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# Matrix Exponential of Secondary Type Matrices

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***Abstract***— In this paper, we presented the idea of matrix exponential of secondary type matrices. In particular secondary unitary matrices (s-unitary matrices).

***Keywords***—*Secondary transpose, s-unitary matrices, s-hermitian matrices, matrix exponential.*

## I. INTRODUCTION AND PRELIMINARIES

The investigation of Secondary symmetric and Secondary Orthogonal Matrices was started by Anna Lee [1] and [2]. In this paper, we presented the idea of matrix exponential of secondary type matrices. In particular s-unitary matrices. We signify the space of $n \times n$ Complex matrices by $C_{n\times n}$. The Secondary transpose of A is characterized by $A^S = VA^T V$ and $A^\Theta = VA^* V$, where 'V' is the settled disjoint stage matrix with units in its optional askew. In Mathematics, the framework exponential is a capacity on square networks undifferentiated from the conventional exponential capacity. Uniquely, the grid exponential gives the association between framework Lie polynomial math and the relating Lie gathering. Let X is $n \times n$ real or complex matrix. The matrix exponential of X signified by $e^X$ or exp(x) is the $n \times n$ matrix given by the power arrangement $e^X = \sum_{K=0}^{\infty} \frac{X^K}{K!}$. The above arrangement dependably focalizes, so the exponential of X is all around characterized. In the event that X is $1\times 1$ a grid the lattice exponential of X compared with the normal exponential of X thought of as a number.

**Definition 1.1 [4].** Let $A \in C_{n\times n}$

(a). The matrix A is called s-hermitian if $A^\theta = A$.
(b). The matrix A is called s-skew-hermitian if $A^\theta = -A$.
(c). The matrix A is called s-orthogonal if $A^S = A^{-1}$.
(d). The matrix A is called s-unitary if $A^\theta = A^{-1}$.

**Definition 1.2 [3].** The geometric mean and spectral means of positive definite matrices A and B are defined by $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$, $A \bullet B = (A^{-1}\#B)^{1/2} A (A^{-1}\#B)^{1/2}$.

### 1.1 Properties of matrix exponential [4]

Let X and Y be $n \times n$ complex matrices and let a and b be an arbitrary number. We denote the n x n identity matrix by I and the zero matrix by $\vartheta$. The matrix exponential satisfies the following properties

a. $e^0 = I$
b. $e^{aX+bX} = e^{(a+b)X}$
c. $e^X e^{-X} = I$
d. If Y is invertible then $e^{YXY^{-1}} = Ye^X Y^{-1}$
e. det ( $e^X$ ) = $e^{tr(X)}$
f. exp ( $X^T$ ) = ( $e^X$ )$^T$ and exp ( $X^s$ ) = ( $e^X$ )$^s$

 

g. exp ( $X^*$ ) = ( $e^X$ )$^*$ and exp ( $X^\Theta$ ) = ( $e^X$ )$^\Theta$

## II. MAIN RESELTS

**Theorem 2.1.** For s-hermitian n by n matrices X and Y, there exist s-unitary matrices U and V such that $e^{X/2}e^Y e^{X/2} = e^{UXU^\Theta + VYV^\Theta}$ or equivalently $e^{X+VYV^\Theta} = U(e^{X/2}e^Y e^{X/2})U^\Theta$.

**Theorem 2.2.** Let X and Y be a s-hermitian matrices and there exist a s-unitary matrices $U_i$ and $V_i$ such that $e^{2X} \# e^{2Y} = e^{U_1XU_1^\Theta + V_1YV_1^\Theta}$ and $e^{2X} \bullet e^{2Y} = e^{U_2XU_2^\Theta + V_2YV_2^\Theta}$. In particular for $p > 0$,

$$Tr\ e^{X+UYU^\Theta} = Tr\left(e^{pX} \# e^{pY}\right)^{2/p} \le Tr\ e^{X+Y} \le Tr\left(e^{pX} \bullet e^{pY}\right) = Tr\ e^{X+VYV^\Theta}$$

for some s-unitary matrices $U$ and $V$, depending on p and X, Y.

*Proof.* We consider the matrix exponential equation of the geometric and spectral geometric means $e^{2X} \# e^{2Y} = e^Z$ and $e^{2X} \bullet e^{2Y} = e^W$. By Riccati lemma,

$$e^Z e^{-2X} e^Z = e^{2Y}$$

Now $e^{X/2}e^Y e^{X/2} = e^{UXU^\Theta + VYV^\Theta}$, there exist s-unitary matrices U and V such that

$$e^Z e^{-2X} e^Z = e^{2Z/2} e^{-2X} e^{2Z/2}$$
$$= e^{2UXU^\Theta - 2VYV^\Theta}$$

Since the exponential map on the space of s-hermitian matrices is bijective onto the convex cone of positive definite matrices, we have

$2UZU^\Theta - 2VZV^\Theta = 2Y \Rightarrow UZU^\Theta - VZV^\Theta = Y$ and hence $UZU^\Theta = VZV^\Theta + Y$ or

$$Z = U^\Theta\left(VXV^\Theta + Y\right)U$$

$$Z = U^\Theta\left(VXV^\Theta\right)U + U^\Theta YU$$

$$Z = \left(U^\Theta V\right)X\left(V^\Theta U\right) + U^\Theta YU$$

$$Z = \left(U^\Theta V\right)X\left(U^\Theta V\right)^\Theta + U^\Theta YU$$

$$Z = WXW^\Theta + U^\Theta YU$$

where $W = U^\Theta V$. For the spectral geometric mean, $e^W = e^{2X} \bullet e^{2Y}$. We apply

$$A \bullet B = U\left(A^{1/2}BA^{1/2}\right)^{1/2} U^\Theta$$

$$e^W = e^{2X} \bullet e^{2Y}$$
$$= U\left(e^{2X/2}e^{2Y}e^{2X/2}\right)^{1/2} U^\Theta$$
$$= U\left(e^X e^{2Y} e^X\right)^{1/2} U^\Theta \text{ for some s-unitary}$$
$$= U(e^{X/2}e^Y e^{X/2})U^\Theta$$
$$= Ue^{V_1XV_1^\Theta + V_2YV_2^\Theta} U^\Theta$$
$$= e^{U_1XU_1^\Theta + U_2YU_2^\Theta}$$

for some s-unitary, and $W = U_1XU_1^\Theta + U_2YU_2^\Theta$

$$e^{2X} \# e^{2Y} = e^{U_1XU_1^\Theta + UYU^\Theta}$$

$$Tr\left(e^{U_1XU_1^\Theta + UYU^\Theta}\right) = Tr\left(e^{2X} \# e^{2Y}\right)$$

$$Tr\ e^{X+UYU^\Theta} = Tr\left(e^{pX} \# e^{pY}\right)^{2/p}$$

By known theorem, $Tr\left(e^{pX} \# e^{pY}\right)^{2/p} \le Tre^{X+Y}$

$$\le Tr(e^{pX/2}e^{pY}e^{pX/2})^{1/p}$$

 

Since $e^{pX} \bullet e^{pY}$ is similar to $(e^{pX/2}e^{pY}e^{pX/2})^{1/2}$ , $(e^{pX} \bullet e^{pY})^{2/p}$ is similar to $(e^{pX/2}e^{pY}e^{pX/2})^{1/p}$. Therefore $Tr(e^{pX/2}e^{pY}e^{pX/2})^{1/p} = Tr(e^{pX} \bullet e^{pY})^{2/p}$ for every $p > 0$

$$Tre^{X+UYU^{\Theta}} = Tr\left(e^{pX} \# e^{pY}\right)^{2/p} \le Tre^{X+Y} \le Tr\left(e^{pX} \bullet e^{pY}\right) = Tre^{X+VYV^{\Theta}}$$

**Proposition 2.3.** Let A and B be positive definite matrices. Then $(MAM^{\Theta})\#(MBM^{\Theta}) = M(A\#B)M^{\Theta}$ and $(UAU^{\Theta}) \bullet (UBU^{\Theta}) = U(A \bullet B)U^{\Theta}$ for any invertible matrix M and $U \in U(n)$.

*Proof.* $(MAM^{\Theta})\#(MBM^{\Theta}) = M(A\#B)M^{\Theta}$. Now $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$, where $A = MAM^{\Theta}, B = MBM^{\Theta}$

$$\begin{aligned}(MAM^{\Theta})\#(MBM^{\Theta}) &= (MAM^{\Theta})^{1/2}[(MAM^{\Theta})^{-1/2}(MBM^{\Theta})(MAM^{\Theta})^{-1/2}]^{1/2}(MAM^{\Theta})^{1/2}\\
&= M^{1/2}A^{1/2}(M^{\Theta})^{1/2}\{[M^{-1/2}A^{-1/2}(M^{\Theta})^{-1/2}]MBM^{\Theta}[M^{-1/2}A^{-1/2}(M^{\Theta})^{-1/2}]\}^{1/2}M^{1/2}A^{1/2}(M^{\Theta})^{1/2}\\
&= M^{1/2}A^{1/2}(M^{\Theta})^{1/2}[M^{-1/4}A^{-1/4}(M^{\Theta})^{-1/4}M^{1/2}B^{1/2}(M^{\Theta})^{1/2}M^{-1/4}A^{-1/4}(M^{\Theta})^{-1/4}]M^{1/2}A^{1/2}(M^{\Theta})^{1/2}\\
&= M^{(2-1+2-1+2)/4}A^{1/2}[A^{-1/4}B^{1/2}A^{-1/4}]A^{1/2}(M^{\Theta})^{(2-1+2-1+2)/4}\\
&= MA^{1/2}[A^{-1/2}BA^{-1/2}]^{1/2}A^{1/2}M^{\Theta}\end{aligned}$$

$(MAM^{\Theta})\#(MBM^{\Theta}) = M(A\#B)M^{\Theta}$. Now

$A \bullet B = (A^{-1}\#B)^{1/2}A(A^{-1}\#B)^{1/2}$, where $A = UAU^{\Theta}, B = UBU^{\Theta}$

$$(UAU^{\Theta}) \bullet (UBU^{\Theta}) = [(UAU^{\Theta})^{-1}\#(UBU^{\Theta})]^{1/2}(UAU^{\Theta})[(UAU^{\Theta})^{-1}\#(UBU^{\Theta})]^{1/2}$$

$$\begin{aligned}(UAU^{\Theta})^{-1}\#(UBU^{\Theta}) &= [(UAU^{\Theta})^{-1}]^{1/2}\{[(UAU^{\Theta})^{-1}]^{-1/2}(UBU^{\Theta})[(UAU^{\Theta})^{-1}]^{-1/2}\}[(UAU^{\Theta})^{-1}]^{1/2}\\
&= [(U^{\Theta})^{-1}A^{-1}U^{-1}]^{1/2}\{[(U^{\Theta})^{-1}A^{-1}U^{-1})^{-1/2}(UBU^{\Theta})[(U^{\Theta})^{-1}A^{-1}U^{-1})^{-1/2}\}^{1/2}[(U^{\Theta})^{-1}A^{-1}U^{-1}]^{1/2}\\
&= (U^{\Theta})^{-1/2}A^{-1/2}U^{-1/2}[(U^{\Theta})^{1/2}A^{1/2}(U^{1/2}U)B(U^{\Theta}(U^{\Theta})^{1/2})A^{1/2}U^{1/2}]^{1/2}(U^{\Theta})^{-1/2}A^{-1/2}U^{-1/2}\\
&= (U^{\Theta})^{-1/2}A^{-1/2}U^{-1/2}[(U^{\Theta})^{1/2}A^{1/2}U^{3/2}B(U^{\Theta})^{3/2}A^{1/2}U^{1/2}]^{1/2}(U^{\Theta})^{-1/2}A^{-1/2}U^{-1/2}\\
&= (U^{\Theta})^{-1/2}A^{-1/2}U^{-1/2}(U^{\Theta})^{1/4}U^{3/4}[A^{1/2}BA^{1/2}]^{1/2}(U^{\Theta})^{3/4}U^{1/4}(U^{\Theta})^{-1/2}A^{-1/2}U^{-1/2}\\
&= (U^{\Theta})^{-1/4}A^{-1/2}U^{1/4}[A^{1/2}BA^{1/2}]^{1/2}(U^{\Theta})^{-1/4}A^{-1/2}U^{1/4}\\
&= A^{-1/2}[A^{1/2}BA^{1/2}]^{1/2}A^{-1/2}\\
&= (A^{-1})^{1/2}[(A^{-1})^{-1/2}B(A^{-1})^{1/2}]^{1/2}(A^{-1})^{1/2}\\
&= (A^{-1}\#B)^{1/2}\end{aligned}$$

$$\begin{aligned}(UAU^{\Theta}) \bullet (UBU^{\Theta}) &= (A^{-1}\#B)^{1/2}(UAU^{\Theta})(A^{-1}\#B)^{1/2}\\
&= U[(A^{-1}\#B)^{1/2}A(A^{-1}\#B)^{1/2}]U^{\Theta}\end{aligned}$$

$$(UAU^{\Theta}) \bullet (UBU^{\Theta}) = U(A \bullet B)U^{\Theta}$$

**Theorem 2.4.** Let A and B be positive definite matrices. Then $A \bullet B = U(A^{1/2}BA^{1/2})^{1/2}U^{\Theta}$.

*Proof.* $X^{\Theta}X$ and $XX^{\Theta}$ are s-unitarily similar for any invertible matrix X. Setting $X = A^{1/2}(A^{-1}\#B)^{1/2}$ there exists a s-unitary matrix U such that

$$\begin{aligned}A \bullet B &= X^{\Theta}X\\
&= UXX^{\Theta}U^{\Theta}\\
&= UA^{1/2}(A^{-1}\#B)^{1/2}(A^{-1}\#B)^{1/2}A^{1/2}U^{\Theta}\\
&= UA^{1/2}\{(A^{-1})^{1/2}[(A^{-1})^{-1/2}B(A^{-1})^{1/2}]^{1/2}(A^{-1})^{1/2}\}^{1/2}\{(A^{-1})^{1/2}[(A^{-1})^{-1/2}B(A^{-1})^{1/2}]^{1/2}(A^{-1})^{1/2}\}^{1/2}A^{1/2}U^{\Theta}\\
&= UA^{1/2}\{A^{-1/2}[A^{1/2}BA^{1/2}]^{1/2}A^{-1/2}\}^{1/2}\{A^{-1/2}[A^{1/2}BA^{1/2}]^{1/2}A^{-1/2}\}^{1/2}A^{1/2}U^{\Theta}\\
&= UA^{1/2}\{A^{-1/4}[A^{1/4}B^{1/2}A^{1/4}]^{1/2}A^{-1/4}\}\{A^{-1/4}[A^{1/4}B^{1/2}A^{1/4}]^{1/2}A^{-1/4}\}A^{1/2}U^{\Theta}\\
&= UA^{1/2}\{[(A^{-1/4}A^{1/8})B^{1/4}(A^{1/8}A^{-1/4})][(A^{-1/4}A^{1/8})B^{1/4}(A^{1/8}A^{-1/4})]\}A^{1/2}U^{\Theta}\end{aligned}$$

 

$$=UA^{1/2}\{[A^{-1/8}B^{1/4}A^{-1/8}][A^{-1/8}B^{1/4}A^{-1/8}]\}A^{1/2}U^{\Theta}$$
$$=U\{[A^{1/2}A^{-1/8}B^{1/4}A^{-1/8}][A^{-1/8}B^{1/4}A^{-1/8}A^{1/2}]\}U^{\Theta}$$
$$=U\{[A^{3/8}B^{1/4}A^{-1/8}][A^{-1/8}B^{1/4}A^{3/8}]\}U^{\Theta}$$
$$=U\{[A^{3/8}A^{-1/8}B^{1/4}][B^{1/4}A^{-1/8}A^{3/8}]\}U^{\Theta}$$
$$=U\{A^{1/4}(B^{1/4}B^{1/4})A^{1/4}\}U^{\Theta}$$
$$=U\{A^{1/4}B^{1/2}A^{1/4}\}U^{\Theta}$$
$$A\bullet B=U(A^{1/2}BA^{1/2})^{1/2}U^{\Theta}$$

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