partial \theta_i^2}(\Gamma(\theta_i + 1)) - \left(\frac{1}{\Gamma(\theta_i+1)} \frac{\partial}{\partial \theta_i}(\Gamma(\theta_i + 1))\right)^2, i = 1,2, \dots, k,$

$E(w_i) = d_i$ and $Var(w_i) = d_{ii} - (d_i)^2, i = 1,2, \dots, k$ by using

$d_i = \frac{1}{\Gamma(\theta_i+1)} \frac{\partial}{\partial \theta_i}(\Gamma(\theta_i + 1)), d_{ii} = \frac{1}{\Gamma(\theta_i+1)} \frac{\partial^2}{\partial \theta_i^2}(\Gamma(\theta_i + 1))$

Since $Cov(w_i, w_j) = 0$ for $i \neq j$, the variance-c0variance matrix, Σ of $w_1, w_2, \dots, w_k$ will be

$$\Sigma = \text{diag}(d_{11}, d_{22}, \dots, d_{kk}) - diag(d_1^2, d_2^2, \dots, d_k^2)$$

$$= D - D_1 D_1',$$

where $D = \text{diag}(d_{11}, d_{22}, \dots, d_{kk})$, $D_1 = (d_1, d_2, \dots, d_k)'$

under the hypothesis $H: \theta_1 = \theta_2 = \cdots = \theta_k \Leftrightarrow d_1 = d_2 = \cdots = d_k \Leftrightarrow CD_1' = 0$

WHERE

$$C = \begin{pmatrix} 1\ 0\ 0\ \ldots\ldots 0 & -1 \\ 0\ 1\ 0\ \ldots\ldots 0 & -1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ 0\ 0\ 0\ \ldots\ldots 1 & -1 \end{pmatrix}$$

is $(k-1)\times k$ contrast matrix.

By multivariate central limit theorem, $(w_1, w_2, \ldots, w_k)$ is asymptotic distributed as k-variate normal with mean vector D_1 and variance-covariance matrix $\Sigma = D - D_1 {D_1}'$, that is

$$\underline{w} \xrightarrow{d} N_k(D_1, \Sigma) \text{ and } C\underline{w} \xrightarrow{d} N_{k-1}(CD_1,\ C\Sigma C')$$

when H is true, $C\underline{w} \xrightarrow{d} N_{k-1}(0,\ C\Sigma C')$

The asymptotic UMPI level-α test is given by (2.2) and the quadratic form involved in it can be computed by using (2.5) in which

$w_i = \sum_{i=1}^{n} x_{ij}$, $i = 1,2, \ldots .k$, $\overline{w}_i = \frac{w_i}{n}$, $i = 1,2, \ldots .k$, $\overline{\overline{w}} = \frac{1}{nk}\sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}$.

IV. CONCLUSION

No UMPU is available for testing equality of parameters of several exponential family of distributions. For some of them invariant tests have been tried but in the class of invariant tests, no UMP test have been established for many of them. So in this paper asymptotically UMPI test for testing the equality of parameters of K family of exponential distributions has been obtained. If someone consider other than invariance restriction on the test and develop the UMP test in this restricted class, it will enhance the future research work.

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